THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

EDITED BY

ELTON JAMES MOULTON, Editor-in-Chief

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINK[illegible] WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 [illegible]WNED AND PUBLISHED BY REPRESENTATIVES OF [illegible]EEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 47

1940

PUBLISHED BY THE ASSOCIATION

MENASHA, WIS., AND EVANSTON, ILL.

OVER AND UNDER FUNCTIONS AS RELATED TO DIFFERENTIAL EQUATIONS*

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This paper discusses certain classes of functions that have been called *Over* and *Under* functions and which have had extensive use in recent work on differential equations. The underlying fundamental principles for this work go back at least as far as the formulation of the Dedekind-Cut postulate. Here, as in the Dedekind-Cut, one describes two non-empty sets of elements which, with reference to a specified relation of order, have the property that each element in one set is preceded by every element in the other. The existence of a least upper bound for the one set and a greatest lower bound for the other is then postulated or proved. These bounds are examined for new and useful properties. The principles and procedures here indicated have been used so often in all fields of mathematics that it is difficult to select a specific place in the literature for use as a starting point for a discussion. However, the particular topics that we wish to describe can very appropriately be introduced by a sketch of the definition of the Perron integral. This integral was defined by Perron† in 1914 and has been studied extensively since that time. His work was preceded by De la Vallée-Poussin's proof‡ of an important theorem on the relation of the Lebesgue integral to *Over* and *Under* functions. In 1915, Perron§ made use of *Over* and *Under* functions to establish fundamental existence theorems for ordinary differential equations of the type $y'=f(x, y)$. Müller,|| and others have extended this work to systems of ordinary differential equations. In 1923, Perron¶ used *Over* and *Under* functions in a new and fruitful attack on boundary value problems of the Dirichlet type for partial differential equations.** Recently, F. Riesz, Radó, and others have made extensive studies of *super* and *sub* harmonic functions.†† These functions are similar to the *Over* and *Under* functions of Perron and play important rôles in potential theory.‡‡ In the following sections of this paper, we describe the *Over* and *Under* functions that arise in the above mentioned

* Read before the Association at Madison on September 4, 1939, by invitation of the Program Committee.

† Sitzungsbereichte der Heidelberger Akademie der Wissenschaften, Abt. A, 1914, 14 Abh.

‡ Cours d'analyse infinitésimale, third edition, Paris, 1914, vol. 1, pp. 269–271.

§ Mathematische Annalen, vol. 76, 1915, pp. 471–484.

|| Mathematische Zeitschrift, vol. 26, 1927, pp. 619–645.

¶ Mathematische Zeitschrift, vol. 18, 1923, pp. 42–54.

** The Dirichlet problem is to find a function $U(x, y)$ which is continuous together with its partial derivatives of first and second orders in a domain G and on its boundary R, satisfies Laplace's differential equation $\partial^2U/\partial x^2+\partial^2U/\partial y^2=0$ in G, and is equal to a given continuous function $f(x, y)$ on the boundary R.

†† For a treatment of these functions and references to the literature, see Radó, Ergebnisse der Mathematik, vol. 5, 1937, pp. 1–56.

‡‡ See a paper by Carathéodory, American Journal of Mathematics, vol. 59, 1937, pp. 709–731; Radó's article (*loc. cit.*); and recent works of Kellogg, Evans, and others.

applications and indicate how these are used to secure many of the important results. In each case, our emphasis is placed on the differential equations that are involved.

1. Equations of type $y'=f(x)$. Let $f(x)$ be a real function of the real variable x that is defined on the interval $(a, b): a \leqq x \leqq b$. Infinite values for $f(x)$ are not excluded. By an *Under function of* $f(x)$ *on* (a, b), is meant any function $\phi(x)$ which is defined and satisfies the four conditions:
1) $\phi(x)$ is finite and continuous on (a, b),
2) $\phi(a)=0$,

3) $\overline{D}\phi(x) = \text{Upper derivative of } \phi(x) = \text{Upper} \lim\limits_{h\to 0} \dfrac{\phi(x+h)-\phi(x)}{h} \neq +\infty$ on (a, b),

4) $\overline{D}\phi(x) \leqq f(x)$ on (a, b).

Similarly, an *Over function of* $f(x)$ *on* (a, b) is any function $\psi(x)$ which satisfies:

1) $\psi(x)$ is finite and continuous on (a, b),
2) $\psi(a)=0$,

3) $\underline{D}\psi(x) = \text{Lower derivative of } \psi(x) = \text{Lower} \lim\limits_{h\to 0} \dfrac{\psi(x+h)-\psi(x)}{h} \neq -\infty$ on (a, b),

4) $\underline{D}\psi(x) \geqq f(x)$ on (a, b).

If a function $f(x)$ has both *Over* and *Under* functions, it can be shown that the relation

$$\phi(x) \leqq \psi(x) \tag{1}$$

holds between any *Under* function $\phi(x)$ and any *Over* function $\psi(x)$. It then follows that for such a function $f(x)$, the collection of all *Under* functions is bounded above and hence has a finite upper bound function $g(x)$. Similarly, the collection of all *Over* functions is bounded below and has a finite lower bound function $G(x)$. It follows from (1) that

$$g(x) \leqq G(x). \tag{2}$$

The quantity $g(b)$ is called the *lower* Perron integral of $f(x)$ on (a, b). Similarly, $G(b)$ is called the *upper* Perron integral of $f(x)$ on (a, b). If $g(b)=G(b)$, $f(x)$ is said to be Perron integrable on (a, b) and its Perron integral, $\int_a^b f(x)dx$, is the common value of $g(b)$ and $G(b)$.

The elementary properties of Perron integrals are well known and are essentially the same as those of Riemann and Lebesgue integrals.* For bounded functions, the Perron and Lebesgue integrals are the same. That is to say, when one exists, the other also exists and the two are equal in value. For unbounded

* For a treatment of the Perron integral, we may refer to Kamke, Das Lebesguesche Integral, Teubner, Leipzig, 1925, pp. 126–151.

functions, this identity with the Lebesgue integral is not valid. The Perron and Denjoy integrals are identical in all cases except for mode of definition. The properties of Perron integrals are not developed here, as the purpose of this paper is not this so much as it is to indicate broad uses of *Over* and *Under* functions in connection with differential equations.

It should be observed that *Over* and *Under* functions do not exist for all functions $f(x)$. For example:

$$f(x) = 1/(1 - x), \qquad x \neq 1, \qquad f(1) = 0,$$

has $\phi(x) \equiv 0$ for an *Under* function on (0, 1) but has no *Over* function on this interval. [Any *Over* function $\psi(x)$ would have

$$\underline{D}\psi(x) \geqq 1/(1 - x) \geqq 1/(1 - h), \qquad 0 \leqq h < 1, \qquad h \leqq x < 1.$$

A theorem of Kamke's* states that if $\psi(x)$ is finite and continuous on (a, b) and $\underline{D}\psi(x) \geqq c$, a constant, on (a, b) then

$$\psi(x_1) - \psi(x_2) \geqq c(x_1 - x_2) \quad \text{for} \quad a \leqq x_1 \leqq x_2 \leqq b.$$

Application of this theorem yields

$$\psi\left(\frac{1 + h}{2}\right) - \psi(h) \geqq \frac{(1 - h)}{2(1 - h)} \geqq \frac{1}{2} \cdot$$

Since $1-h$ can be arbitrarily small, $\psi(x)$ could not be continuous at $x=1$.] This function has *Over* functions on (1, 2) but not *Under* functions on this interval. Hence on the interval $0 \leqq x \leqq 2$ it has neither *Over* nor *Under* functions. This then is an example of a finite-valued function which is not Perron integrable, [neither is it Riemann or Lebesgue integrable, of course].

A *fundamental theorem of integral calculus* for Perron integrals states:

If $F(x)$ is the indefinite Perron integral of a function $f(x)$ in the interval (a, b), the derivative $F'(x)$ exists almost everywhere (i.e., except for a set of points of measure zero) on (a, b) and is equal to $f(x)$.

A consequence of this theorem is the following:

If $F(x)$ has a finite derivative $f(x)$ at each point of (a, b), then this derivative is Perron integrable and its indefinite Perron integral equals $F(x) - F(a)$, i.e.,

$$(\text{Perron}) \int_a^x f(t)dt = F(x) - F(a).$$

The Perron integral thus yields solutions for a wide class of differential equations of the form

$$y' = f(x), \tag{3}$$

* *Loc. cit.*, p. 129, Theorem 3.

where the function $f(x)$ is Perron integrable on (a, b). In this connection, *Over* and *Under* functions yield the Perron integral and through it, existence theorems for differential equations of type (3). It is worthy of note that the *Over* and *Under* functions offer means of approximating the Perron integral and hence the solutions of equation (3) in cases where direct evaluation—by inverting the derivative through the fundamental theorem of integral calculus—is not feasible. Here one seeks *Over* and *Under* functions which differ by an amount within the limit of error. Techniques for obtaining such *Over* and *Under* functions can be developed from their definitions. Further mention of such techniques is made in a later section of this paper.

As a final comment on the Perron integral we mention *generalized Over* and *Under* functions. These functions have all the properties of *Over* and *Under* functions except that property 4) is required to hold except for a possible set of points of measure zero. That is to say

4′) $\overline{D}\phi(x) \leqq f(x)$, $\underline{D}\psi(x) \geqq f(x)$ *almost everywhere on* (a, b).

Such functions are useful in establishing relations between Perron and Lebesgue integrals.

2. Equations of type $y' = f(x,y)$. Let the real function $f(x,y)$ be defined over a domain

$$T: \begin{cases} a \leqq x \leqq b, & \text{where } v(x) \text{ may be identically } +\infty, \\ u(x) \leqq y \leqq v(x), & \text{and } u(x) \text{ may be identically } -\infty, \end{cases}$$

and let α be a number such that $u(a) \leqq \alpha \leqq v(a)$. A function $\phi(x)$ is called an *Under* function of $f(x, y)$ on T through (a, α) if it satisfies the conditions:

1) $\phi(x)$ is finite and continuous on (a, b),
2) $\phi(a) = \alpha$,
3) the left and right hand derivatives, $D_{\pm}\phi(x)$, exist on (a, b), where

$$D_{+}\phi(x) = \lim_{h \to 0} \frac{\phi(x+h) - \phi(x)}{h}, \qquad D_{-}\phi(x) = \lim_{h \to 0} \frac{\phi(x-h) - \phi(x)}{-h}, \qquad h > 0,$$

4) $D_{\pm}\phi(x) < f(x, \phi(x))$ on (a, b).

Similarly, an *Over* function $\psi(x)$ is defined as one which satisfies:

1) $\psi(x)$ is finite and continuous on (a, b),
2) $\psi(a) = \alpha$,
3) $D_{\pm}\psi(x)$ exist on (a, b),
4) $D_{\pm}\psi(x) > f(x, \psi(x))$ on (a, b).

We observe that left and right hand derivatives enter here instead of the upper and lower derivatives used in the preceding case. We also observe that condition 4) now uses the strong inequalities.

THEOREM I. *The inequality*

$$\phi(x) < \psi(x) \tag{4}$$

holds on $a<x\leqq b$ *between any Under function* $\phi(x)$ *and any Over function* $\psi(x)$.

Proof. Let $h(x)=\psi(x)-\phi(x)$, then $h(a)=0$. We have $D_{\pm}h(x)=D_{\pm}\psi(x)-D_{\pm}\phi(x)>f(x,\psi(x))-f(x,\phi(x))$. At $x=a$, $D_{+}h(x)>f(a,\psi(a))-f(a,\phi(a))>0$. Hence $h(x)>0$ in a neighborhood of $x=a$, since $\lim_{x\to a} h(x)/x=D_{+}h(a)>0$, $x>0$. Let $x=c$ be the first point of $a<x\leqq b$ for which $h(x)=0$, [since $h(x)$ is continuous on (a, b), a first such point will exist]. Now $D_{-}h(c)>f(c,\psi(c))-f(c,\phi(c))>0$, since $\psi(c)=\phi(c)$. But

$$D_{-}h(c) = \lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c} \frac{h(x)}{x-c},$$

for $x<c$. Hence $D_{-}h(c)\leqq 0$, since $h(x)>0$ and $x-c<0$. This, however, contradicts $D_{-}h(c)>0$ and yields the theorem.

Let $f(x, y)$ have at least one *Over* function and at least one *Under* function on T through (a, α). Since inequality (4) holds for a fixed $\phi(x)$ and *any* $\psi(x)$, and also for a fixed $\psi(x)$ and *any* $\phi(x)$, it follows that the set of all *Under* functions has a finite upper bound function $g(x)$ and the set of all *Over* functions has a finite lower bound function $G(x)$. We have $g(a)=G(a)=\alpha$ and it follows from (4) that

$$g(x) \leqq G(x) \tag{5}$$

holds on (a, b).

Perron* has shown that under rather general assumptions, $g(x)$ and $G(x)$ are solutions on $a\leqq x\leqq b$ of the differential equation

$$y' = f(x, y). \tag{6}$$

For this purpose it is assumed that $u(x)$ and $v(x)$ are continuous on (a, b), $u(a)=v(a)=\alpha$, their backward and forward derivatives exist and satisfy

$$\left.\begin{aligned} D_{\pm}u(x) &\leqq f(x, u(x)) \\ D_{\pm}v(x) &\geqq f(x, v(x)) \end{aligned}\right\} \quad \text{on } (a, b).$$

The function $f(x, y)$ is assumed continuous on

$$T: \begin{cases} a \leqq x \leqq b, \\ u(x) \leqq y \leqq v(x). \end{cases}$$

With these hypotheses, it is possible to show the existence of *Over* and *Under* functions for $f(x, y)$ on T through (a, α). This is done by extending the definition of $f(x, y)$ to the domain

* Mathematische Annalen, vol. 76, 1915, pp. 471–484.

$$T': \begin{cases} a \leqq x \leqq b, \\ -\infty < y < \infty, \end{cases}$$

by letting

$$f(x, y) \equiv f(x, u(x)), \quad \text{when} \quad y < u(x),$$
$$f(x, y) \equiv f(x, v(x)), \quad \text{when} \quad y > v(x).$$

The extended function $f(x, y)$ is then continuous and bounded on T'. If ϵ is an arbitrary positive number, then

$$\begin{aligned} \phi(x) &= u(x) - \epsilon(x - a), \\ \psi(x) &= v(x) + \epsilon(x - a), \end{aligned} \tag{7}$$

are *Under* and *Over* functions, respectively. This follows from

$$D_{\pm}\phi(x) = D_{\pm}u(x) - \epsilon \leqq f(x, u(x)) - \epsilon = f(x, \phi(x)) - \epsilon < f(x, \phi(x)),$$
$$D_{\pm}\psi(x) = D_{\pm}v(x) + \epsilon \geqq f(x, v(x)) + \epsilon = f(x, \psi(x)) + \epsilon > f(x, \psi(x)).$$

In this case then the functions $g(x)$, the upper bound function for all *Under* functions, and $G(x)$, the lower bound function for all *Over* functions, exist. It follows from (5) and (7) that

$$u(x) \leqq g(x) \leqq G(x) \leqq v(x). \tag{8}$$

Perron proved that $g(x)$ and $G(x)$ are solutions on $a \leqq x \leqq b$ of the differential equation (6). His proof is accomplished in three steps. First it is shown that $g(x)$ and $G(x)$ are continuous on (a, b). It is next shown that $g(x)$ can be uniformly approximated on (a, b) by *Under* functions and $G(x)$ can be similarly approximated by *Over* functions. Finally, it is shown that $g(x)$ and $G(x)$ satisfy the differential equation (6).

It is not difficult to show that if $y(x)$ is any solution of equation (6) through (a, α), then

$$g(x) \leqq y(x) \leqq G(x) \qquad \text{on } (a, b).$$

Since this is true, $g(x)$ is called the *minimal* solution and $G(x)$ is called the *maximal* solution through (a, α). The case where $g(x) \equiv G(x)$ on (a, b) yields a unique solution through (a, α). Sufficient conditions for this uniqueness have been developed by a number of men. Perhaps the best known of these are the Lipschitz condition, and the conditions derived by Osgood.* A theorem of G. Mie† states that if (c, β) is any point of

$$T'': \begin{cases} a \leqq x \leqq b, \\ g(x) \leqq y \leqq G(x), \end{cases}$$

* Monatshefte für Mathematik und Physik, vol. 9, 1898, p. 331.

† Mathematische Annalen, vol. 43, 1893, p. 553.

there is a solution of the differential equation which passes through (a, α) and (c, β). This theorem states roughly that the region T'' is completely filled with solutions of the equation (6). Hence the *maximal* and *minimal* solutions may be obtained as upper and lower limit functions of the family of solutions through the point (a, α).

Perron's existence theorem as described above differs essentially from those previously given by Peano and others. The Peano type theorem which would yield the same conclusion as that of Perron would require that the function $f(x, y)$ be *continuous* and *bounded* on the domain

$$T': \begin{cases} a \leqq x \leqq b, \\ -\infty < y < \infty. \end{cases}$$

These hypotheses are more restrictive than the ones used by Perron. For example, the linear equation $y' = y$ does not meet the boundedness hypothesis on any domain T'. Another and more illustrative example is

$$y' = \log y, \qquad y(0) = e.$$

If $u(x) = e$, $v(x) = e^{x+1}$, all of the hypotheses of the Perron theorem are met for any real interval (a, b). On the other hand, the function $\log y$ is not bounded for all real y and hence the other theorem does not apply. If the domain T' is replaced by a finite domain $\bar{T}$, and continuity of $f(x, y)$ is required on this, the boundedness condition is automatically satisfied for a closed sub-domain, but the Peano theorem yields existence of the solution in a *neighborhood* of $x = a$ only. Iterated application of this theorem may or may not yield existence on the given interval (a, b) [dependent partially on the techniques used]. A single application of the Perron theorem gives this existence on the entire interval (a, b). Mention is made of the fact that the so-called Cauchy-Polygons which play a central rôle in the Cauchy-Lipschitz proofs of existence theorems are very similar to the *Over* and *Under* functions that are being discussed. They do not, of course, necessarily meet the conditions 4) on the derivatives.

It is clear that the functions $u(x)$, $v(x)$ play a central rôle in Perron's theorem. Successful application of this theorem depends primarily on the determination of these functions. In a number of ways it is unfortunate that Perron described these functions by means of differential inequalities which give little suggestion of methods for securing them. This criticism becomes more valid when it is observed that $f(x, y)$ enters these inequalities and in turn is described for a domain which is dependent on $u(x)$ and $v(x)$. It is convenient to replace the conditions

$$\begin{aligned} D_{\pm}u(x) &\leqq f(x, u(x)), \\ D_{\pm}v(x) &\geqq f(x, v(x)), \end{aligned} \qquad \text{on } (a, b)$$

by the conditions

$$\begin{aligned} &du/dx = f_1(x, u), \\ (9) \qquad &dv/dx = f_2(x, v), \quad \text{where} \quad f_1(x, z) \leqq f(x, z) \leqq f_2(x, z) \end{aligned}$$

for all x on (a, b) and all z between the lower bound of $u(x)$ on (a, b) and the upper bound of $v(x)$ on (a, b). The loss of generality due to the requirement that $u(x)$ and $v(x)$ have derivatives is very slight and is heavily offset by the advantages gained through the indicated replacement. Equations (9) offer a mechanism for determining $u(x)$ and $v(x)$. The function $f(x, y)$ is known and there are many devices for obtaining the functions $f_1(x, u)$, $f_2(x, v)$ from it in such a way that the conditions of (9) are met and, furthermore, in such a way that the differential equations of (9) can actually be solved. With these functions, one merely solves the differential equations (9) subject to the condition $u(a)=v(a)=\alpha$ to secure $u(x)$ and $v(x)$. The following example illustrates this procedure:

$$y' = 1 + y^2, \qquad 0 \leqq x \leqq 1/3, \qquad y(0) = 1,$$

$$u(x) = \frac{1}{1-x}, \qquad v(x) = \frac{1+2x}{1-2x}, \qquad u(0) = v(0) = 1,$$

$$u' = u^2, \qquad v' = 1 + 2v + v^2 = (1+v)^2,$$

$$f_1(x, u) = u^2, \qquad f_2(x, v) = 1 + 2v + v^2,$$

$$z^2 \leqq 1 + z^2 \leqq 1 + 2z + z^2 \quad \text{on} \quad 1 \leqq z \leqq 5.$$

An application of Perron's theorem yields a solution $y(x)$ of the given differential equation for which

$$\frac{1}{1-x} \leqq y(x) \leqq \frac{1+2x}{1-2x} \quad \text{on} \quad 0 \leqq x \leqq 1/3.$$

Furthermore, this solution exists on any closed sub-interval of $0 \leqq x < \frac{1}{2}$. Since $u(x)$ has a pole at $x=1$ while $v(x)$ becomes infinite at $x=\frac{1}{2}$, it follows that the solution $y(x)$ also becomes infinite on $\frac{1}{2} \leqq x \leqq 1$. Actually, $y(x) = \tan(x+\pi/4)$ and its pole occurs at $x=\pi/4=.78+$. It is to be noted that $u(x)$ and $v(x)$ can be obtained easily by integration of their differential equations, using no more than integration of rational functions, whereas $y(x)$ itself involves integration of derivatives of inverse trigonometric functions. The indicated procedure gives information about the tangent function without using more than algebraic integrations. It is clear that $f_1(x, u)$ and $f_2(x, v)$ in this example are obtained by simple modifications of the function $f(x, y)$.

The relation $u(x) \leqq y(x) \leqq v(x)$ on (a, b) that results from Perron's theorem clearly offers opportunity for approximation of the solution function $y(x)$. The functions $f_1(x, u)$, $f_2(x, v)$ may be varied to yield u and v functions which give close approximations to $y(x)$. Approximations once obtained may often be improved by repetition of the procedure. It sometimes happens that the Taylor's series for $u(x)$ and $v(x)$ can be used to dominate formal series for $y(x)$. In this

way, convergence of these formal series to solutions of the equation may be established. These important possibilities for approximation through Perron's method have been largely overlooked—probably due to the lack of means for securing the functions $u(x)$ and $v(x)$. If equations (9) are used, these possibilities come into view and offer promise of development.

Müller* and others have extended Perron's theorem and method of proof to systems of first order differential equations. For a discussion of this work, reference is made to a paper by W. M. Whyburn.†

3. The Dirichlet problem. Let G be a finite, connected, open domain in the plane and let R be its boundary. Let $f(x, y)$ be continuous for (x, y) on the boundary R. A function $\phi(x, y)$ which is defined and continuous throughout the closed domain $G+R$ is called an *Under function* if it satisfies:

1) $\phi(x, y) \leqq f(x, y)$ on R,
2) $\phi(x, y) \leqq h_{\phi K}(x, y)$ throughout the interior of every circle K in G, where $h_{\phi K}(x, y)$ is harmonic in K [satisfies Laplace's differential equation, $\partial^2 U/\partial x^2 + \partial^2 U/\partial y^2 = 0$], and is equal to $\phi(x, y)$ for all points of $G+R$ except the interior of K.

Similarly, an *Over function* is defined as a function $\psi(x, y)$ continuous on $G+R$ which satisfies:

1) $\psi(x, y) \geqq f(x, y)$ on R,
2) $\psi(x, y) \geqq h_{\psi K}(x, y)$ for every circle K in G.

Clearly any constant which does not exceed the lower bound of $f(x, y)$ on R is an *Under* function, and any constant which is not less than the upper bound of $f(x, y)$ on R is an *Over* function. Thus functions of both types exist. It follows readily that

$$\psi(x, y) - \phi(x, y) \geqq 0, \quad \text{in} \quad G + R$$

holds between any *Over* function $\psi(x, y)$ and any *Under* function $\phi(x, y)$. The set of all *Under* functions has a finite upper bound function $g(x, y)$ and the set of all *Over* functions has a finite lower bound function $G(x, y)$. Under quite general assumptions on G and R, Perron‡ has shown that $G(x, y)$ and $g(x, y)$ are solutions of the Dirichlet problem. Perron's approach to this problem was essentially new and inaugurated a new line of investigation in the field.

F. Riesz, T. Radó, and others, have studied classes of functions which have been called *super*-and *sub*-harmonic functions. An excellent treatment of these functions, together with references to the literature, is given by Radó in *Ergebnisse der Mathematik*, vol. 5, 1937, pp. 1–56. These functions differ very

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* *Loc. cit.*, pp. 619–645.

† University of California at Los Angeles, Publications in Mathematical and Physical Sciences, vol. 1, 1935, p. 115–134.

‡ Mathematische Zeitschrift, vol. 18, 1923, pp. 42–54.

little from the *Over* and *Under* functions introduced by Perron. If we let G and R have the same meanings as in the foregoing discussions of *Over* and *Under* functions, a function $W(x, y)$ which is defined and continuous in G [$W(x, y)$ may be upper semicontinuous and may assume negatively infinite values at certain points without seriously changing the treatment] is *sub-harmonic* in G if for any sub-domain G' of G with boundary R', it is true that $W(x, y) \leqq h(x, y)$ in G' whenever $h(x, y)$ is harmonic in G', continuous in $G'+R'$, and $h(x, y) \geqq W(x, y)$ on R'.

A *super-harmonic* function may be defined as a function whose negative is *sub-harmonic*.

Super-and *sub*-harmonic functions have been used extensively in potential theory. References in this connection may be made to Radó's paper in *Ergebnisse*, to a paper by Carathéodory in *American Journal of Mathematics*, vol. 59, 1937, pp. 709–731, and to works of Kellogg, Evans, and others.

4. The Dedekind-Cut. We shall conclude this paper with a few comments on the Dedekind-Cut. A somewhat weakened form of this postulate for real numbers follows:

If S_1 and S_2 are two non-empty sets of real numbers such that the inequality $p \leqq q$ holds between any number p of S_1 and any number q of S_2, then there exists a least number P such that every p in S_1 satisfies $p \leqq P$ and a greatest number Q such that every q in S_2 satisfies $q \geqq Q$. Furthermore, $P \leqq Q$.

If the additional hypothesis is made that for any $\epsilon > 0$, there exists p in S_1, and q in S_2 such that $q - p < \epsilon$, then $P = Q$ and the Cut defines a unique number.

If instead of real numbers we let S_1 be composed of *Under* functions and S_2 be composed of *Over* functions of the type here discussed, the rôles of P and Q are played by the *minimal* and *maximal* solution functions. The case of uniqueness appears here just as in the Dedekind-Cut. Essentially, then, the use of *Over* and *Under* functions may be regarded as an extension of the Dedekind-Cut. In line with this notion, mention is made of a paper by E. R. Hedrick and W. M. Whyburn* in which the subject of integration is approached from the Dedekind-Cut point of view. The sets of functions used are composed of *step* functions, but these may readily be replaced by continuous functions of the type used by Perron as *Over* and *Under* functions.

Over and *Under* functions constitute a highly valuable tool in work on differential equations. There is every reason to believe that their usefulness will increase as further advances are made.

* American Journal of Mathematics, vol. 55, 1933, pp. 390–398.

THE TRIANGLE: ITS DELTOIDS AND FOLIATES

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1. Introduction. The appearance recently (this MONTHLY, vol. 45, p. 434, and vol. 46, p. 85) of two papers on the so-called Simson line envelope indicates that the following derivation and treatment of it might be of some interest. In addition to the references given, the reader is referred to Chapters 15 and 19 of Morley and Morley, *Inversive Geometry*. The analytic geometry we use was developed by Professor Morley over a period of years and used consistently in that book. When applied to the geometry of the triangle, it frequently surprises one by its economy of expression, and the following discussion further emphasizes that feature. Moreover, we are led directly to a family of foliates whose intimate connection with the triangle does not seem to have been explicitly noticed heretofore.

2. The Simson line. In the plane of a given triangle we establish a complex coördinate system (Z) which has the circumcircle of the triangle as base-circle $z\bar{z}=1$. The vertices of the triangle are turns t_1, t_2, t_3, conveniently considered as roots of the cubic

$$t^3 - s_1t^2 + s_2t - s_3 = 0;$$

and the equations of the three sides are given by

$$z + s_3\bar{z}/t_i = s_1 - t_i, \tag{2.1}$$

for $i=1, 2, 3$ (Morley, pp. 186–187).

Associated with equations (2.1) are the image equations

$$z + s_3\bar{w}/t_i = s_1 - t_i, \tag{2.2}$$

for $i=1, 2, 3$ (Morley, p. 153). Each establishes a (1, 1) correspondence, called a reflection, between the points of the plane; that is, any two points z and w which satisfy such a relation are images, the one of the other, in the line considered. The locus of fixed points of the reflection is given by the respective equation (2.1). Hence, if the turn τ represents a point on the circumcircle of the triangle, its images z_i' in the three sides are given by equation (2.2) for $w=\tau$, *i.e.*, $\bar{w}=1/\tau$, and

$$z_i' = s_1 - t_i - s_3/\tau t_i.$$

Now since the feet of the perpendiculars from τ to the sides are the midpoints of the segments from z_i' to τ, we have

$$z_i = \tfrac{1}{2}(z_i' + \tau),$$

for $i=1, 2, 3$.

These three feet z_i are collinear, for the clinants of the lines determined by any two are equal; *e.g.*, the clinant of the line joining z_2 and z_3 is

$$c = (z_2 - z_3)/(\bar{z}_2 - \bar{z}_3) = s_3/\tau.$$

The equation of this line, the Simson line of τ, is

$$\tau^3 - (2z - s_1)\tau^2 + (2\bar{z} - \bar{s}_1)s_3\tau - s_3 = 0. \tag{2.3}$$

3. The deltoid. We repeat that for a fixed τ, equation (2.3) represents the Simson line of τ. On the other hand, for variable τ it is the equation of a one-parameter family of lines. The envelope [**1**] of this family was called a deltoid by Professor Morley [**2**]; it is otherwise known as a three-cusped hypocycloid.

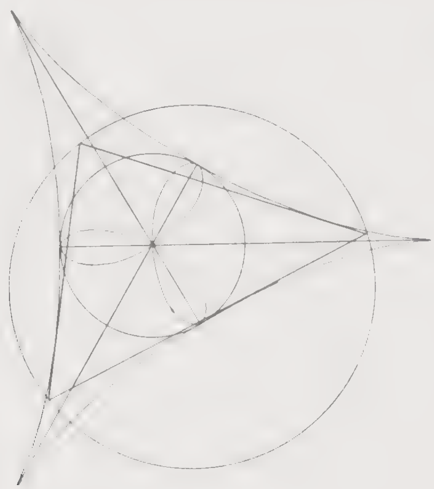

FIG. 1

Equation (2.3) shows that there is a region of the plane where three Simson lines lie on any point; that is, the envelope is a curve of class 3. We shall see later that it is of order 4.

Transforming to a new coördinate system (X) by the homology (*i.e.*, a rigid motion followed by a stretch)

$$x = (2z - s_1)/s_3^{1/3}, \tag{3.1}$$

we find that (2.3) takes the simpler form (*cf.* Morley, equation (1), p. 244)

$$t^3 - xt^2 + \bar{x}t - 1 = 0, \tag{3.2}$$

where the old and new parameters are turns connected by the relation $\tau = ts_3^{1/3}$. The envelope of the family (3.2) is obtained by the usual method, and has the equation

$$4(x^3 + \bar{x}^3) - x^2\bar{x}^2 - 18x\bar{x} + 27 = 0. \tag{3.3}$$

In the inversive theory the deltoid is a special bicubic, for it determines with any circle six points [**3**]. It has three finite cusps at $x = 3$, 3ω, $3\omega^2$, where $\omega^3 = 1$.

The parameters of the cusp tangents are given by (3.2) for these values of x; they are $t=1$, ω, ω^2 respectively, and the equations are

$$x = \bar{x}, \qquad x = \omega^2\bar{x}, \qquad x = \omega\bar{x}.$$

These cusp tangents meet at $x=0$, the nine-point center of the triangle; and the nine-point circle $x\bar{x}=1$, in coördinate system (X), is a tritangent circle of the deltoid, touching it at $x=-1$, $-\omega$, $-\omega^2$.

4. The parametric equation. Using equation (3.2), we find that the point x common to two Simson lines (*i.e.*, two tangents to the deltoid) whose parameters are t and t' is

$$x = t + t' + 1/tt'. \tag{4.1}$$

If, in this last equation, we consider t as fixed and t' as variable, the point x runs along a segment on the fixed tangent t of the deltoid. To show this it is only necessary to remark that (1) the clinant of the vector $x-t$ is constant (it is $1/t$, in fact); and (2) the modulus of $x-t$ is less than or equal to a constant (in this case 2), since

$$|x - t| = \left| t' + \frac{1}{tt'} \right| \leqq |t'| + |1/tt'| = 2.$$

Thus, x traces a segment of length 4 in coördinate system (X), which is twice the circumradius; the midpoint of the segment is $x_m=t$, a point on the nine-point circle; and the end-points $e_i=t\pm 2/t^{1/2}$ are points on the deltoid.

When $t'=t$, equation (4.1) gives the point of contact of the segment and the deltoid. Thus, for fixed t,

$$x = 2t + \frac{1}{t^2} \tag{4.2}$$

is a point on the deltoid. And for variable t, equation (4.2) is a parametric representation of the deltoid. It maps the nine-point circle on the deltoid, and is equivalent to the cartesian representation

$$X = 2\cos\phi + \cos 2\phi,$$
$$Y = 2\sin\phi - \sin 2\phi,$$

where $x=X+iY$ and $t=\cos\phi+i\sin\phi$. Here the X-axis is one of the cusp tangents, and the origin is at the nine-point center.

5. The Simson line generalized. Poncelet in his *Traité des propriétés projectives des figures* (1822), page 270, generalized the Simson line in the following theorem: *Si, d'un point quelconque d'une circonférence de cercle circonscrite à un triangle donné, on abaisse, sous un même angle d'ailleurs arbitraire, des obliques sur les directions des trois côtés de ce triangle, leurs pieds seront situés sur une seule et même ligne droite.*

In the notation of coördinate system (Z), the equations of these *obliques*

from a point τ on the circumcircle of the triangle may be easily found. Their intersections with the respective sides are three points z_i collinear on a line σ whose clinant is $-s_3t_0^2/\tau$, where $t_0=e^{i\phi_0}$ is the turn associated with the angle which each line from τ makes with the corresponding side of the triangle. The equation of σ, after subjecting (Z) to the homology

$$x = [(1 - t_0^2)z + s_1t_0^2]/s_3^{1/3}t_0^{4/3},$$

may be written in the new coördinate system (X')

$$t^3 - xt^2 + \bar{x}t - 1 = 0, \tag{5.1}$$

where $\tau = ts_3^{1/3}t_0^{4/3}$ (*cf.* equation (3.2)).

Thus, the envelope of σ, for a constant t_0 and a variable t, is the deltoid

$$x = 2t + \frac{1}{t^2}\cdot \tag{5.2}$$

Its center is $x=0$ or, in the coördinate system (Z),

$$z = \frac{-s_1t_0^2}{1 - t_0^2}\cdot \tag{5.3}$$

Also, the tritangent circle of the deltoid has a radius which is $\frac{1}{2}|\csc\phi_0|$ times the circumradius. This radius is a minimum when $\phi_0=\pi/2$, *i.e.*, for the Simson line envelope.

The deltoid given by equation (5.2), *i.e.*, the envelope of σ, is a function of t_0. Hence for varying t_0 we have a family of deltoids associated with the triangle. Any member of this family may be obtained from the Simson line envelope (4.2) by a properly chosen homology. The locus of the centers of this family is given parametrically by equation (5.3); it is the line

$$\frac{z}{s_1} + \frac{\bar{z}}{\bar{s}_1} = 1,$$

in coördinate system (Z), perpendicular to the Euler line of the triangle at the nine-point center.

We remark also that the cusps of (5.2) are, in system (X'), $x=3,\ 3\omega,\ 3\omega^2$; hence, in system (Z), the cusps are

$$z = \frac{s_1 - 3\tau}{1 - \tau^3/s_3}, \tag{5.4}$$

where for brevity, we write τ^3 for s_3/t_0^2. Now for variable t_0, *i.e.*, for variable τ, equation (5.4) is a parametric representation of the cusp locus of the family of deltoids (5.2). It is, we note, a rational bicubic, and its properties have been studied by others [4].

6. **A family of foliates.** Our method of analysis lends itself particularly well to the study of a certain family of foliates. The connection of this family with

the deltoid has been remarked before, but its rather obvious place in the geometry of the triangle seems to have been neglected. Its curves are pedal curves of a point with respect to the Simson lines of a triangle. Of these, the pedal curve of the nine-point center is perhaps the most striking—it is the three-leafed rose.

Associated with equation (3.2) of a Simson line, in coördinate system (X), there is the image equation

$$t^3 - xt^2 + \bar{y}t - 1 = 0.$$

The image, in the Simson line, of a fixed point $y=a$ is given by

$$x_a = t + \bar{a}/t - 1/t^2,$$

and the foot x of the perpendicular from a is the midpoint of the segment from a to x_a, *i.e.*, $2x=x_a+a$; hence,

$$2x = a + \bar{a}/t + t - 1/t^2. \tag{6.1}$$

This, for variable t and fixed a, is a parametric equation of the pedal curve of a with respect to the Simson lines of the triangle.

The last equation may be simplified by the translation $w=x-a$ to a new coördinate system (W), a translation which shifts the origin to the point from which the perpendiculars are dropped. Then, the locus is

$$2w = -a + \bar{a}/t + t - 1/t^2. \tag{6.2}$$

The clinant of the Simson line being $1/t$, in both systems (X) and (W), it is evident that $w/\bar{w}=-1/t$, for the join of the new origin to w is perpendicular to the Simson line. Eliminating t from equation (6.2) by this last relation, we have the equation of the pedal curve in w and $\bar{w}$, *viz.*,

$$w^3 + \bar{w}^3 + 2w^2\bar{w}^2 + \bar{a}w^2\bar{w} + aw\bar{w}^2 = 0. \tag{6.3}$$

Inversively, this is a bicubic with a triple point at $w=0$; projectively, we should call it a unicursal quartic with a triple point. The clinants c_i of the tangents at the triple point are given by

$$c^3 + \bar{a}c^2 + ac + 1 = 0.$$

These curves are more familiar in polar coördinates, and to this end we write $w=\rho\tau=\rho$ $(\cos\theta+i\sin\theta)$, and similarly $a=\rho_0\tau_0$. Then, observing that

$$\tau/\tau_0 + \tau_0/\tau = 2\cos(\theta-\theta_0),$$

equation (6.3) becomes

$$\rho = -\rho_0\cos(\theta-\theta_0) - \cos 3\theta. \tag{6.4}$$

The following types of curves of this family are of special interest [5].

I. The *oblique trifolium* $(a=\tau_0)$ is the locus of the feet of perpendiculars on the Simson lines of a triangle from a point on the nine-point circle. This curve has

a triple point at τ_0 and cuts the nine-point circle at three other points, which form a right triangle. Of the tangents at the triple point, two are perpendicular. The equation in polar coördinates, with pole at the triple point and prime axis parallel to a cusp tangent of the deltoid, is

$$\rho = -\cos(\theta - \theta_0) - \cos 3\theta.$$

II. The *right trifolium* ($a=1, \omega, \omega^2$). The point from which perpendiculars are dropped to the Simson lines is on the nine-point circle and diametrically opposite to its contact with the deltoid; that is, it is the midpoint of the segment of a cusp tangent of the deltoid. The curve has three leaves and one axis of symmetry, the cusp tangent at the point a. Two of the leaves are of equal size and lie outside the nine-point circle; the third lies inside the circle entirely and touches it at one of its contacts with the deltoid. If $a=1$, the polar equation with pole at the triple point is

$$\rho = -2\cos\theta\cos 2\theta.$$

III. The *double folium* ($a=-1, -\omega, -\omega^2$) is the locus of feet of perpendiculars from one of the contact points of the deltoid and its tritangent circle—the nine-point circle of the triangle. The self-symmetrical leaf of the right trifolium (II) has disappeared in this case and the triple point has become a cusp on the curve. The tangents to the double folium at this point are perpendicular, and one is the axis of symmetry. This axis of symmetry is a cusp tangent of the deltoid and also of the double folium. The polar equation is

$$\rho = 4\cos\theta\sin^2\theta.$$

IV. The *Münger oval* ($a=3, 3\omega, 3\omega^2$) is the pedal curve of a cusp of the deltoid [5; 6]. It is a single oval tangent to the deltoid and the nine-point circle, and has the corresponding cusp tangent as axis of symmetry. The two symmetrical leaves of the right trifolium (II) have disappeared in this case, and the tangents at the triple point have become coincident and perpendicular to the axis of symmetry. The polar equation, for $a=3$, is

$$\rho = -4\cos^3\theta.$$

V. The *three-leafed* rose ($a=0$) is the pedal curve of the nine-point center with respect to the Simson lines of the triangle. Its three axes of symmetry are cusp tangents of the deltoid, and it touches the deltoid at its contacts with the nine-point circle (Fig. 1). Its polar equation is

$$\rho = -\cos 3\theta,$$

and the parametric equation, in coördinate system (X),

$$2x = t - 1/t^2, \tag{6.5}$$

maps the nine-point circle on it.

7. A trinodal quartic. Another curve associated with the triangle, projectively a trinodal quartic with three axes of symmetry, has an immediate derivation from equation (4.2) [5; 7]. If x is a point of the deltoid (4.2),

$$x = 2t + 1/t^2,$$

then y, the image of x in the perpendicular from the nine-point center onto the tangent at x, is given by

$$y + \bar{x}/t = 0,$$

which is the image equation of that perpendicular.

Consequently the locus of y, the image of x, has the parametric equation

$$y = -t - 2/t^2, \tag{7.1}$$

since $x = 2t + 1/t^2$. Its equation in y and $\bar{y}$ is

$$2(y^3 + \bar{y}^3) + 4y^2\bar{y}^2 - 27y\bar{y} + 27 = 0;$$

and its cartesian parametric equations are

$$X = -\cos\phi - 2\cos 2\phi,$$
$$Y = -\sin\phi + 2\sin 2\phi.$$

Each node lies on a cusp tangent of the deltoid; and the nine-point circle is a tritangent circle, touching the quartic at -1, $-\omega$, $-\omega^2$, where it also touches the deltoid.

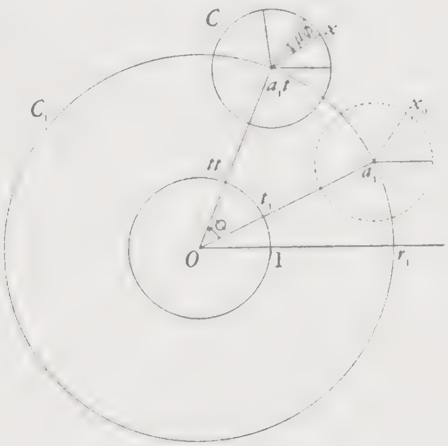

FIG. 2

8. Circles on circles. Consider in the plane of coördinate system (X) two points a_1 and $x_0 = a_1 + a_2$, where $a_1 = r_1 t_1 = r_1 e^{i\phi_1}$, and similarly for a_2. Now (Fig. 2) let C_1 be a circle with center at the origin and radius r_1, *i.e.*, C_1 passes through a_1; and let C_2 be a circle with center at a_1 and radius r_2. If the center of C_2 moves about C_1 with a constant angular velocity $\dot{\phi}$ while C_2 rotates about its

center with a constant angular velocity $-\mu\dot{\phi}$ $(\mu>0)$, the locus of a point x fixed on the circumference of C_2 is given parametrically by

$$x = a_1 t + a_2/t^{\mu}, \tag{8.1}$$

where $t=e^{i\phi}$, and $x=x_0$ when $\phi=0$.

A special case of this motion occurs when $\mu=r_1/r_2$. Then C_2 may be considered as rolling on the inside of a circle C with radius $r=r_1+r_2$ and concentric with C_1. For the arc traced by x on C_2 has length $(\mu+1)r_2\phi$; and this is equal to the corresponding arc on C, which subtends the angle ϕ at the origin, when $\mu=r_1/r_2$.

Thus the motion described by equation (8.1) includes the following:

I. The deltoid (4.2) when $a_1=2$, $a_2=1$, $\mu=2$. This, as is known, is equivalent to the rolling of a circle inside another with radius three times as large, for here $\mu=r_1/r_2=2$.

II. The three-leafed rose (6.5) when $a_1=-a_2=\frac{1}{2}$, $\mu=2$.

III. The trinodal quartic (7.1) when $a_1=-1$, $a_2=-2$, $\mu=2$.

IV. The segment

$$x = t_1 t + t_2/t,$$

when $a_i=t_i$ and $\mu=1$. This segment, whose midpoint is $x=0$, may also be generated by a point fixed on a circle which rolls inside another with radius twice as large. It might be called a hypocycloid with two cusps. (*Cf.* equation (4.1).)

V. The familiar four-cusped hypocycloid when $a_1=3$, $a_2=1$, $\mu=3$.

The reader, if interested, can readily adjust the argument of this section to include the motion arising when C_2 rotates in the same sense as its center a_1. The curves then described by a point fixed on C_2 include the epicycloids. Another problem of interest, for which the method is quite pertinent, is that of the trochoidal curves.

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ON A GENERALIZATION OF THE ARBELOS

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The Arbelos or Shoemaker's Knife is the figure consisting of the three semicircles on the same side of a line OP_0Q_0, diameters being OP_0, OQ_0, and P_0Q_0. While Archimedes is given credit for proving some of the properties of the figure, the author of its most famous proposition, called by Pappus "The Ancient Theorem," is unknown. The Ancient Theorem is:

THEOREM I. *Given two circles c_0 and $\bar{c}$ externally tangent to one another, and a third circle c having as diameter the sum of their collinear diameters. Then if the series of circles $c_0c_1c_2 \cdots c_i \cdots$ be all drawn tangent to c and $\bar{c}$, and successively to one another, the distance from the center of c_n to the line of centers of $c_0\bar{c}$ is n times the diameter of c_n.*

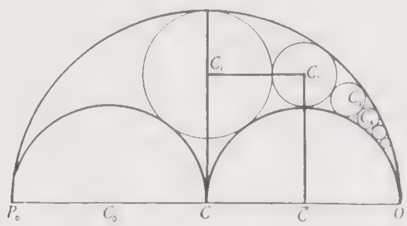

FIG. 1

Steiner* proved a number of theorems concerning the arbelos, most of them being corollaries of the Ancient Theorem. Among them was the following one:

THEOREM II. *Given two equal circles c_0 and $\bar{c}$ externally tangent to one another, and a third circle c having as its diameter the sum of their collinear diameters. Then if the series of circles $c_0c_1c_2$ be drawn tangent to c and $\bar{c}$, and successively to one another, then the centers of the circles c, c_1, c_2, and $\bar{c}$ form a rectangle.*

It is obvious that Theorem II is a special case of Theorem III, which in its turn will later be generalized.

THEOREM III. *Given two circles c_0 and $\bar{c}$ externally tangent to one another, and a third circle c having as diameter the sum of their collinear diameters; and given that the radius of $\bar{c}$ is k times the radius of c_0, k an integer. Then if the series of circles $c_0c_1c_2 \cdots c_i \cdots$ be all drawn tangent to c and $\bar{c}$, and successively to one another, then the centers of the circles c, c_k, c_{k+1}, and $\bar{c}$ form a rectangle.*

We will adopt the following notation. The points of tangency of circles c and c_i, $\bar{c}$ and c_i, and c and $\bar{c}$ are P_i, Q_i, and O respectively. The point of tangency of circles c_i and c_{i+1} is T_i. The center and radius of c, of $\bar{c}$, and of c_i are C and r, $\overline{C}$ and $\bar{r}$, and C_i and r_i respectively. In Figure 2 we show the case where $k=3$.

* Steiner, Einige geometrische Betrachtungen, Crelle's Journal, vol. 1, 1826, pp. 161–184 and pp. 252–288.

In any arbelos

$$r_0 + \bar{r} = r, \tag{1}$$

and in our case we have

$$\bar{r} = kr_0. \tag{2}$$

If in addition we choose our scale, which we may do without any loss of generality, so that

$$2r\bar{r} = r + \bar{r}, \tag{3}$$

it can easily be verified that

$$r = \frac{1 + 2k}{2k}, \qquad \bar{r} = \frac{1 + 2k}{2(1 + k)}, \qquad r_0 = \frac{1 + 2k}{2k(1 + k)}. \tag{4}$$

Let O be the center of a unit circle, and invert the arbelos with respect to

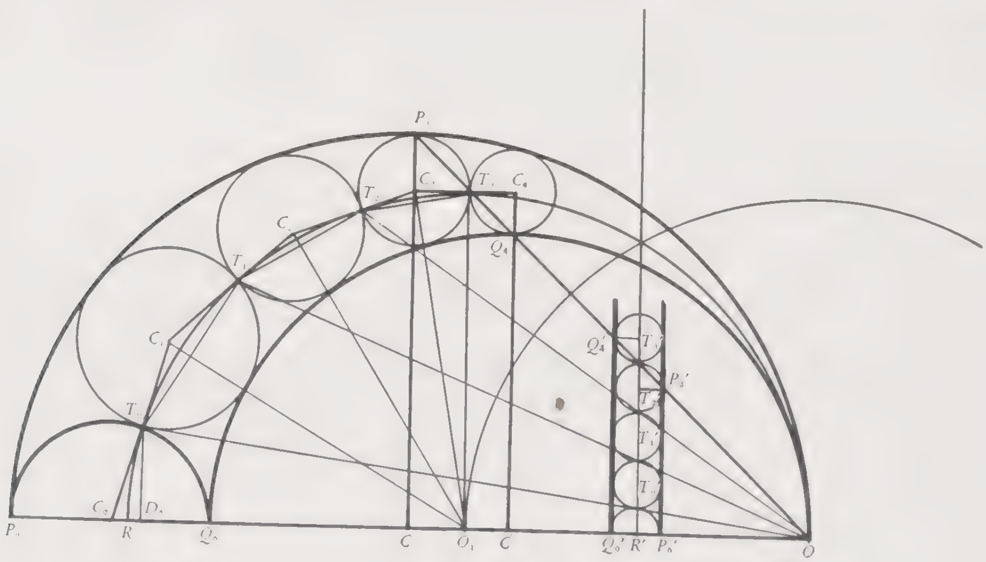

FIG. 2

that circle. The circles c and $\bar{c}$ will invert into the lines $P_0'P_i'$ and $Q_0'Q_i'$ perpendicular to OC. From (4) it follows that

$$OQ_0' = \frac{1 + k}{1 + 2k}, \qquad OP_0' = \frac{k}{1 + 2k}. \tag{5}$$

The circles c_i invert into equal circles c_i' whose line of centers is the inverse of the circle passing through the points of tangency T_i. The radius r' of the circles c_i' and OR', R' being the center of c_0', are from (5)

$$r' = \frac{1}{2(1 + 2k)}, \qquad OR' = \tfrac{1}{2}. \tag{6}$$

Since we take k an integer

$$R'T'_k = (2k+1)r' = \tfrac{1}{2} = OR', \tag{7}$$

and we therefore have angle $P_0OT'_k = 45°$.

From similar triangles it is seen that P_k and Q_{k+1} must be the intersections of the perpendiculars to OP_0 at C, with circle c; and at $\overline{C}$ with circle $\bar{c}$, respectively. Since C, C_k, and P_k are collinear and $\overline{C}$, C_{k+1}, and Q_{k+1} are collinear, the lines CC_k and $\overline{C}C_{k+1}$ are parallel and perpendicular to the line $C\overline{C}$. The line of centers C_kC_{k+1} is perpendicular to the common tangent at T_k. Since the common tangent at T_k makes the same angle with OP_k as the common tangent to circles c'_k and c'_{k+1} at T'_k makes with OP_k, angle $C_kT_kP_k$ equals 45°, and C_kC_{k+1} is therefore parallel to $C\overline{C}$. Thus Theorem III is proved.

Theorem III can also be proved by Theorems I and IV.

THEOREM IV. *Given two circles c_0 and $\bar{c}$ externally tangent to one another, and a third circle c having as diameter the sum of their collinear diameters; and given that the radius of $\bar{c}$ is k times the radius of c_0. Then if the series of circles $c_0c_1c_2 \cdots c_i \cdots$ be all drawn tangent to c and $\bar{c}$, and successively to one another, the radius r_n of c_n is given by the formula*

$$r_n = \frac{1+2k}{2(k+k^2+n^2)}. \tag{8}$$

The power of O with respect to the circle c'_n is found by (6) to be $(k+k^2+n^2)(1+2k)^{-2}$. If the unit circle with center at O is the circle of inversion, and if c'_n is the given circle, then Theorem IV is an immediate consequence of the well known theorem:

THEOREM V. *The radius of the inverse of a given circle which does not pass through the center of inversion is equal to the radius of the given circle multiplied by the square of the length of the radius of inversion, and divided by the absolute value of the power of the center of inversion with respect to the given circle.*

If we call the foot of the perpendicular dropped from C_n on the line $C_0\overline{C}$, the point D_n, then using Theorems I and IV we have

$$C_kD_k = 2k\,\frac{1+2k}{2(k+k^2+k^2)} = 1; \tag{9}$$

$$C_{k+1}D_{k+1} = 2(k+1)\,\frac{1+2k}{2(k+k^2+(k+1)^2)} = 1 = C_kD_k; \tag{10}$$

$$C_kD_k + r_k = 1 + \frac{1+2k}{2(k+k^2+k^2)} = \frac{1+2k}{2k} = r; \tag{11}$$

$$C_{k+1}D_{k+1} - r_{k+1} = 1 - \frac{1+2(k+1)}{2(k+k^2+(k+1)^2)} = \frac{1+2k}{2(1+k)} = \bar{r}; \tag{12}$$

therefore $D_k = C$ and $D_{k+1} = \overline{C}$, and Theorem III is again proved.

The condition that C_mD_m should equal C_nD_n is

$$\frac{2m(1+2k)}{2(k+k^2+m^2)} = \frac{2n(1+2k)}{2(k+k^2+n^2)}, \tag{13}$$

which reduces to

$$mn = k(k+1). \tag{14}$$

LEMMA 1. *If $m<n$ and equation* (14) *is satisfied, $D_mC_mC_nD_n$ is a rectangle and $CC_mC_n\overline{C}$ is a trapezoid.*

Since

$$CC_n + \overline{C}C_n = (r - r_n) + (\bar{r} + r_n) = r + \bar{r}, \tag{15}$$

we have the following:

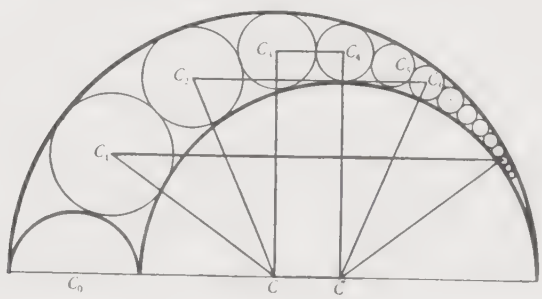

FIG. 3

LEMMA 2. *The centers C_n are situated on an ellipse whose foci are C and $\overline{C}$.*

Since

$$\sin C_0CC_m = \frac{D_nC_m}{CC_m} = \frac{2mr_m}{r - r_m} = \frac{2mk}{m^2+k^2}, \tag{16}$$

we have

LEMMA 3. *The angle C_0CC_m has its sine equal to twice the product of m and k divided by the sum of their squares.*

Lemmas 1, 2 and 3 yield the following:

THEOREM VI. *Given two circles c_0 and $\bar{c}$ externally tangent to one another, and a third circle c having as diameter the sum of their collinear diameters; and given that the radius of $\bar{c}$ is k times the radius of c_0, k an integer. Then if the series of circles $c_0c_1c_2 \cdots c_i \cdots$ be all drawn tangent to c and $\bar{c}$, and successively to one another, and if $mn = k(k+1)$, $m<n$, then the centers of the circles c, c_m, c_n, and $\bar{c}$ form an isosceles trapezoid whose base angles have their sine equal to twice the product of m and k divided by the sum of the squares of m and k.*

If $m=k$ Theorem VI obviously reduces to Theorem III.

Equation (6) shows that $OR=2$ and therefore the circle which passes through the points of tangency T_i is a unit circle. Let the center of this unit circle be called O_1. The area of the quadrant of the unit circle $O_1RT_0T_1 \cdots T_kO_1$ is less than that of the polygon $O_1C_0C_1C_2 \cdots C_kT_kO_1$ which we will call A_k'. Since

$$C_nC_{n+1} = r_n + r_{n+1}, \qquad O_1T_n = 1, \tag{17}$$

we have

$$2A_k' = (r_0 + r_1) + (r_1 + r_2) + \cdots + r_k = r_0 + 2\sum_{n=1}^{n=k} r_k. \tag{18}$$

Substituting the value of r_n in Theorem IV, we have

$$A_k' = \frac{1 + 2k}{4(k^2 + k)} + \frac{1}{2}\sum_{n=1}^{n=k} \frac{1 + 2k}{k^2 + k + n^2}. \tag{19}$$

Since as k increases A_k' approaches $\pi/4$, it follows that

$$\pi = \lim_{k\to\infty} 4A_k' = \lim_{k\to\infty}\left(2\sum_{n=1}^{n=k} \frac{1 + 2k}{k^2 + k + n^2} + \frac{1 + 2k}{k^2 + k}\right), \tag{20}$$

where $4A_k'$ is the area of a polygon of $4k+2$ sides circumscribed about a unit circle.

The area A_k of the inscribed polygon $O_1D_0T_0T_1 \cdots T_nO_1$ can also be found.

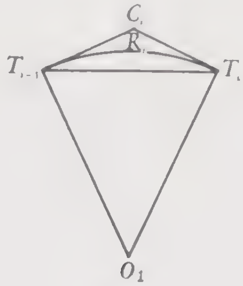

FIG. 4

If we denote the areas of triangles $OT_{n-1}T_n$ and $OT_{n-1}'T_n'$ by Δ_n and Δ_n' we have

$$\Delta_n' = \Delta' = \frac{1}{4(k + 1)}, \tag{21}$$

$$\frac{\Delta_n}{\Delta'} = \frac{OT_{n-1}\cdot OT_n}{OT_{n-1}'\cdot OT_n'} = \frac{1}{(OT_{n-1}')^2(OT_n')^2}, \tag{22}$$

$$\Delta_n = \frac{16(2k + 1)^4}{[(2n - 1)^2 + (2k + 1)^2][(2n + 1)^2 + (2k + 1)^2]}. \tag{23}$$

From (23) we can find the area of $OD_0T_0T_1 \cdots T_kO$. Subtracting from the area of $OD_0T_0T_1 \cdots T_kO$ the area of the triangle OT_kO_1, we get after suitable simplification*

$$A_k = \frac{4(2k+1)^3}{2k^2+2k+1} \sum_{n=1}^{n=k} \frac{1}{(2n-1)^2+(2k+1)^2} + \frac{2k}{2k^2+2k+1}. \tag{24}$$

Since $4A_k$ is the area of an inscribed polygon of $4k+2$ sides whose vertices are the points of tangency of the circumscribed polygon whose area is $4A'$, we know that

$$\pi = \lim_{k\to\infty} 4A_k \text{ and } 4A_k < \pi < 4A'_k. \tag{25}$$

From a theorem due to Huyghens, the area of the segment $T_{i-1}R_iT_i$ is less than two-thirds of the area of the triangle $T_{i-1}C_iT_i$.† Consequently the area of the sector $T_{i-1}O_1T_i$ is less than the sum of two-thirds the area of polygon $O_1T_{i-1}C_iT_i$ and one-third the area of triangle $T_{i-1}O_1T_i$. Hence if we call $\pi'_k = 4A'_k$, $\pi_k = 4A_k$, and $\pi_k^* = (2\pi'_k + \pi_k)/3$ we have $\pi'_k > \pi_k^* > \pi > \pi_k$.

The sequences π'_k, π_k, and π_k^* are of interest since they are rational and were derived without the use of the calculus. For ease and speed in calculating π these sequences do not approach the use of certain infinite series such as Machin's. The following table illustrates how π'_k, π_k, and π_k^* converge to $\pi = 3.1415926536 \cdots$.

k	π'_k	π_k	π_k^*
1	3.5	2.56	3.1867
2	3.2629	2.9182	3.1482
5	3.16577	3.09408	3.14187
10	3.14819	3.12846	3.14161
20	3.143321	3.138140	3.141594

* The method used was that employed by the author in solving essentially the same problem in his article, A simple approximation for π, this MONTHLY, vol. 45, pp. 373–375.

† In fact it can be proved that the limit, as angle $T_{i-1}O_1T_i$ approaches 0, of the area of segment $T_{i-1}R_iT_i$ divided by the area of triangle $T_{i-1}C_iT_i$ is two-thirds.

HIGHER DIMENSIONAL DETERMINANTS

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1. Introduction. The study of higher-dimensional determinants was begun by Cayley [1] in 1846, and continued by numerous other writers whose works are listed in bibliographies of Lecat [2] and Rice [3]. Recent non-trivial applications of these determinants have created a need for a new exposition of the fundamental properties of higher determinants, a need which this article is intended to fill.

Essentially one new notion, called "signancy of index," arises in the theory of higher determinants.

The author has tried to avoid the use of terms and notations not used in ordinary determinant theory, and from which much of the past published work in this field suffers.

Although most of the material here occurs in the existing literature, some of the results are believed to be new. Attention is called in particular to Theorems 10 and 11, which state that the coefficients in the expansion of an ordinary determinant, whose elements are linear forms, are 3-dimensional determinants, and the derivative of an ordinary determinant, whose elements are differentiable functions of a variable x, is a 3-dimensional determinant. Thus higher determinants arise in a simple way in the study of ordinary determinants.

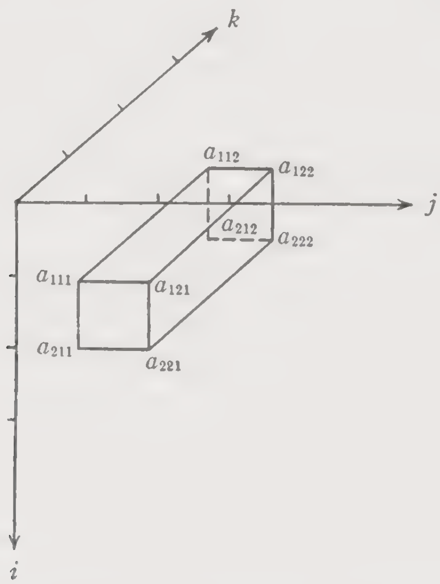
In what follows the elements of the matrices and determinants are assumed to belong to a given field of numbers; such as the class of all rational numbers, the class of all real numbers, *etc.*

2. Preliminary definitions. Let $i, j, \cdots, m$ be the p cartesian coördinates in a p-dimensional euclidean space. The points of intersection of the hyperplanes $i=1, \cdots, i=n, j=1, \cdots, j=n, \cdots, m=1, \cdots, m=n$ form what we shall call the p-way *lattice* L of *order* n. The lattice L is thus the set of points $(i, j, \cdots, m)$ where $i, j, \cdots, m=1, 2, \cdots, n$. A *p-way matrix* A of order n is defined to be a hypercubical array of numbers obtained by placing a number at each of the *vertices* $(i, j, \cdots, m)$ of the lattice L. We shall say that L is the *lattice of* A. Let the number at the point $(i, j, \cdots, m)$ in the array be denoted by $a_{ij\cdots m}$. The matrix A is then written as $(a_{ij\cdots m})$, and the numbers $a_{ij\cdots m}$ $[i, j, \cdots, m=1, 2, \cdots, n]$ are called the *elements* of A. The coördinates $i, j, \cdots, m$ are called the *indices* of A and they are said to *range* over $1, 2, \cdots, n$. The *diagonal of the lattice* L is the set of vertices $(1, \cdots, 1), \cdots, (n, \cdots, n)$ of L whose coördinates are equal. The *diagonal of* A is the set of elements on the diagonal of L. The vertices of L lying in the hyperplane $i=\alpha$ are said to form a *layer* L_α of L. The layers $L_1, \cdots, L_n$ are called the *i-layers* of L, and are said to be *parallel*. There are in a like manner layers associated with the other coördinates $j, \cdots, m$ of the space. Similarly, the elements of A at the vertices of L_α form the *i-layer* A_α of A.

Example 1. The 2-way matrix $A=(a_{ij})$ of order n is the following square array, the index i being associated with the rows, and the index j with the columns of A, as is usual in the literature. The i-layer A_α of A is seen to be the α row $(a_{\alpha 1}, a_{\alpha 2}, \cdots, a_{\alpha n})$ of the matrix.

$$\begin{array}{c|cccc}
 & 1 & 2 & \cdots & n \quad \to j \\
\hline
1 & a_{11} & a_{12} & \cdots & a_{1n} \\
2 & a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \cdot & \cdot & \cdots & \cdot \\
n & a_{n1} & a_{n2} & \cdots & a_{nn} \\
\downarrow i & & & &
\end{array}$$

Example 2. The 3-way matrix $A=(a_{ijk})$ of order 2 is the cubical array



In this case the indices i, j, k range over 1, 2. The i-layers A_1, A_2 of A are given by

$$\left\| \begin{matrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{matrix} \right\|, \qquad \left\| \begin{matrix} a_{211} & a_{212} \\ a_{221} & a_{222} \end{matrix} \right\|,$$

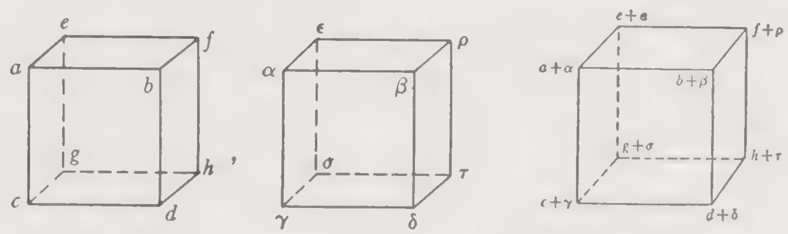
respectively.

We return to the general p-way matrix $A=(a_{ij\cdots m})$ of order n. We let $(b_{ij\cdots m})$ be a second p-way matrix of order n, denoted by B. At each point $(i, j, \cdots, m)$ of the p-way lattice L of order n we place the *sum* $a_{ij\cdots m}+b_{ij\cdots m}$. The matrix so obtained is said to be the sum of A and B, written $A+B$.

Example 3. In this and examples that follow we shall drop the coördinate axes from our matrix displays. The sum of the 2-way matrices (a_{ij}) and (b_{ij}) of order n is the matrix:

$$\left\| \begin{matrix} (a_{11}+b_{11}) & \cdots & (a_{1n}+b_{1n}) \\ \cdot & \cdots & \cdot \\ (a_{n1}+b_{n1}) & \cdots & (a_{nn}+b_{nn}) \end{matrix} \right\|.$$

Example 4. The sum of the first two of the adjacent 3-way matrices is the third one.



An unordered collection of n numbers will be called a *set of order* n. An ordered array of n numbers will be said to be a *sequence of order* n. Two sets S and S' are *equal* if they contain the same n numbers. Two sequences, R and R' are equal if for each i the ith number in R is the same as the ith number in R'.

3. Definition of generalized determinants. We let A be the p-way matrix $(a_{ij\cdots m})$ of order n. We let $P_1, P_2, \cdots, P_n$ denote the respective vertices $(i_1, j_1, \cdots, m_1), (i_2, j_2, \cdots, m_2), \cdots, (i_n, j_n, \cdots, m_n)$ of the lattice L of A. We let the sequences of coördinates $i_1j_1 \cdots m_1, i_2j_2 \cdots m_2, \cdots, i_nj_n \cdots m_n$ be denoted by $R, S, \cdots, V$ respectively. We let the elements of A at $P_1, P_2, \cdots, P_n$ be denoted by $a_R, a_S, \cdots, a_V$ respectively. We let the set of points $P_1, \cdots, P_n$ be chosen so that no two points in the set lie in the same layer of A. The product $\Pi = a_Ra_S \cdots a_V$ then contains an element from each layer of A, and no two elements from the same layer. We let t be a positive integer not greater than p, and $i, j, \cdots, q$ be t of the coördinates in our space. By permutations of the i-layers amongst themselves, the j-layers amongst themselves, . . . , and finally the m-layers amongst themselves, the element a_R can be brought to the vertex $(1, 1, \cdots, 1)$, while the element a_S is brought to the vertex $(2, 2, \cdots, 2), \cdots,$ and finally a_V is brought to the vertex $(n, n, \cdots, n)$. We let $p_i, p_j, \cdots, p_m$ be a set of permutations that thus brings the elements $a_R, \cdots, a_V$ to the diagonal of A, where p_i denotes a permutation of the i-layers, p_j a permutation of the j-layers, *etc*. We define Π_1 to be $+\Pi$ or $-\Pi$ according as p_i is even or odd. We define Π_2 to be $+\Pi_1$ or $-\Pi_1$ according as p_j is even or odd. Finally, we define Π_t to be $+\Pi_{t-1}$ or $-\Pi_{t-1}$ according as p_q is even or odd. *The determinant* $|A|$ *of* A *with the indices* $i, j, \cdots, q$ *signant is defined* to be the sum of all products Π_t that can be formed from A. The products Π_t are called *terms* in the expansion of $|A|$. Since A is p-way of order n, the determinant $|A|$ is said to be *p-way of order n*.

It can be seen that the sign placed in front of a product Π to obtain a signed

product Π_t is independent of the order of the elements $a_R, a_S, \cdots, a_V$ in Π if, and only if, t is even. We illustrate with an example. If i is the only index signant in the expansion of $|A|$ where A is a 3-way matrix (a_{ijk}) of order 2, the product $\Pi = a_{121}a_{212}$ yields a product $\Pi_1 = +\Pi$ since p_i is even (in this case we can take p_i to be identity). If we write Π as $a_{212}a_{121}$, $\Pi_1 = -\Pi$ since p_i is now odd (in this case we may take p_i to be the cycle $[12]$). Thus in this case one does not arrive at a unique product Π_1. It is therefore assumed throughout higher-dimensional determinant theory that an *even number of indices are signant.* The remaining indices are said to be *non-signant.* The layers associated with signant indices are said to be *signant layers*; thus, if i is signant, the i-layers are signant.

In an ordinary 2-way determinant both indices are signant.

Example 5. We let $A = (a_{ijk})$ be a three-way matrix of order two. With A can be associated the following four determinants (See Figure 1).

1. *No indices signant* (determinant usually then called a "permanent").

$$|A| = a_{111}a_{222} + a_{121}a_{212} + a_{211}a_{122} + a_{221}a_{112}. \tag{1.1}$$

2. *i, j, signant.*

$$|A| = a_{111}a_{222} - a_{121}a_{212} - a_{211}a_{122} + a_{221}a_{112}. \tag{1.2}$$

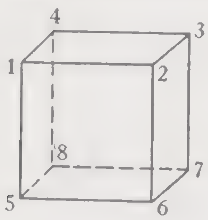
3. *i, k, signant.*

$$|A| = a_{111}a_{222} + a_{121}a_{212} - a_{211}a_{122} - a_{221}a_{112}. \tag{1.3}$$

4. *j, k, signant.*

$$|A| = a_{111}a_{222} - a_{121}a_{212} + a_{211}a_{122} - a_{221}a_{112}. \tag{1.4}$$

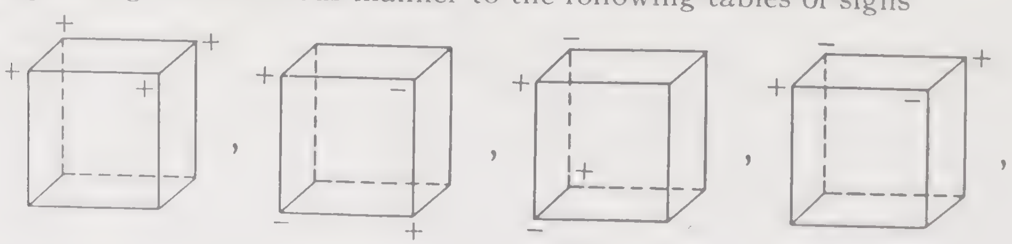
Thus the various expansions of the adjacent matrix



are

$$1\cdot 7 + 2\cdot 8 + 3\cdot 5 + 4\cdot 6,$$
$$1\cdot 7 - 2\cdot 8 + 6\cdot 4 - 5\cdot 3,$$
$$1\cdot 7 - 4\cdot 6 + 8\cdot 2 - 5\cdot 3,$$
$$1\cdot 7 - 2\cdot 8 + 3\cdot 5 - 4\cdot 6,$$

corresponding in an obvious manner to the following tables of signs



respectively.

4. Theorems of structure. From the definition of general determinants given in §3, we derive immediately the following theorems:

THEOREM 1. *There are* $(n!)^{p-1}$ *terms in the expansion of a p-way determinant of order n.*

THEOREM 2. *Interchange of two parallel signant layers changes the sign of the determinant. Interchange of two parallel non-signant layers does not change the sign of the determinant.*

THEOREM 3. *If the i-layers of a determinant are interchanged with the corresponding j-layers, the determinant is unchanged in value, provided that the i and j indices are both signant, or both non-signant.*

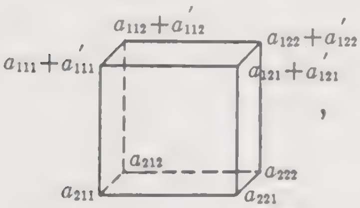
THEOREM 4. *If the elements of a layer vanish the determinant is zero.*

THEOREM 5. *If two signant layers of a determinant are identical the determinant is zero.*

THEOREM 6. *If the elements of a layer of a determinant are multiplied by m, the determinant is multiplied by m.*

THEOREM 7. *If a layer* A_1 *of a matrix A is written as a sum of* B_1+C_1, *each determinant* $|A|$ *is a sum of determinants* $|B|$ *and* $|C|$ *with the same signant indices as* $|A|$, *where the matrix B differs from A only in that* A_1 *is replaced by* B_1, *and C differs from A only in that* A_1 *is replaced by* C_1.

Example 6. The first determinant below is the sum of the other two,



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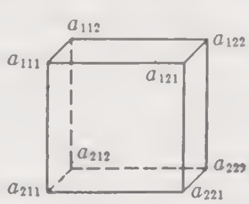

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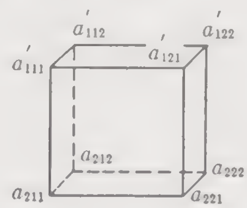


,

where the same indices are signant throughout.

THEOREM 8. *The value of a determinant is not changed if the elements of a signant layer are multiplied by a constant, and added to the corresponding elements of a parallel layer.*

THEOREM 9. *If the i-layers of a p-way determinant* $|A|$ *of order n with i non-signant are identical, the determinant is* $n!D_{p-1}$, *where* D_{p-1} *is the* $(p-1)$*-way determinant of the layer* $i=1$ *of A with the same signant indices as* $|A|$.

COROLLARY. *The determinant* $|a_{ijk}|$ *of order n with j, k signant and identical i-layers satisfies the equality*

$$|a_{ijk}| = n!|a_{1jk}|,$$

where $|a_{1jk}|$ *is an ordinary 2-way determinant.*

Example 7. In the matrix A of Figure 1, we let $a_{1jk}=a_{2jk}$ for $j, k=1, 2$. The expansion (1.4) of Section 3 becomes

$$|A| = a_{111}a_{122} - a_{121}a_{112} + a_{111}a_{122} - a_{121}a_{112} = 2!\begin{vmatrix} a_{111} & a_{112} \\ a_{121} & a_{122} \end{vmatrix} = 2!\,|a_{1jk}|.$$

We remark that the above result follows rather immediately from the fact that the determinant (1.4) can be written as

$$\begin{vmatrix} a_{111} & a_{212} \\ a_{121} & a_{222} \end{vmatrix} + \begin{vmatrix} a_{211} & a_{112} \\ a_{221} & a_{122} \end{vmatrix}.$$

This can be generalized as follows:

Let D be the 3-way determinant of a matrix (a_{ijk}) of order n, where j, k are signant in D. It is readily seen that

$$D = \sum \begin{vmatrix} a_{\alpha 11} & a_{\beta 12} & \cdots & a_{\delta 1n} \\ a_{\alpha 21} & a_{\beta 22} & \cdots & a_{\delta 2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{\alpha n1} & a_{\beta n2} & \cdots & a_{\delta nn} \end{vmatrix},$$

where the summation is over all permutations $[\alpha, \beta, \cdots, \delta]$ of the numbers $[1, 2, \cdots, n]$.

5. How 3-way determinants arise in the study of 2-way determinants. An ordinary determinant, whose elements are linear forms, can be written as

$$D = \begin{vmatrix} \left(\sum_{k=1}^{m} a_{11k}x_k\right) & \left(\sum_{k=1}^{m} a_{12k}x_k\right) & \cdots & \left(\sum_{k=1}^{m} a_{1nk}x_k\right) \\ \left(\sum_{k=1}^{m} a_{21k}x_k\right) & \left(\sum_{k=1}^{m} a_{22k}x_k\right) & \cdots & \left(\sum_{k=1}^{m} a_{2nk}x_k\right) \\ \cdot & \cdot & \cdots & \cdot \\ \left(\sum_{k=1}^{m} a_{n1k}x_k\right) & \left(\sum_{k=1}^{m} a_{n2k}x_k\right) & \cdots & \left(\sum_{k=1}^{m} a_{nnk}x_k\right) \end{vmatrix}.$$

Evidently, we may write

$$D = \sum_{[\alpha, \cdots, \delta]} \begin{vmatrix} a_{11\alpha} & a_{12\beta} & \cdots & a_{1n\delta} \\ a_{21\alpha} & a_{22\beta} & \cdots & a_{2n\delta} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1\alpha} & a_{n2\beta} & \cdots & a_{nn\delta} \end{vmatrix} x_\alpha x_\beta \cdots x_\delta,$$

where the sum is over all distinct sequences of n values $[\alpha, \cdots, \delta]$ which can be chosen from $[1, 2, \cdots, m]$, repetition of values allowed.

We let $[\alpha, \cdots, \delta]$ now denote a set, not a sequence, of n values of k, not necessarily distinct. We denote this set by Γ. We can write

$$(2) \qquad D = \sum_{\Gamma} x_\alpha x_\beta \cdots x_\delta \left\{ \sum_{P} \begin{vmatrix} a_{11\lambda} & a_{12\mu} & \cdots & a_{1n\xi} \\ a_{21\lambda} & a_{22\mu} & \cdots & a_{2n\xi} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1\lambda} & a_{n2\mu} & \cdots & a_{nn\xi} \end{vmatrix} \right\},$$

where the outer summation is over all distinct sets Γ, and the inner over all distinct sequences $P=[\lambda, \mu, \cdots, \xi]$, which can be formed from the set Γ. For a fixed set Γ the second summation in (2) is, except for a positive rational factor, the 3-way determinant of a matrix (a_{ijk}) with i, j signant, where i and j range over $1, 2, \cdots, n$, and k ranges over $\alpha, \beta, \cdots, \delta$. We have proved the following theorem valid for fields of numbers whose characteristic is subject to minor restrictions.

THEOREM 10. *The coefficients in the expansion of an ordinary 2-way determinant, whose elements are linear forms, are 3-way determinants.*

The Hessian of a cubic written in the usual symmetric form

$$\sum_{i,j,k=1}^{n} a_{ijk}x_i x_j x_k$$

is $6^n D$, where D is the determinant used at the beginning of this section with $m=n$. We have the following corollary:

COROLLARY. *The coefficients in the expansion of the Hessian of a cubic form C are, except for positive rational factors, the various 3-way determinants whose k-layers are non-signant and are k-layers of the matrix (a_{ijk}) of C.*

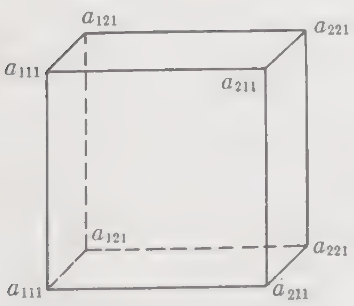
Example 8. If C is binary the Hessian H of C satisfies the formula

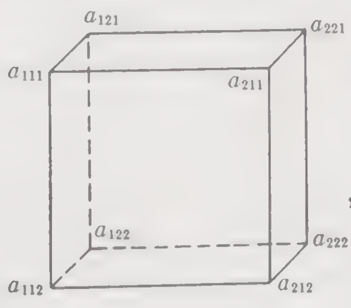
$$\frac{H}{36} = \begin{vmatrix} a_{111} & a_{121} \\ a_{211} & a_{221} \end{vmatrix} x_1^2 + \left[\begin{vmatrix} a_{111} & a_{122} \\ a_{211} & a_{222} \end{vmatrix} + \begin{vmatrix} a_{112} & a_{121} \\ a_{212} & a_{221} \end{vmatrix} \right] x_1 x_2$$
$$+ \begin{vmatrix} a_{112} & a_{122} \\ a_{212} & a_{222} \end{vmatrix} x_2^2 .$$

We may write

$$\frac{H}{36} = \frac{1}{2!} A_{11}x_1^2 + A_{12}x_1x_2 + \frac{1}{2!} A_{22}x_2^2 ,$$

where A_{11}, A_{12}, and A_{22} are respectively the 3-way determinants



,


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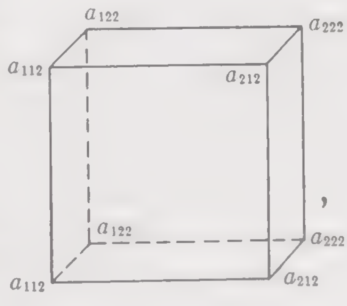

with the first two indices signant.

Theorem 10 can be generalized since the coefficients in the expansion of a

p-way determinant D whose elements are linear forms are $(p+1)$-way determinants with the same signant indices as D.

Let us now consider the determinant $|A|$ of the nth order matrix $A=[a_{ij}(x)]$ whose elements $a_{ij}(x)$ are differentiable functions of x, the field being one for which the differentiation operation is defined. The derivative of $|A|$ with respect to x is the sum

$$
(3)\quad \begin{vmatrix} a'_{11}(x) & a_{12}(x) & a_{13}(x) & \cdots & a_{1n}(x) \\ a'_{21}(x) & a_{22}(x) & a_{23}(x) & \cdots & a_{2n}(x) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a'_{n1}(x) & a_{n2}(x) & a_{n3}(x) & \cdots & a_{nn}(x) \end{vmatrix} + \begin{vmatrix} a_{11}(x) & a'_{12}(x) & a_{13}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a'_{22}(x) & a_{23}(x) & \cdots & a_{2n}(x) \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1}(x) & a'_{n2}(x) & a_{n3}(x) & \cdots & a_{nn}(x) \end{vmatrix}
$$

$$
+ \cdots + \begin{vmatrix} a_{11}(x) & \cdots & a_{1,n-1}(x) & a'_{1n}(x) \\ a_{21}(x) & \cdots & a_{2,n-1}(x) & a'_{2n}(x) \\ \cdot & \cdots & \cdot & \cdot \\ a_{n1}(x) & \cdots & a_{n,n-1}(x) & a'_{nn}(x) \end{vmatrix}.
$$

We construct a 3-way matrix $B=(b_{ijk})$ such that $b_{ij1}=a_{ij}'(x)$, whence the first k-layer of B is the matrix whose elements are the derivatives of the elements of A, and we let $b_{ij2}=b_{ij3}= \cdots =b_{ijn}=a_{ij}(x)$, whence the layers $k=2$, $k=3, \cdots$, $k=n$ of B are equal to A. It is readily seen that (3) yields the equality

$$
\frac{d|A|}{dx} = \frac{1}{(n-1)!}|B|,
$$

where $|B|$ is the 3-way determinant of B with i and j signant. The indices i and j are signant in $|A|$, so that $|B|$ and $|A|$ have the same signant indices. We have proved the following theorem:

THEOREM 11. *Except for a positive rational factor the derivative of an ordinary 2-way determinant $|A|$ whose elements are differentiable functions of x is the 3-way determinant D of a matrix formed by n parallel layers, one of which is the matrix of the derivatives of the elements of A, and the remaining are equal to A, the signancy in D being the same as in $|A|$.*

Theorem 11 may be generalized since the derivative of a determinant of a p-way matrix A is $1/(n-1)!$ times a $(p+1)$-way determinant.

6. Characteristic properties of Weierstrass [7]. The "characteristic properties of Weierstrass" can be extended as in the following theorem whose proof is rather immediate.

THEOREM 12. *The determinant D of the matrix $A=(a_{ij\cdots m})$ with at least two indices signant is the polynomial in the elements of A which satisfies the following properties:*

a) It is linear and homogeneous in the elements of each layer of A.

b) It changes sign when two parallel signant layers are interchanged, and is unchanged when two parallel non-signant layers are interchanged.

c) It has the value 1 for $(\delta_{ij\cdots m})$, where $\delta_{ij\cdots m}=1$ when $i=j=\cdots=m$, and $\delta_{ij\cdots m}=0$ otherwise.

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Editorial Note. The editor would welcome advice from readers of the MONTHLY as to the proper department to which to assign the following.

E.J.M.

WORK PROBLEM APPLIED. IMPORTANT AMBIGUOUS CASE

When a workman W, who is 5 ft. 11.38 in. tall, standing at A places his shovel at P on the ground 4 ft. 8.57 in. from A, and leans at an angle of 72°27.4″ on the shovel which is just 4 ft. 2.19 in. long, how far up on his anatomy will the end of the shovel come? Also find the area of WPA. Note that there are two answers to this problem, which is very important to the workman.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, New York

The Department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

THE PYTHAGOREAN DIOPHANTINE EQUATION $x_1^2+x_2^2+\cdots+x_k^2=1$

W. E. BLEICK, Cooper Union

The following very simple proofs are believed to be new. We consider the cases $k=2$, $k=4$, and $k=n$.

Case I. $k=2$. A necessary and sufficient condition that x and y be rational numbers satisfying $x^2+y^2=1$ is that the complex number $x+iy$ be the square of a complex number $(m+in)/(m^2+n^2)^{1/2}$ where m/n is rational. To prove that the condition is necessary we extract the square root of $x+iy$ under the condition that $x^2+y^2=1$. Then $(x+iy)^{1/2}=[(1+x)+iy]/[(1+x)^2+y^2]^{1/2}$ which is of the required form. To prove that the condition is sufficient we square $(m+in)/(m^2+n^2)^{1/2}$ and show that

$$x^2+y^2=\left(\frac{m^2-n^2}{m^2+n^2}\right)^2+\left(\frac{2mn}{m^2+n^2}\right)^2=1.$$

Case II. $k=4$. A necessary and sufficient condition that w, x, y, and z be rational numbers satisfying $w^2+x^2+y^2+z^2=1$ is that the quaternion $wI+xi+yj+zk$ be the square of a quaternion $(mI+ni+pj+qk)/(m^2+n^2+p^2+q^2)^{1/2}$ where the ratios m/n, n/p, and p/q are rational. The proof is similar to that given above.

Case III. $k=n$. A necessary and sufficient condition that $x_1, x_2, \cdots, x_n$ be rational numbers satisfying $x_1^2+x_2^2+\cdots+x_n^2=1$ is that

$$x_1=\frac{y_1^2-\sum_{s=2}^{n} y_s^2}{\sum_{s=1}^{n} y_s^2},\quad x_i=\frac{2y_1y_i}{\sum_{s=1}^{n} y_s^2},\qquad (i=2,3,\cdots,n),$$

where the ratios $y_1/y_2, y_2/y_3, \cdots, y_{n-1}/y_n$ are rational. To see that the condition is necessary we notice that these relations are true if

$$y_1=1+x_1,\qquad y_i=x_i,\qquad (i=2,3,\cdots,n).$$

The condition is sufficient since

$$\sum_{i=1}^{n} x_i^2=\frac{\left(y_1^2-\sum_{s=2}^{n} y_s^2\right)^2+\sum_{i=2}^{n}(2y_1y_i)^2}{\left(\sum_{s=1}^{n} y_s^2\right)^2}=1.$$

Note by the editor. The sufficiency for the condition in the case $k=n$ has been known for many years (Dickson, *History of the Theory of Numbers*, vol. 2, p. 318). Its necessity seems not to have been previously mentioned in the literature. R. J. W.

ON THE EXPLICIT SOLUTION OF SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

J. S. FRAME, Brown University

The following method of solving simultaneous linear differential equations with constant coefficients is not included in most textbooks on the subject,* yet it is relatively easy to apply, and demands only an elementary knowledge of matrix theory (addition and multiplication of matrices, and the solution of the characteristic determinantal equation). It may prove useful not only to pure mathematicians, but to those using differential equations in physics, chemistry, or engineering.

The set of n linear differential equations

$$dx_i/dt = \sum_{j=1}^{n} a_{ij}x_j, \qquad (i = 1, 2, \cdots, n), \tag{1}$$

may be written in the matrix form

$$dX/dt = AX, \tag{2}$$

where A is a square $n\times n$ matrix, with elements a_{ij}, and X and dX/dt are the n-row, 1-column matrices whose elements are x_i and dx_i/dt. The formal solution

$$X = e^{At}C = (I + At + A^2t^2/2! + A^3t^3/3! + \cdots)C, \tag{3}$$

where C is an $n\times 1$ matrix of constants and I is the unit matrix, although known to converge, is of little help in actual computation. But if r is a characteristic root of multiplicity m of the equation

$$\det (A - rI) = 0, \tag{4}$$

then the matric equation

$$(A - rI)^mC = 0 \tag{5}$$

will have m linearly independent solutions C. The formula

$$X = e^{rt}[I + (A - rI)t + (A - rI)^2t^2/2! + \cdots + (A - rI)^{m-1}t^{m-1}/(m - 1)!]C \tag{6}$$

gives for each of the m matrices C an explicit solution of (2). For we see immediately that the matrix X in (6) satisfies the equation

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* The author did not find it in over a dozen standard texts he consulted. The nearest approach to it was found in a 48 page discussion of the same subject in F. R. Moulton, Differential Equations, 1930, pp. 246–294. But the present treatment seems much shorter and simpler than that.

$$dX/dt - rX = (A - rI)X - (A - rI)^m C(t^{m-1}/(m-1)!). \tag{7}$$

When we apply (5) and simplify, we obtain the given equation (2). The actual computation of the solution (6) would be as follows. First we find the characteristic roots in (4). For each root r of multiplicity m we calculate the matrices $(A-rI)$, $(A-rI)^2, \cdots, (A-rI)^m$. As a check, we note that the product of all the matrices $(A-r_iI)^{m_i}$ is the zero matrix, since the matrix A satisfies its characteristic equation. This fact gives us a nice way of finding solutions C_i of (5), corresponding to the root r_i. We form the product of the matrices $(A-r_jI)^{m_j}$ for all $j \neq i$. Each column of this product gives a solution C_i, and there are just m_i columns which are linearly independent. Finally we substitute in (6), using the matrices $(A-rI)^k$ and C which have already been computed, and adding together the solutions corresponding to all the different roots r.

An example will serve to illustrate the theory. Given the system of differential equations

$$\begin{aligned} dx_1/dt &= x_1 - 2x_2 + 3x_3 - 2x_4, \\ dx_2/dt &= x_1 + 5x_2 - x_3 - x_4, \\ dx_3/dt &= 2x_1 + 3x_2 + 2x_3 - 2x_4, \\ dx_4/dt &= 2x_1 - 2x_2 + 6x_3 - 3x_4, \end{aligned} \qquad (A - rI) = \begin{Vmatrix} 1-r & -2 & 3 & -2 \\ 1 & 5-r & -1 & -1 \\ 2 & 3 & 2-r & -2 \\ 2 & -2 & 6 & -3-r \end{Vmatrix}. \tag{8}$$

The characteristic equation

$$\det(A - rI) \equiv r^4 - 5r^3 + 6r^2 + 8r - 8 = 0, \tag{9}$$

has roots 2, 2, 2, -1, and we find the matrices $(A-rI)^k$ as follows:

$$\begin{array}{cccc} (A-2I), & (A-2I)^2, & (A-2I)^3, & (A+I), \\ \begin{Vmatrix} -1 & -2 & 3 & -2 \\ 1 & 3 & -1 & -1 \\ 2 & 3 & 0 & -2 \\ 2 & -2 & 6 & -5 \end{Vmatrix}, & \begin{Vmatrix} 1 & 9 & -13 & 8 \\ -2 & 6 & -6 & 2 \\ -3 & 9 & -9 & 3 \\ -2 & 18 & -22 & 11 \end{Vmatrix}, & \begin{Vmatrix} -2 & -30 & 42 & -25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -30 & 42 & -25 \end{Vmatrix}, & \begin{Vmatrix} 2 & -2 & 3 & -2 \\ 1 & 6 & -1 & -1 \\ 2 & 3 & 3 & -2 \\ 2 & -2 & 6 & -2 \end{Vmatrix}. \end{array}$$

In this case, where there are only two distinct roots r, the columns of $(A+I)$ are solutions of $(A-2I)^3C=0$, and the columns of $(A-2I)^3$ are solutions of $(A+I)C=0$. The solution (6) of (8) is given by

$$\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = M \cdot \begin{Vmatrix} 2c_1 - 2c_2 + 3c_3 \\ c_1 + 6c_2 - c_3 \\ 2c_1 + 3c_2 + 3c_3 \\ 2c_1 - 2c_2 + 6c_3 \end{Vmatrix} e^{2t} + \begin{Vmatrix} -2c_4 \\ 0 \\ 0 \\ -2c_4 \end{Vmatrix} e^{-t}.$$

where

$$M = \left\| \begin{array}{cccc} 1 - t + \ t^2/2 & -2t + \ 9t^2/2 & 3t - 13t^2/2 & -2t + \ 8t^2/2 \\ t - 2t^2/2 & 1 + 3t + \ 6t^2/2 & -t - \ 6t^2/2 & -\ t + \ 2t^2/2 \\ 2t - 3t^2/2 & 3t + \ 9t^2/2 & 1 \quad - \ 9t^2/2 & -2t + \ 3t^2/2 \\ 2t - 2t^2/2 & -2t + 18t^2/2 & 6t - 22t^2/2 & 1 - 5t + 11t^2/2 \end{array} \right\|.$$

A PARADOX IN COMPLEX NUMBERS AND INTEGRATION

C. O. OAKLEY, Haverford College

1. The paradox. From Peirce's table of integrals we find the formulas:*

124 $$\int \sqrt{x^2 - a^2}\, dx = \tfrac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log\left(x + \sqrt{x^2 - a^2}\right)\right],$$

125 $$\int \sqrt{a^2 - x^2}\, dx = \tfrac{1}{2}\left[x\sqrt{a^2 - x^2} + a^2 \arcsin (x/a)\right].$$

Now by the definition of $i = \sqrt{-1}$ we may write

$$2\int \sqrt{x^2 - a^2}\, dx = 2i \int \sqrt{a^2 - x^2}\, dx,$$

(1) $$= i(x\sqrt{a^2 - x^2} + a^2 \arcsin (x/a)),$$

$$= x\sqrt{x^2 - a^2} + ia^2 \arcsin (x/a),$$

$$-\log\left(x + \sqrt{x^2 - a^2}\right) = i \arcsin (x/a),$$

on multiplying i into the right-hand member of (1) and reducing; whence

$$-\log\left(x + i\sqrt{a^2 - x^2}\right) = i \arcsin (x/a).$$

Since $\log (a+ib) = \log \sqrt{a^2+b^2} + i \arctan (b/a)$, we finally obtain

$$-(\log a + i \arcsin (x/a)) = i \arcsin (x/a),$$

or

$$-\log a = 2i \arcsin (x/a).$$

This last is an obvious impossibility since the right-hand member is dependent upon x whereas the left-hand member is not.

2. Explanation and an application. This double-edged hoax may be cleared up by taking into careful account the following two points:

(a) $\sqrt{-N} = \pm i\sqrt{N}$, according as $N(\text{real}) \gtrless 0$;

(b) the constant of integration.

The ideas and formulas in this paradox may be made to serve a useful purpose in the theory of circular and hyperbolic functions and their relation to the

* These are typical and adequate for the purpose in mind; there are, of course, many other pairs which could be treated similarly.

exponential function. For example consider the circle $x^2+y^2=a^2$, the hyperbola $x^2-y^2=a^2$, the point $P(x, y)$ on either curve, and the three fixed points $O(0, 0)$, $A(a, 0)$, $B(0, a)$. Then the usual geometric definitions are, for sectorial measure u, $\sin u$, and $\sinh u$:

$$u = \frac{\text{Sector } OAP}{\triangle OAB};$$

$$\sin u = \frac{\triangle OAP}{\triangle OAB} = \frac{y}{a}, \quad \text{(point } P \text{ on the circle)};$$

$$\sinh u = \frac{\triangle OAP}{\triangle OAB} = \frac{y}{a}, \quad \text{(point } P \text{ on the hyperbola)}.*$$

In a perfectly direct and straightforward way it follows that for the hyperbola

$$a^2u = xy - 2\int_a^x \sqrt{x^2 - a^2}\,dx$$

$$= a^2 \log\left(\frac{x}{a} + \frac{y}{a}\right);$$

whence

$$\frac{y}{a} = \sinh u = \tfrac{1}{2}(e^u - e^{-u}),$$

which gives us the desired relationship.

However, the details in the corresponding procedure for the circular function $\sin u$ require much more care; and, since they seem not to have received complete expository treatment, it may be of interest to carry these out at this time. By a correct application of principles (a) and (b) in the steps below we have:

$$a^2u = xy + 2\int_x^a \sqrt{a^2 - x^2}\,dx,$$

$$= xy - 2i\int_x^a \sqrt{x^2 - a^2}\,dx,$$

$$= xy - i\left[-x\sqrt{x^2 - a^2} + a^2 \log\left(\frac{x + \sqrt{x^2 - a^2}}{a}\right)\right],$$

$$= xy - x\sqrt{a^2 - x^2} - ia^2 \log\left(\frac{x + i\sqrt{a^2 - x^2}}{a}\right).$$

$$= xy - xy - ia^2 \log\left(\frac{x}{a} + i\,\frac{y}{a}\right).$$

* In the case of the circle, u reduces to angle AOP.

Hence

$$u = -i \log\left(\frac{x}{a} + i\frac{y}{a}\right),$$

from which it readily follows that

$$\frac{y}{a} = \sin u = -\frac{i}{2}(e^{iu} - e^{-iu}).$$

THE PRODUCT OF n INTEGERS IN TERMS OF GREATEST COMMON DIVISORS AND LEAST COMMON MULTIPLES

R. A. ROSENBAUM, Yale University

In an article entitled *Notes on Greatest Common Divisor and Least Common Multiple of Integers* (this MONTHLY, vol. 19, 1912) B. F. Yanney generalizes to the product of n integers the well known relation

$$ab = DL \tag{1}$$

where D, L are, respectively, the greatest common divisor, least common multiple of the integers a and b.

He derives the expression

$$\Pi_k D_i \Pi({}_kL_i)^{k-1} \geqq (a_1a_2 \cdots a_n)^{(n-1)!/(n-k)!k!} \geqq \Pi({}_kD_i)^{k-1}\Pi_k L_i \tag{2}$$

where ${}_kD_i$, ${}_kL_i$ represent, respectively, the greatest common divisor, least common multiple of any k of the n integers $a_1, a_2, \cdots, a_n$. There are, for example, ${}_nC_k$ terms in $\Pi_k D_i$. From (2) he obtains, by setting $k=2$,

$$(a_1a_2 \cdots a_n)^{n-1} = \Pi_2 D_i \Pi_2 L_i. \tag{3}$$

Relation (3) might have been more simply derived directly from (1).

A different generalization of (1), giving the first power of the product of the n integers, is obtainable if we admit least common multiples *of* greatest common divisors, or vice versa:

$$a_1a_2 \cdots a_n = \prod_{j=1}^{n} L({}_jD_i) = \prod_{j=1}^{n} D({}_jL_i), \tag{4}$$

where $L({}_jD_i)$ is the least common multiple of the ${}_nC_j$ greatest common divisors of any j of the a's. The definition of $D({}_jL_i)$ is analogous. The method of proof of (4) is similar to the ordinary one of (3):

If the prime p_k appears to the power α_{ik} in a_i, and if the α's are arranged in descending order,

$$\alpha_{i_1k} \geqq \alpha_{i_2k} \geqq \cdots \geqq \alpha_{i_nk},$$

it is easily seen that p_k will appear to the power α_{i_jk} in $L({}_jD_i)$ and $D({}_{n-j+1}L_i)$, and only in these terms.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Mathematics in Action, Book II. By W. W. Hart and Lora D. Jahn. New York, D. C. Heath, 1939. 10+374 pages. $0.96.

Le Matematiche nella Storia e nella Cultura. Lezioni publicate per cura di Attilio Frajese. By Federigo Enriques. Bologna, Nicole Zanichelli, 1938. 339 pages. 20 Lire.

Vector Analysis. With an Introduction to Tensor Analysis. By J. H. Taylor. New York, Prentice-Hall, 1939. 9+180 pages. $2.85.

Descriptive Geometry. By A. V. Miller and K. S. Shields. Boston, D. C. Heath, 1939. 10+192 pages. $2.25.

General Analysis. Part II: The Fundamental Notions of General Analysis. By E. H. Moore. (Memoirs of the American Philosophical Society, vol. 1, part 2, 1939). Philadelphia, American Philosophical Society, 1939. 6+255 pages. $3.00.

First Course in Theory of Numbers. By H. N. Wright. New York, John Wiley & Sons; London, Chapman & Hall, 1939. 7+108 pages. $2.00.

Cowles Commission for Research in Economics. Reports of fifth annual research conference on Economics and Statistics held at Colorado Springs, July 3 to 28, 1939. University of Chicago, 1939. 93 pages.

Tables for Converting Rectangular to Polar Coördinates. By J. C. P. Miller. London, Scientific Computing Service Limited, 1939. 16 pages. 2s.

Elementary Number Theory. By J. V. Uspensky and M. A. Heaslet. New York and London, McGraw-Hill Book Company, 1939. 10+484 pages. $4.00.

REVIEWS

Descriptive Geometry. By James T. Larkins, Jr. New York, Prentice-Hall, Inc., 1939. 8+317 pages. $2.50.

The author of this new book on the classical subject of descriptive geometry states his principal objective as follows: *Descriptive Geometry should stimulate the student's judgment and ability to visualize.* The secondary objectives to be attained are to make the subject-matter appealing, comprehensive, understandable, interesting, and broadly practical. These specifications of the author offer the basis on which the book should be judged in comparison with other texts dealing with the same subject.

Opportunity for the development of the student's aptitude to visualize is consistently provided throughout the text. This objective is obtained by accompanying the orthographic presentation and verbal discussions with carefully planned and effectively drawn pictorial views.

Understandability is achieved in a large measure by dividing the text into

four basic sections. Each section is then developed about one central fundamental idea in orthographic projection. This scheme permits a strong association of related principles and, therefore, a clear grasp of the subject.

The extent to which the student's judgment in visualization is stimulated is rather difficult to appraise. The reviewer defines judgment as the skill to choose correctly and expeditiously on the basis of previously acquired knowledge and experience. The apparent lack of a convenient number of well-graded problems deprives the author of effective means not only to test the student's judgment but his grasp of the fundamentals as well.

With regard to the secondary objectives, all are exceptionally well reached with the important exception of that relating to *broadly practical*. The general tone of the book is academic rather than practical. This characteristic should by no means detract from the usefulness of the text, because a thorough and clear understanding of fundamentals will enable the competent student to make his application more reliably than if he concentrated only on practical solutions.

The subject-matter is presented and developed in accord with long established conventional teaching methods that are found in most text-books on this subject. It is regrettable that the author follows the classical precedent of presenting the projections in the *first angle*, when almost all engineering drawing in the United States is projected in the *third angle*. Pictorial projections—oblique, isometric, and perspective—are missing. Some mention about warped surfaces should have been included to round out the subject.

The narrative discussion of geometric constructions and proofs, the uniformly good line drawings, and the happy choice of pleasing type all combine to make this book appealing and understandable. The author may have had good reason for omitting photographs of practical engineering examples in which the principles of descriptive geometry are applied. The appeal to interest has been weakened by omitting practical problems and related technical illustrations.

The experienced teacher of descriptive geometry should get good results with this text, provided he has a good file of original problems and illustrative photographs.

FRANK KEREKES

College Algebra. By F. W. Sparks and P. K. Rees. New York, McGraw-Hill Book Company, 1939. 11+312 pages. $2.25.

The authors present the usual subject-matter of college algebra in a clear and concise style, but in an unusual order. To catch and hold the interest of the well-prepared student throughout the long review of high school algebra, the authors devote alternate chapters to old and new material. The new topics in general require a knowledge of the review material. Although the order adopted has its advantages, two important points for criticism exist. (1) The topics do not increase gradually in difficulty. For example, the first difficult topic, partial fractions, follows immediately after the review of high school algebra. After higher degree equations, simultaneous quadratics, and inequalities, comes the

easy topic of progressions. (2) Due to the unusual order, the authors are forced to assume theorems which they would otherwise be in a position to prove. Thus, since partial fractions precede higher degree equations, they must assume the theorem concerning identical polynomials. In fact, there is a general tendency throughout the text to omit or to state in the form of unproved theorems the more difficult portions of the requisite theory.

Definitions are stressed, and the material covered is well explained. The typography is excellent throughout the text, except that the columns in the general determinants are much too close together. For those seeking a moderately difficult college algebra text, this one is well worth a careful examination.

J. J. CORLISS

Selections Illustrating the History of Greek Mathematics. With an English translation by Ivor Thomas. *I: From Thales to Euclid.* (Loeb Classical Library.) Cambridge, Mass., Harvard University Press, 1939. 16+506 pages.

This volume, like all volumes in the Loeb Classical Library, gives the ancient texts on the lefthand pages and their English translation on the right; and, like many of the later volumes, it contains a large amount of annotation. The texts are intended to illustrate the history of Greek mathematics from Thales to Euclid. More than half of them, however, unfortunately have to be taken from writers later than Euclid, because very little of the earlier mathematical work remains in its original form. Only Zeno, Plato, Aristotle, and Euclid, can be quoted now as they actually wrote; for the rest, students of these matters have to weigh and interpret the statements of later writers who never imagined the degree of accuracy demanded by historians today.

For most of the thinkers and subjects concerned, I cannot judge whether Mr. Thomas has made the best selection possible, because I am unacquainted with most Greek mathematical literature; but his selections from Zeno, Plato, and Aristotle, whom I have read, seem to me very judicious. I could not want any change, except perhaps two more passages from Aristotle. One of these I will mention below; the other is that chapter in which Aristotle gives his opinion about the nature and ontological status of the *subject-matter* of mathematics (*Metaph.* M 3). Mr. Thomas has included so much of the Greek philosophy of mathematics that this important philosophical passage seems to demand inclusion too.

I have made no attempt to judge the soundness of the text; but the translation seems to be very accurate and easy to read. The selections are grouped neither wholly by subjects nor wholly by persons, a compromise for which Mr. Thomas asks indulgence, but which he declares to be more satisfactory than either of the two 'logical' ways. We may wonder why he puts Zeno after Democritus, when his own statements imply that Zeno was the older. We may also wonder why he separates 'Pythagorean Arithmetic' from 'Pythagorean Geometry' by inserting between them his sections on 'Thales' and on 'Proclus's

Summary' of the history of Greek mathematics. But on the whole we shall agree that the arrangement is as satisfactory as the texts permit.

Mr. Thomas expects three things to strike the modern reader in Greek mathematics: its rigor, its perfection of form, and the dominating position it assigned to geometry. To me it seems that there is a fourth particular, which should strike all modern mathematicians by reason of its sharp contrast with present notions; and that is the *dogmatism* of Greek mathematics. Euclid's 'postulates' and 'common notions' are presented as pieces of assured knowledge, so that all the consequent theorems are assured knowledge too. There is no trace of the modern notion of alternative and conflicting postulate-sets, of the modern refusal to assert the truth or falsehood of any primitive proposition. And here comes in my second query about Mr. Thomas's selections from Aristotle; he has, I think, concealed this important difference between Greek mathematics and our own by not quoting anything from the classic formulation of the Greek dogmatism in the first six chapters of Aristotle's *Posterior Analytics*.

RICHARD ROBINSON

Outline of the History of Mathematics. By Raymond C. Archibald. Oberlin, Ohio, Mathematical Association of America, Inc., 1939. 66 pages. $0.50.

This is the fourth edition of the *Outline* which originally consisted of two lectures delivered at the University of Minnesota in September, 1931 at a Summer School for Engineering Teachers. The first edition was reviewed in this MONTHLY, vol. 39, 1932, p. 422. Extensive revisions have been made in this edition, and the "Literature List and Notes" has been appreciably extended.

CAROLINE A. LESTER

TRIVIA MATHEMATICA

According to reports from the recent Columbus meetings of the Association and the Society, a new publication, *Trivia Mathematica*, is about to emerge from its chrysalis. At one of those profound meetings which last far into the night, learned savants sired and damned this new idea. Reports, though meager, place Brown and Princeton as focal points, with Cornell and M.I.T. as vortices of the hyperboles. Apparently without reMORSE, Princeton has added HURWITZ to those of divine Providence to FLOOD Cambridge and Ithaca with puns. No one has been judged best at this game, we understand, but doubtless by general consent they place WIENERworst.

E.J.M.

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.

NATIONAL CONVENTION OF KAPPA MU EPSILON

The fourth national convention was held at Charleston, Illinois on April 28–29, 1939, with Illinois Beta chapter as host to the convention. The fraternity was organized in 1931 and national conventions are held every two years. Previous conventions were held at Tahlequah, Oklahoma in 1933, at Pittsburg, Kansas in 1935, and at State College, Mississippi in 1937.

All eighteen chapters were represented at the last convention, with a total of eighty-five students and twenty-three faculty members from the seventeen chapters outside Charleston.

The convention opened Friday evening with a lecture by Professor R. D. Carmichael, Dean of the Graduate School, University of Illinois, who spoke on the "Consonance of Thought and Things." Following the lecture, a reception was held at Pemberton Hall for delegates and visitors.

Saturday forenoon was devoted to a program meeting of six student and two faculty presentations:

The circle contact problem, James McDonough, Illinois Alpha.

Pythagoras and the Pythagoreans, J. R. Davenport, Nebraska Alpha.

Geometrical solution of the quadratic equation, J. L. Campbell, Kansas Alpha.

An overview of mathematics, Professor Paul A. Devore, Missouri Beta.

Mathematics and religion, Eleanor Wilson, Alabama Alpha.

A little known theorem concerning symmetric determinants, Doris Kirgis, Iowa Alpha.

Trigonometric relationships found in integer right triangles, Robert Bemer, Michigan Alpha.

The struggle toward infinity, Dr. C. V. Newsom, New Mexico Alpha.

A picnic lunch was served at the college picnic grounds, after which the business session convened with reports of officers, chapter reports, reports of committees, and election of officers for the next biennium. The elected officers are:

President: Dr. C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

Vice-President: Dr. E. H. Taylor, State Teachers College, Charleston, Illinois.

Secretary: E. Marie Hove, State Teachers College, Wayne, Nebraska.

Treasurer: Dr. H. Van Engen, State Teachers College, Cedar Falls, Iowa.

Historian: Dr. Kathryn Wyant, Athens College, Athens, Alabama.

After the business meeting, an excursion was made to Shiloh Park, the last home of Lincoln's parents. The convention closed with a banquet for 110 members in Pemberton Hall. Professor J. A. G. Shirk as president presided at all sessions and acted as toastmaster at the banquet.

The retiring officers were:

President: Professor J. A. G. Shirk, State Teachers College, Pittsburg, Kansas.

Vice-President: Dr. C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

Secretary: E. Marie Hove, State Teachers College, Wayne, Nebraska.

Treasurer: Professor L. E. Pummill, State Teachers College, Springfield, Missouri.

Historian: Orpha Ann Culmer, State Teachers College, Florence, Alabama.

MATHEMATICS CLUB CONTESTS

The reports received from the various clubs bring out the fact that among the extra activities carried on by the organization may be a form of competition or contest for the benefit of its members or students of mathematics of the college. At least four clubs sponsor annual prizes for the best papers presented by members at regular meetings. The *Harvard Mathematical Club* sponsors the *Rogers Prize.* The award was won last year by Edwin Hewitt for his paper "Gödel's undecidable propositions"; a second prize went to J. C. Abbott who spoke on "Transfinite numbers." The *Sherk Memorial Prize* for the best paper at meetings of the *Mathematics Club* of the *University of*

Buffalo was awarded to Jack Lotsof who chose for his topic "Curve of pursuit." Merle Boyd of the *Mathematics Club* of *Boston University* received a prize for his paper "Dimensional equations," and Kenneth Schroeder of the *Mathematics Club* of the *University of Wisconsin Extension Division in Milwaukee* won the annual *Euler Prize Essay Contest* for his discussion of "A duodecimal arithmetic."

Several chapters of *Pi Mu Epsilon* sponsored competitions involving examinations in mathematics. *Alpha of Nebraska* at the *University of Nebraska* held an examination in freshman mathematics and calculus. The former was won by Merle Andrew and the latter by Roland Fricke, who incidentally holds the distinction of winning the freshman mathematics examination a year ago and of being the first to win both competitions since these have been held. First prize in the prize examination of *Missouri Alpha* at the *University of Missouri* was won by R. L. Powell, and second prize by Eugene Cowan. A calculus prize contest of *Alpha of California* at the *University of California at Los Angeles* was won by Harold Schniad. *Epsilon of New York* in coöperation with *Alpha Mu Gamma* at *St. Lawrence University* made its award to John Burgess. The *Mathematics Club* at *Brooklyn College* holds an integration contest each semester between the several classes in the calculus. Individual honors were secured by Harry Feingold, Lucille Kraus, Esther Conwell, and Paul Rosenbluth.

The members of the *Mathematics-Physics Club* of *Haverford College* had the following problem proposed for solution: "There is a battleship with fifteen guns which must be fired each once and only once each day in order to keep them clean. They are to be fired each day in five groups of three at a time. Find an arrangement such that each gun shall be fired once and only once each week in the same group with any other gun." The winner of this contest was Robert B. Dickson.

Hunter College Mathematics Club members were awarded prizes if they submitted or answered problems in the Problems and Solutions Department of this MONTHLY. Prizes were also offered to the three students submitting the best solutions to the missing word story "The Mathematical Saga of Linnie R. E. Quashun" (see this MONTHLY, vol. 46, p. 234).

Pi Mu Epsilon of *Brooklyn College* has sponsored for the last three years the Metropolitan Intercollegiate Mathematics Contest for colleges in the New York area. The material is that of analytic geometry and differential and integral calculus. Each contest consists of about fifteen questions, each given with its own time limit and each marked right or wrong immediately following expiration of this time limit, the marker's decision being subject to review by a committee on appeals. A team consists of five to seven undergraduates, the best five of whose scores make up the team score. A plaque with names of the winning teams is held for a year by each winning team, and permanent possession is to go to the first team to win it three times. The leading teams in order have been: 1937, Columbia University, Brooklyn College, Cooper Union Institute of Technology; 1938, Cooper Union, Brooklyn, Columbia; 1939, Cooper Union, Brooklyn, Manhattan College. The contest has been a close one each year, a tie for first place last year being broken only by considering points scored by the sixth and seventh contestants on each team. A medal is awarded to the high scoring individual each year. Any clubs in the metropolitan area wishing to participate, and clubs wishing a copy of the questions used in the last competition should address Professor L. S. Kennison, Director of Pi Mu Epsilon, Brooklyn College.

The *New York University, Washington Square College* chapter of *Pi Mu Epsilon* annually sponsors three contests. The first of these contests, the sixth annual interscholastic mathematics contest for high schools in New York City and vicinity, was held on April 22, 1939. There were 87 high schools represented, with a total of 329 contestants. The highest team score was made by Abraham Lincoln High School of Brooklyn, N.Y., which was awarded the Interscholastic Mathematics Contest Cup. Four sectional cups were awarded for the highest team score in each section, one to Weequahic High School of Newark, N.J., one to Central High School of Valley Stream, N.Y., one to A. B. Davis High School of Mt. Vernon, N.Y., and one to Boys High School of Brooklyn, N.Y. Three medals, one gold, one silver, and one bronze, were awarded to the contestants making the three highest individual scores.

Of particular interest to Junior College members of Mathematics Clubs may be the following problem listed in this first contest:

A dress manufacturer decides that on a certain new type of costume, which he is designing for spring trade, he will use combinations of the colors red, violet, blue, green, orange, yellow, and gray on the different models, with the following stipulations:

(a) the combination of any two colors should be used on one model;
(b) the combination of any two colors should not be used on more than one model;
(c) any two models should have one color in common;
(d) there should be at least one model;
(e) every model has at least three colors;
(f) all of the colors are not used on the same model.

Prove that:

I. any combination of two colors shall appear on one and only one model;
II. any two models have one and only one color in common.

Are the above conditions sufficient to prove the following? III. There are three colors which are not all on the same model. If the conditions are sufficient, state proof of III. If the conditions given are not sufficient, suggest sufficient additional condition or conditions and then state the proof of III.

The second contest, open to freshmen of *Washington Square College*, won by Lyell Grewer and Melvin Lax, was based on the topics: the nature and significance of non-euclidean geometry; mathematics in the arts and as an art; polar coördinates. Each contestant was supplied with the following list of references:

1. The nature and significance of non-euclidean geometry:
   Bell, E. T. *The Search for Truth*, chapter XIV; Cajori, Florian, *A History of Mathematics*, pp. 302–309; Carslaw, H. S. *Non-Euclidean Plane Geometry*; Cooley, Gans, Kline and Wahlert, *Introduction to Mathematics*, chap. XVIII and XIX; Manning, H. P. *Non-Euclidean Geometry*; Young, J. W. A. *Monographs on Topics of Modern Mathematics.*
2. Mathematics in the arts and as an art:
   Cooley, Gans, Kline and Wahlert, *Introduction to Mathematics*, pp. 151–159, 367–378, 406–410, 551–555; Ellis, Havelock, *The Dance of Life*, chap. V; Jeans, J. H. *Science and Music*; Miller, D. C. *The Science of Musical Sounds*; Redfield, John, *Music*, chap. I–VII; Ruskin, John, *The Elements of Drawing and Perspective*, the section on perspective; Russell, Bertrand, *Mysticism and Logic*, the essay on "The Study of Mathematics;" Sullivan, J. W. N. *The Limitations of Science*, chap. VI.

The third contest, based on "Topics in Function Theory," was open to any student taking a course in the college in the spring term, 1939. The prize of twenty dollars was won by Harold Cooperman. The following outline was supplied each person participating:

1. Real Number System: (a) rational numbers; (b) sequences; (c) Cantor's and Dedekind's theories of irrational numbers; (d) geometric representation of real numbers; (e) base of a system of notation.
2. Continuity and Discontinuity of Functions: (a) definition of function; (b) definition of limit; (c) laws of operations with limits; (d) continuity and discontinuity of a function; (e) properties of continuous functions.
3. Derivatives and their Properties: (a) definition of a derivative; (b) continuous functions having no derivative; (c) definition of partial derivatives; (d) properties of partial derivatives; (e) total differentiability; (f) order of differentiation.

The questions asked on the examination were based only on related material found in the following books: Cooley, Gans, Kline and Wahlert, *Introduction to Mathematics*, chap. II, III (§§3, 4), IX, XIV, XV, XVI, XVII; Courant, Richard, *Differential and Integral Calculus*, vol. I, chap. I, II; Dantzig, Tobias, *Number, the Language of Science*; Fine, H. B., *The Number System of Algebra*; Fine, H. B., *College Algebra*, Part I; Fite, W. B., *Advanced Calculus*, chap. I, II, III; Hall, H. S. and Knight, S. R., *Higher Algebra*, chap. VII; Hardy, G. H., *Pure Mathematics*, chap. I, II; IV, V, VI; Hobson, E. W., *The Theory of Functions of a Real Variable*, vol. I, (Second Edition), chap. I, IV (sections 144–155), V; Smith, Charles, *A Treatise on Algebra*, chap. XVIII; Townsend, E. J., *Functions of Real Variables*, chap. I, III, IV.

CLUB TOPICS

Requests for bibliography or a discussion on the following topics have been received from a number of organizations:

An axiomatic treatment of the number system.
Ruled surfaces.
Mathematics correlated with the fine arts.
Life and contributions of outstanding mathematicians of today.
History of Pi Mu Epsilon.

The *Mathematics Club* of the *University of Kansas* writes: "We would appreciate references to articles either surveying mathematics in its entirety or some very comprehensive subdivisions thereof."

The report of the *Mathematics Club* of *Wayne University* shows that this organization considered this question in planning its last year's program: "A consistent and well received effort was made this year to expose some of the basic ideas of more advanced mathematics with which the average undergraduate student frequently remains unfamiliar." Among topics considered at meetings were the following: finite groups, orthogonal functions, modern concept of function, the coloring of maps, and Boolean algebras. Among the books used, they list: Mathewson, L. C., *Elementary Theory of Finite Groups*; Encyclopedia Brittanica; Stone, M. H., *Linear Operations in Hilbert Space*; Kellogg, O. D., *Foundations of Potential Theory*; Sierpinski, W., *General Topology*; Volterra, V., *Theory of Functionals*; Levy, P., *L'Analyse Fonctionnelle.*

This department would welcome additional suggestions for publication on the above question.

UNDERGRADUATE PUBLICATIONS

A magazine making its first appearance in the spring of 1939 is *The Mathematical Review*, published by the *Mathematics Society* of the *City College of New York*. The introduction states that "the material in this issue has been selected with the view to providing both the newcomer in collegiate mathematics and the advanced student with material at once profitable and enjoyable." Contained in this forty page mimeographed edition are the following articles: Beyond calculus, Professor B. P. Gill; A note on orthogonal functions, Aaron Galuten; The Königsberg bridge problem, Phil Weiss; Almost periodic functions, Herbert Mintzer; Squaring the circle, Frank Beckman; Some problems in topology, Eugene Isaacson; Generalization of the number concept, K. Arrow; On theory of plane curves, Harry Soodak; Polynomials of Bernoulli and Euler, S. Katz. Also included are a number of problems and puzzles for solution.

The *Math Mirror* of *Brooklyn College*, now in its seventh year of publication, contains news notes of the various activities of the mathematics clubs and of the mathematics department, and the following articles (by student members): Constructions with straight edges only, Benjamin Liebowitz; A generalization of Euler's ϕ function, Daniel Rosen; Geometric difference series, Emanuel Mehr; A miniature mathematical science, Abraham Barshop. This edition concludes with a set of thirteen problems of which the following are illustrations:

1. Tangents and normals are drawn from the ends of a focal chord of a parabola. Prove that the intersections of the tangents and of the normals both lie on the same line parallel to the axis.

2. Find $\lim_{n\to\infty}[(1-1/9)\ (1-1/25)\cdots(1-1/\{2n+1\}^2)]$.

PROBLEMS AND SOLUTIONS

Edited by Otto Dunkel, Orrin Frink, Jr., and H. S. M. Coxeter

Note by the Editor. I wish to acknowledge gratefully the assistance rendered for several years by Dr. H. L. Olson who is retiring as an editor of this department, and to welcome Dr. Orrin Frink, Jr. as a new editor. E. J. M.

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 401. *Proposed by A. H. Stone, Graduate College, Princeton.*

Fit together twenty-eight squares, of sides 2, 18, 22, 37, 38, 39, 41, 43, 49, 67, 72, 80, 85, 103, 116, 154, 164, 175, 178, 192, 200, 207, 215, 222, 230, 247, 422, 593, to make a single square (of side 1015). Is there any simpler solution to the problem of dissecting a square into several unequal squares?

E 402. *Proposed by Irving Kaplansky, Harvard University.*

If n, r, and a are positive integers, the congruence $n^2 \equiv n \pmod{10^a}$ obviously implies $n^r \equiv n \pmod{10^a}$. (When such a number n has only a digits, it is called an automorphic number.) For what values of r does $n^r \equiv n \pmod{10^a}$ imply $n^2 \equiv n \pmod{10^a}$?

E 403. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Show that the two conics

$$x^2 + 2xy + 3y^2 - 1 = 0, \qquad 2x^2 - 6xy - y^2 + 22 = 0$$

have the same director circle.

E 404. *Proposed by V. Thébault, Le Mans, France.*

Determine the largest and smallest perfect squares which can be written with the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, used once each in both cases.

E 405. *Proposed by James Travers, Harrow, England.*

Construct points P and Q on the respective sides AB and BC of a given triangle ABC, so that $AP = PQ = QC$.

SOLUTIONS

E 357 [1939, 48]. *Proposed by H. T. R. Aude, Colgate University.*

The equation, $x^2 + xy + y^2 = k$, is to be solved in positive integers, with $x < y$. Find the least value of k for which there are exactly five distinct solutions.

Solution by Harold Davenport, University of Manchester, England.

It is quite easy to enumerate all numbers k for which the given equation has

exactly 5 integral solutions, subject to $0<x<y$. We use the following two facts:

I. The *total* number of integral solutions of $x^2+xy+y^2=k$ is zero if 2 divides k to an odd power, or if any prime of the form $6n-1$ divides k to an odd power. In all other cases it is

$$6(\alpha_1+1)(\alpha_2+1)\cdots,$$

where $\alpha_1, \alpha_2, \cdots$ are the powers to which k is divisible by primes of the form $6n+1$.

Proof. By Ex. 2, p. 80 of Dickson's *Introduction to the Theory of Numbers* (Chicago, 1929), the total number of integral solutions is $6\mathrm{E}(k)$, where $\mathrm{E}(k)$ is the excess of the number of divisors of k of the form $3n+1$ over the number of the form $3n-1$. It is easily seen that if k_1, k_2 are relatively prime, then $\mathrm{E}(k_1k_2)=\mathrm{E}(k_1)\ \mathrm{E}(k_2)$. Calculating $\mathrm{E}(p^\alpha)$ for each prime power, we obtain the result stated.

II. If k is neither a square nor three times a square, the total number of solutions is twelve times the number of solutions subject to $0<x<y$. If k is either a square or three times a square, there are 6 more solutions.

Proof. If k is neither a square nor three times a square, there are no solutions with $x=0$ or $x=y$. The solutions can be grouped in sets of twelve:

$$\begin{array}{cccc} (x,y), & (y,x), & (-x,-y), & (-y,-x), \\ (-x,x+y), & (x+y,-x), & (x,-x-y), & (-x-y,x), \\ (-y,x+y), & (x+y,-y), & (y,-x-y), & (-x-y,y). \end{array}$$

The solutions forming any set are all different, and exactly one of them satisfies $0<x<y$. One way of seeing this is to consider in which 30° angle the complex number $x-\rho y$ lies, where ρ is a cube root of unity.

If $k=u^2$, there are, in addition to the above sets, the solutions

$$(u,0),\quad (0,u),\quad (-u,0),\quad (0,-u),\quad (u,-u),\quad (-u,u).$$

If $k=3u^2$, there are the solutions

$$(u,u),\quad (-u,-u),\quad (u,-2u),\quad (-2u,u),\quad (-u,2u),\quad (2u,-u).$$

It follows from I and II that if there are exactly 5 solutions subject to $0<x<y$, then

$$(\alpha_1+1)(\alpha_2+1)\cdots = 10 \quad \text{or} \quad 11.$$

Hence the possible values for k are

$$p_1^4p_2K,\quad p^9K,\quad p^{10}K,$$

where p, p_1, p_2 denote primes of the form $6n+1$, and K is any product of powers of $2^2, 3, 5^2, 11^2, 17^2, 23^2, \cdots$.

The least such number is plainly $7^4\cdot 13=31213$. The five solutions for this value of k are

$$(9, 172), \quad (28, 161), \quad (49, 147), \quad (84, 119), \quad (101, 103).$$

Partially solved by G. W. Wishard, and the proposer (by the method of his solution to Problem E 326 [1939, 110]).

E 361 [1939, 49]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

The medians of a triangle ABC cut the nine-point circle of that triangle again at D, E, and F, respectively. The tangents to this circle at D, E, and F meet the corresponding sides of the orthic triangle (with vertices at the feet of the altitudes of ABC) at the points P, Q, and R respectively. Prove P, Q, and R collinear.

Solution by L. M. Kelly, Boston University.

Let LMN be the orthic triangle, let I be the midpoint of BC, and let AI meet MN in V. Since triangles PMD, PDN are similar, we have $PM/PD = DM/DN = PD/PN$, and therefore

$$PM/PN = \overline{DM}^2/\overline{DN}^2.$$

Since $MI = \frac{1}{2}a = NI$, the angle MDN is bisected by DI, and

$$DM/DN = VM/VN.$$

Since MN is antiparallel to BC, the median AI is a symmedian of triangle AMN, and

$$VM/VN = \overline{AM}^2/\overline{AN}^2 = c^2/b^2.$$

Combining these results, $PM/PN = c^4/b^4$. Similarly $QN/QL = a^4/c^4$, $RL/RM = b^4/a^4$. Hence

$$\frac{PM \cdot QN \cdot RL}{PN \cdot QL \cdot RM} = \frac{c^4a^4b^4}{b^4c^4a^4} = 1,$$

and P, Q, R are collinear, by the theorem of Menelaus.

E 362 [1939, 106]. *Proposed by V. Thébault, Le Mans, France.*

If X and Y are consecutive positive integers, and Z is an integer, and if $X^2 + Y^2 = Z^2$, show that each of the two numbers, $(X + Y \pm Z)^2 + 1$, is the sum of the squares of two consecutive integers, and that of these last four integers the two odd ones are squares, and the even ones are the doubles of squares.

Solution by V. W. Graham, Dublin, Ireland.

Since $(X - Y)^2 = 1$, we have

$$\begin{aligned}(X + Y \pm Z)^2 &= (Z \pm X)^2 + (Z \pm Y)^2 - (Z^2 - 2XY)\\ &= (Z \pm X)^2 + (Z \pm Y)^2 - 1,\end{aligned}$$

which proves the first statement. Now the general solution of $X^2 + Y^2 = Z^2$ is

$$X \text{ or } Y = \lambda(m^2 - n^2), \qquad Y \text{ or } X = 2\lambda mn, \qquad Z = \lambda(m^2 + n^2).$$

Since $Y-X=\pm 1$, we must have $\lambda=1$. Therefore

$$Z+X \quad \text{or} \quad Z+Y=2m^2, \qquad Z+Y \quad \text{or} \quad Z+X=(m+n)^2,$$
$$Z-X \quad \text{or} \quad Z-Y=2n^2, \qquad Z-Y \quad \text{or} \quad Z-X=(m-n)^2.$$

Also solved by H. T. R. Aude, Calvin Foreman, Wm. Forman, L. S. Johnston, E. P. Starke, W. R. Talbot, and the proposer.

E 364 [1939, 106]. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

If $(1+x)^n/(1-x)^3=a_0+a_1x+a_2x^2+\cdots$, show that

$$a_0+a_1+a_2+\cdots+a_{n-1}=\frac{n}{3}(n+2)(n+7)\cdot 2^{n-4}.$$

Solution by F. A. Alfieri, New York, N. Y.

It is evident that $a_0+a_1+a_2+\cdots+a_{n-1}$ is the coefficient of x^{n-1} in the expansion of

$$(a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}+\cdots)(1+x+x^2+\cdots+x^{n-1}+\cdots)$$
$$=\frac{(1+x)^n}{(1-x)^3}\,\frac{1}{(1-x)}=\frac{(1+x)^n}{(1-x)^4}=\frac{\{2-(1-x)\}^n}{(1-x)^4}$$
$$=2^n(1-x)^{-4}-n\cdot 2^{n-1}(1-x)^{-3}+\binom{n}{2}2^{n-2}(1-x)^{-2}$$
$$-\binom{n}{3}2^{n-3}(1-x)^{-1}+(\text{a polynomial of degree } n-4 \text{ in } 1-x).$$

The coefficient of x^{n-1} in

$$2^n\sum_3^\infty\binom{r}{3}x^{r-3}-n\cdot 2^{n-1}\sum_2^\infty\binom{r}{2}x^{r-2}+\binom{n}{2}2^{n-2}\sum_1^\infty rx^{r-1}-\binom{n}{3}2^{n-3}\sum_0^\infty x^r$$

is

$$2^n\binom{n+2}{3}-n\cdot 2^{n-1}\binom{n+1}{2}+\binom{n}{2}2^{n-2}n-\binom{n}{3}2^{n-3}$$
$$=\{8n(n+1)(n+2)-12n^2(n+1)+6n^2(n-1)$$
$$-n(n-1)(n-2)\}2^{n-4}/3$$
$$=n(n+2)(n+7)2^{n-4}/3.$$

Also solved by Wm. Forman (by induction), V. W. Graham, E. P. Starke, C. W. Williams (by differentiation), H. A. Wood (by contour integration), and the proposer.

E 365 [1939, 106]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Prove that

$$\int_0^1 x^n(1-x^n)^n dx = \frac{n^n(n!)}{(n+1)(2n+1)(3n+1)\cdots[(n+1)n+1]}.$$

I. *Solution by J. Rosenbaum, Bloomfield, Conn.*

By the theory of partial fractions,

$$\frac{n!}{t(t+1)(t+2)\cdots(t+n)} = \sum_{r=0}^{n} \frac{X_r}{t+r},$$

where

$$X_r = \frac{n!}{-r(-r+1)\cdots(-1)\cdot 1\cdot 2\cdots(-r+n)} = (-)^r\binom{n}{r}.$$

Putting $t=n^{-1}+1$, we deduce

$$\frac{n^n(n!)}{(1+n)(1+2n)(1+3n)\cdots[1+(n+1)n]} = \sum_0^n (-)^r\binom{n}{r}/[1+(r+1)n]$$

$$= \sum_0^n (-)^r\binom{n}{r}\int_0^1 x^{(r+1)n}dx = \int_0^1 x^n(1-x^n)^n dx.$$

II. *Solution by H. A. Wood, University of Connecticut.*

The substitution $u=x^n$ gives

$$\int_0^1 x^n(1-x^n)^n dx = n^{-1}\int_0^1 u^{n^{-1}}(1-u)^n du = n^{-1}\mathrm{B}(n^{-1}+1, n+1)$$

$$= \frac{\Gamma(n^{-1}+1)\Gamma(n+1)}{n\Gamma(n^{-1}+n+2)}$$

$$= \frac{\Gamma(n+1)\Gamma(n^{-1}+1)}{n(n^{-1}+n+1)(n^{-1}+n)\cdots(n^{-1}+1)\Gamma(n^{-1}+1)}$$

$$= \frac{\Gamma(n+1)}{n(1+n^{-1})(2+n^{-1})(3+n^{-1})\cdots(n+1+n^{-1})}$$

$$= \frac{n^n(n!)}{(n+1)(2n+1)(3n+1)\cdots[(n+1)n+1]}.$$

Also solved by Calvin Foreman, Wm. Forman, V. W. Graham, Harry Siller, E. P. Starke, and C. W. Williams.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at the least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3937. *Proposed by V. Thébault, Le Mans, France.*

A given circle has the fixed chord BC and a variable point A on its circumference; the midpoints of CA, AB are B_1, C_1; the centers of the inscribed and escribed circle for the angle A of triangle ABC are I and I_a; the parallels to AB through I, I_a meet AC in M, M'; and the parallels to AC through I, I_a meet AB in N, N'. Prove that: (1) The altitudes of triangle AB_1C_1 from B_1 and C_1 pass each through a fixed point. (2) The circles tangent to the sides of angle A with centers at the orthocenters of triangles IMN, $I_aM'N'$ envelop a fixed circle. (3) The locus of the midpoints of MN and $M'N'$ is a limaçon of Pascal.

3938. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres (A), (B), (C), (D) with non-coplanar centers, the four spheres (A'), (B'), (C'), (D') are constructed belonging, respectively, to the four coaxal nets determined by the triads of spheres (B), (C), (D); (C), (D), (A); (D), (A), (B); (A), (B), (C). Find four spheres coaxal, respectively, with the four pairs of spheres (A) and (A'), (B) and (B'), (C) and (C'), (D) and (D'), and forming a coaxal pencil.

3939. *Proposed by J. H. Curtiss, Cornell University.*

Define the function $f(x)$ by the relations

$$\begin{aligned} f(x) &= x \sin(1/x), \qquad x > 0, \\ &= 0, \qquad x = 0. \end{aligned}$$

Show that $|f(x_1)-f(x_0)|/|x_1-x_0|^\alpha$ is bounded for $0 \leqq x_0 \leqq 1$, $0 \leqq x_1 \leqq 1$, if and only if $\alpha \leqq 1/2$.

3940. *Proposed by Otto Dunkel, Washington University.*

Denote by $\sigma_r(n)$ the elementary symmetric function of the consecutive integers $1, 2, \cdots, (n-1)$, the sum of the products of r at a time. Show that

$$\sigma_r(n) = n(n-1)\cdots(n-r)P_r(n),$$

where $P_r(n)$ is a polynomial in n of degree $r-1$, and that $P_r(x)$ vanishes for $x=0, 1$, if r is odd and greater than unity.

Develop a method for obtaining consecutively the explicit expressions for $\sigma_r(x)$, or $P_r(x)$.

SOLUTIONS

3849 [1937, 667]. *Proposed by V. Thébault, Le Mans, France.*

Let O be the circumcenter of the tetrahedron $ABCD$, and P an arbitrary point in space. The segments of straight lines PA, PB, PC, PD are divided in the same ratio u, and the points of division are taken as centers of four spheres with radii $v\cdot PA$, $v\cdot PB$, $v\cdot PC$, $v\cdot PD$, respectively. The radical center R of these four spheres is on the straight line OP so that

$$OR:OP = (v^2 - u^2 + u):u.$$

This is a generalization of a proposition by N. A. Court in which $P\equiv G$. See this MONTHLY, 1932, p. 198.

Solution by W. T. Short, Oklahoma Baptist University, Shawnee, Okla.

With P as origin of rectangular coördinates, let the vertices A_i of the tetrahedron be (x_i, y_i, z_i). The coördinates of Q_i, which divides PA_i in the ratio $PQ_i/PA_i=u$, will then be (ux_i, uy_i, uz_i); and the equation of the sphere with center Q_i and radius vPA_i will be

(1) $$\sum (x - ux_i)^2 = v^2 \sum x_i^2, \qquad (i = 1, 2, 3, 4),$$

where the summation for each i is for x, y, z. The radical center R has coördinates which are solutions of the three equations

(2) $$2u\sum x(x_i - x_1) = (u^2 - v^2)\sum (x_i^2 - x_1^2), \qquad (i = 2, 3, 4).$$

Let x_0, y_0, z_0 be the coördinates of the circumcenter O of the given tetrahedron; then these coördinates are determined by the system

(3) $$2\sum x_0(x_i - x_1) = \sum (x_i^2 - x_1^2), \qquad (i = 2, 3, 4).$$

The solutions of the systems (2) and (3) are unique, since we consider only non-degenerate tetrahedrons. Hence the equation

(4) $$\sum [ux - (u^2 - v^2)x_0](x_i - x_1) = 0$$

has the unique solution, for the coördinates of R,

(4) $$ux_R = (u^2 - v^2)x_0, \quad etc.$$

This says that

(5) $$PR/PO = (u^2 - v^2)/u, \quad \text{or} \quad OR/OP = (v^2 - u^2 + u)/u.$$

It will be seen that by extending the summations to n independent variables and setting $i=1, 2, \cdots, n+1$, the above is a proof of the extension to space of n dimensions.

Solved also by L. M. Kelly, and the proposer.

Editorial Note. The remaining two solutions were synthetic; and Kelly stated that a proof may be obtained from the one by Court in the problem reference by little more than the mere substitution of P for G.

3850 [1937, 668]. *Proposed by V. Thébault, Le Mans, France.*

Let BCA_1A_2, CAB_1B_2, ABC_1C_2 be squares constructed interiorly on the sides of a triangle ABC for which V is the angle of Brocard. If $\cot V=2$, the lines which join A, B, C, respectively, to the symmetrics of A_1, B_1, C_1 with respect to A_2, B_2, C_2, meet in a point.

See corrected form in 3921.

Solution by J. W. Clawson, Ursinus College.

Extend P_1P_2 its own length to P_3, $(P=A, B, C)$.

Taking the triangle ABC as the basis for a system of trilinear coördinates, and using the usual notation of elementary trigonometry, we find that A_1 is $(a, -a\cos C, a\sin B-a\cos B)$; and A_3 is $(a, 2a\sin C-a\cos C, -a\sin B-a\cos B)$.

Hence the equation of AA_3 is readily seen to be:

$$\frac{y}{z}=\frac{\cos C-2\sin C}{\cos B+\sin B}=\frac{\cot C-2}{\cot B+1}\cdot\frac{\sin C}{\sin B}.$$

In a similar way, the equations of BB_3 and CC_3 are obtained.

The condition that these three straight lines shall be concurrent is clearly

$$\frac{\cot C-2}{\cot B+1}\cdot\frac{\cot B-2}{\cot A+1}\cdot\frac{\cot A-2}{\cot C+1}=1;$$

or $\cot A\cot B+\cot B\cot C+\cot C\cot A+3=\cot A+\cot B+\cot C$.

But it is well known that $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$, (Hobson, *Trigonometry*, page 45).

Hence the condition for concurrency reduces to

$$\cot A+\cot B+\cot C=4;\ \textit{i.e.},\ \cot V=4.$$

Note. If this solution is correct, the problem should read $\cot V=4$, instead of $\cot V=2$.

Editorial Note. For $\cot V=4$ the straight lines AA_3, BB_3, CC_3 meet in a point; but, as will appear later, the point of intersection is at infinity. If instead of the point A_3 on A_1A_2, we take the point M_a where $A_1M_a=kA_1A_2$, and similarly $B_1M_b=kB_1B_2$, $C_1M_c=kC_1C_2$, the homogeneous barycentric coördinates of M_a are

(1) $$\sin A,\quad \sin B\,[k\sin C-\cos C],\quad -\sin C\,[(k-1)\sin B+\cos B\,];$$

and the equation of AM_a is

(2) $$\frac{y}{k-c_3}=\frac{z}{-(k-1+c_2)},\quad c_1=\cot A,\quad c_2=\cot B,\quad c_3=\cot C.$$

The straight lines BM_b and CM_c meet in A' with the coördinates

(3) $$-(k-c_2)(k-1+c_3),\quad (k-1+c_3)(k-1+c_1),\quad (k-c_1)(k-c_2).$$

The sum of these coördinates is $k^2-k+2-\cot V$, and in order to make (3) absolute areal coördinates, we divide each coördinate by this sum. The points B' and C' for the same k are then obtained by circular permutation. For any given value of k the straight lines AM_a, BM_b, CM_c are parallel if $\cot V=k^2-k+2$. In the problem $k=2$ and then $\cot V=4$, as in the above solution. Denote the area of ABC by S and that of $A'B'C'$ by S'; then, by the area formula in the solution of 3795 [1938, 697],

$$S' = \frac{(2k-1)^2 S}{k^2-k+2-\cot V}, \quad \cot V \neq k^2-k+2. \tag{4}$$

Thus the three straight lines AM_a, BM_b, CM_c meet in a finite point, if and only if $k=1/2$ and $\cot V\neq 7/4$. If $\cot V=7/4$ and $k\neq 1/2$, $S'=4S$.

3851 [1938, 52]. *Proposed by V. Thébault, Le Mans, France.*

Let $N=123\cdots(n-3)(n-2)n$ be a number written in the system with the base $n+1$, where the digits in that system are placed in increasing order of magnitude omitting 0 and $(n-1)$. Form the product $P=N\cdot L$, where $L=\alpha\beta$ has two digits such that the sum of the digits $\gamma=\alpha+\beta$ is a number less than n and prime to n. The product may be written with n distinct digits taken from the $n+1$ figures $0, 1, 2, 3, \cdots, n$ and suitably arranged; the digit missing will be $n-\gamma$.

Solution by E. P. Starke, Rutgers University.

(1) Any number is congruent to the sum of its digits, mod n. Thus the $(n+1)$-digit number $M=\alpha\gamma\gamma\gamma\cdots\gamma\beta$ is a multiple of n.

(2) If M be divided by n, no two of the partial divisions can produce the same remainders. For if the ith and jth remainders, r_i and r_j, $j<n$, could be equal, the $(j-i+1)$-digit number $r_i\gamma\gamma\gamma\cdots\gamma(\gamma-r_j)$ would be divisible by n, whereas by (1) it is congruent to $\gamma(j-i)$ mod n with $j-i<n$ and with γ relatively prime to n.

(3) For the divisions of the two-digit numbers $\theta\gamma$, $\theta=0, 1, 2, \cdots, (n-1)$, by n, different remainders imply different quotients. Thus the digits of M/n are all distinct.

(4) The product of the given number N by n is the n-digit number $R=111\cdots 1$. Thus P is the quotient of $R\cdot L=M$ by n. Now by (1), $N+(n-1)\equiv\sum_{k=0}^{n}k$ or $N\equiv\Sigma k+1$ (mod n). Thus $P=N\cdot L\equiv\gamma\Sigma k+\gamma$. Thus the sum of digits of P is congruent to $\gamma\Sigma k+\gamma$. By (3) the n digits of P are all distinct; let d be the missing digit. Then $\gamma\Sigma k+\gamma+d\equiv\Sigma k$ or $(\gamma-1)\Sigma k+\gamma+d\equiv 0$. From $\Sigma k=n(n+1)/2$, we see that $\Sigma k\equiv 0$ if n is odd; while if n is even, γ(prime to n) is odd and $(\gamma-1)$ is even. Thus $(\gamma-1)\Sigma k\equiv 0$ for every n. Then $\gamma+d\equiv 0$ and $d=n-\gamma$.

3853 [1938, 52]. *Proposed by N. A. Court, University of Oklahoma.*

Two tangent spheres (A), (D) are each touched by the spheres (B) and (C). The two lines joining an arbitrary point of (A) to the points of contact of this

sphere with (B) and (C) meet the latter two spheres again in two points coplanar with the two points of contact of (D) with (B) and (C).

I. *Solution by the Proposer.*

Let X, Z, P be the points of contact of the pairs of spheres (A) and (B), (A) and (D), (B) and (D); let L be an arbitrary point of (A) and let B', D' be the second points of intersection of the lines LX, LZ with the spheres (B), (D), respectively; let the line $B'P$ meet (D) again in D''. The points X, Z, P are centers of similitude of the respective pairs of spheres; we have therefore the pairs of parallel radii $AL \| BB'$, $AL \| DD'$, $BB' \| DD''$; hence the three points D, D', D'' are collinear. Thus the line PB' passes through the diametric opposite D'' of D' on the sphere (D). Similarly, if C' is the second point of intersection of the sphere (C) with the line LY joining L to the point of contact of (C) with (A), the line QC' joining the point C' to the point of contact Q of the spheres (C) and (D) will pass through the point D''. Hence the proposition.

II. *Solution by L. M. Kelly, Northeastern University, Boston, Mass.*

Consider the triangle LXY. The line $B'C'$ intersects the line XY in the point J and divides it in the ratio $JY/JX = LB' \cdot YC'/XB' \cdot LC'$. Again in the skew quadrilateral $ACDB$, XY and PQ will be coplanar and intersect in J' on BC; see Court's *Solid Geometry*, p. 111. Then $J'Y/J'X = AB \cdot YC/XB \cdot AC$. But from similar triangles $LX/XB' = AX/XB$, or $LB'/XB' = AB/XB$; and, similarly, $YC'/LC' = YC/AC$. Hence $J'Y/J'X = JY/JX$, and $J' \equiv J$. Therefore, the points B', C', P, Q are coplanar.

Solved also by W. T. Short.

3854 [1938, 53]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Prove that

$$(x + \sqrt{y})^4 + (x - \sqrt{y})^4 = z^4$$

has no solution in which x, y, and z are positive integers.

Solution by Andrew G. Clark, Colorado State College.

The equation with the terms of its left member expanded becomes $2x^4 + 12x^2y + 2y^2 = z^4$ which evidently can be expressed as $2(3x^2 + y)^2 = z^4 + (2x)^4$. It is well known that this form has no solutions in positive integers other than the trivial solutions $z^2 = 4x^2 = 4y$. (*Cf.* L. E. Dickson, *Introduction to the Theory of Numbers*, p. 43.)

Solved also by E. P. Starke, Elijah Swift, C. W. Trigg, and W. Wernick.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The National Council of Teachers of Mathematics will have its 20th annual meeting in St. Louis, February 22–23, 1940. The theme of the meetings will be "Mathematics for the Other-Than-College-Preparatory Student." All phases of this theme will be discussed in the four divisions of the meetings: I. General Meetings, II. Elementary Schools Program, III. Secondary Schools Program, and IV. Teacher Education Program. Questions and suggestions should be directed to 525 West 120th Street, New York City, N.Y.

The suggestion has been made that a section of the Mathematical Association of America should be organized in Central New York. This is a preliminary notice that a meeting will be held on Saturday, May 11, 1940, at Colgate University, Hamilton, New York, which may result in the organization of such a section. The tentative plan for the meeting includes the presentation of papers, a discussion of the proposal for organizing a section with action upon the proposal, and a dinner which will be held early enough for those coming a distance to return home at a reasonable hour. Further information may be obtained from Professor C. W. Munshower, Colgate University, Hamilton, New York.

The first meeting of the Maine Mathematical Conference was held at Colby College, Waterville, Maine, on Saturday, December 9, 1939, with a morning session, a noontime luncheon, and an afternoon session. Papers were presented by Professor N. R. Bryan on "Reorganization in the subject-matter of college algebra," by Professor I. J. Schoenberg on "A new freshman course at Colby for non-science majors," by Professor C. T. Holmes on "Bowdoin mathematical courses," by Professor S. H. Kimball on "Transfer of training," and by Professor R. L. Korgen on "A new course in logic." An interesting feature of the meeting was the inspection of the Landau Mathematical Collection which was recently lent to and set up in the Colby College Library.

Dr. Lury Barber of the University of Toronto has been appointed lecturer in actuarial sciences at the University of Manitoba.

Dr. W. C. McDaniel has accepted a position at Southern Illinois Normal University.

Associate Professor Paul Muehlmann of Xavier University, Cincinnati, has been promoted to a professorship.

Assistant Professor R. C. Stephens of Knox College has been promoted to an associate professorship. He is on leave of absence during the current academic year, being at Princeton University.

Assistant Professor E. C. Stopher of Ashland College has been promoted to an associate professorship.

Dr. Abraham Wald, formerly of Vienna, has been appointed lecturer at Columbia University for the academic years 1939–41.

Dr. C. C. Wylie, associate professor of astronomy at the State University of Iowa, has been promoted to a professorship.

The Editor wishes to express his appreciation to the following persons who refereed papers or otherwise assisted in the work of editing the MONTHLY for the year 1939:

A. A. Albert; R. W. Barnard; Walter Bartky; R. W. Brink; W. B. Carver; E. W. Chittenden; E. G. H. Comfort; A. H. Copeland; N. A. Court; H. S. M. Coxeter; D. R. Curtiss; J. H. Curtiss; H. T. Davis; L. L. Dines; H. L. Dorwart; L. R. Ford; W. B. Ford; T. C. Fry;

J. W. Givens; Lois W. Griffiths; V. G. Grove; W. L. Hart; Ernst Hellinger; T. H. Hildebrandt; C. A. Hutchinson; Mark Ingraham; R. A. Johnson; B. W. Jones; E. P. Lane; R. E. Langer; C. G. Latimer; Walter Leighton; N. J. Lennes; Mayme I. Logsdon; H. F. Mac Neish; J. R. Musselman;

W. T. Reid; H. L. Rietz; J. B. Rosser; R. G. Sanger; H. A. Simmons; W. J. Trjitzinsky; A. W. Tucker; Bryant Tuckerman; H. S. Wall; J. H. Weaver; F. M. Weida; Marie J. Weiss; M. E. Wescott; L. R. Wilcox; K. P. Williams; F. E. Wood.

THE LOCKE COLLECTION OF CALCULATING MACHINES

L. Leland Locke has placed his collection of old calculating machines in the Smithsonian Institution. This collection contains many unique and interesting machines, among them the first two machines made in the United States, the first direct multiplication machine, and the first attempt at recording the multiplier. As the Museum already possesses the work of Barbour, the addition of the machines of Baldwin, Grant, Warren, and Verea covers the work of all contributors to the Art, with the exception of Teasdell, or Teasdale, from the inception of the Patent Office through 1876. There has never been any evidence uncovered that the latter progressed beyond patent specifications.

CHANGE IN NAME OF *THE AMERICAN PHYSICS TEACHER*

American Journal of Physics is the new title of the bi-monthly publication known since its inception in 1933 as "The American Physics Teacher," according to an action taken recently by the American Association of Physics Teachers concerning its official journal. Remaining under the editorship of Professor Duane Roller, of Hunter College, and under the publication management of the American Institute of Physics, the journal will continue to stress the educational, historical, socio-economic and philosophic aspects of physics, and the instruction of students who take physics as part of a liberal education as well as those who specialize in the science.

MICROFILM SETS OF PERIODICALS

The Committee on Scientific Aids to Learning, President Conant of Harvard, chairman, has made a grant to cover the cost of making a microfilm master negative, on the most expensive film, of sets of volumes of scientific and learned journals.

This permits the non-profit Bibliofilm Service to supply microfilm copies at the sole positive copy cost, namely 1 cent per page for odd volumes, or a special rate of $\frac{1}{2}$ cent per page for any properly copyable 10 or more consecutive volumes.

The number of pages will be estimated on request to: American Documentation Institute, care offices of Science Service, 2101 Constitution Ave., Washington, D.C.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N.H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va., April 20.
Illinois, Bloomington, May 3–4.
Indiana, Richmond.
Iowa, Mt. Vernon, April 19–20.
Kansas, Wichita, March 30.
Kentucky
Louisiana-Mississippi, Oxford, Miss.
Maryland-District of Columbia-Virginia
Michigan, Ann Arbor, April 26–27.
Minnesota
Missouri
Nebraska, Omaha.
Northern California, Berkeley, January 27.
Ohio, Columbus, April 4 or 6.
Oklahoma
Philadelphia
Rocky Mountain
Southeastern, Athens, Ga., March 29–30.
Southern California, Compton, March 2.
Southwestern, Tucson, Arizona.
Texas
Wisconsin, Milwaukee.

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics.

THE NOVEMBER MEETING OF THE MICHIGAN SECTION

The fall meeting of the Michigan Section of the Mathematical Association of America was held at Kalamazoo College and Western State Teachers College, Kalamazoo, Michigan, on Saturday, November 18, 1939. Professor A. E. Lampen, chairman of the Section, presided at the morning session, which was held at Kalamazoo College, and at the luncheon and afternoon sessions, which were held at Western State Teachers College.

The attendance was about sixty-five, including the following twenty-six members of the Association: H. M. Ackley, N. H. Anning, W. L. Ayres, J. W. Baldwin, F. A. Beeler, Harold Blair, W. M. Borgman, R. V. Churchill, A. H. Copeland, Max Coral, P. S. Dwyer, J. P. Everett, C. H. Fischer, K. W. Folley, R. E. Gaskell, J. D. Hill, E. E. Ingalls, D. K. Kazarinoff, A. E. Lampen, A. L. Nelson, E. D. Rainville, C. C. Richtmeyer, E. R. Sleight, Alvin Sugar, T. O. Walton, Margarete C. Wolf.

A proposal to meet with the Indiana, Illinois, and Wisconsin Sections in the spring of 1940 was discussed. Though some such meeting at a future date was looked on with favor, it was felt wise to carry out the previously determined plan to meet with the Schoolmasters Club of Michigan at Ann Arbor next April.

The following papers were read:

1. "Rank correlation for the case of equal variates" by Max Woodbury, University of Michigan, introduced by the Secretary.

2. "The solar halo complex of June 12" by Professor H. M. Ackley, Western State Teachers College.

3. "A summation formula in two variables" by J. I. Northam, Michigan State College, introduced by Professor V. G. Grove.

4. Undergraduate papers:

a. "The use of counters in arithmetical operations" by Dorothy Rafter, Albion College, Class of 1940, introduced by Professor Sleight.

b. "A few mathematical recreations" by Cornelius Groenewoud, Hope College, Class of 1940, introduced by Professor Lampen.

c. "A trisection problem" by Norman Sleight, Albion College, Class of 1940, introduced by Professor Sleight.

5. "Geometrical equivalences of the generalized hypothesis of the continuum" by Professor K. W. Folley, Wayne University.

6. "Certain simple operations on linear differential operators" by Dr. E. D. Rainville, University of Michigan.

7. "Polynomials with coefficients in a division algebra" by A. C. Holtz, Wayne University, introduced by Professor Nelson.

8. "Some famous problems in the theory of numbers" by Dr. Alvin Sugar, University of Detroit.

9. "On the analytic nature of minimal surfaces" by Dr. Max Coral, Wayne University.

10. "The discriminant of the quartic" by Professor N. H. Anning, University of Michigan.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Woodbury pointed out that the usual formula for the coefficient of rank correlation is incorrect if the mid-rank or the bracket-rank method is used in the case of equal variates (ties in rank). He derived a formula appropriate for this case of equal variates by finding the average of the values of the rank correlation coefficients for all possible arbitrary assignments of ranks to the equal variates. The average value can be found by calculating the rank correlation coefficient by the mid-rank method and subtracting a small correction.

2. Professor Ackley described a very remarkable halo which occurred at Kalamazoo on June 12. Present in the complex were a brilliant parhelic circle about the zenith of 33° radius, a 22° circular halo (about the sun) with an elliptical halo tangent to it above and below, a considerable arc of a 46° halo near the horizon, and a vertical pillar above and below the sun extending nearly to the points of tangency of the 22° circle and the ellipse. Notably absent were parhelia (sun dogs) and tangent arcs of Lowitz, although conditions were favorable to their visibility. Professor Ackley discussed the mathematical theory of the halo and showed that this particular complex can be explained by assuming a homogeneous cloud of fortuitously falling ice crystals of only one type, the *columnar* hexagonal, with a strong tendency, however, for a considerable number to "coast" down in a stable condition with principal axes horizontal. Meteorological conditions at the time, together with the fact that if *tabular* hexagonal crystals were also present parhelia and Lowitz arcs would have been formed, support this view.

3. Mr. Northam discussed the extension into two variables of three forms of Lubbock's summation formulas. Approximations of a double sum were obtained by taking hk times the sum of every hth term for every kth value of one variable. To these approximations were added certain corrective terms involving finite differences, and also remainder terms involving derivatives which might yield an upper bound for the error.

4. a. Miss Rafter gave a historical account of counters as used in arithmetical operations. By means of a counter board designed according to a chapter in Robert Recorde's *Arithmetic* (about 1543), she showed how the operations of addition, subtraction, and multiplication were performed.

b. Mr. Groenewoud (1) derived formulas for use in construction of magic squares, and (2) developed a method for making any fraction equal to the sum of unit fractions.

c. The problem of constructing an angle three times as large as a given acute angle is well known and not difficult. In this paper Mr. Sleight reversed the problem, and explained the difficulties involved. He also explained, as a background, the analytic conditions for a ruler-compass construction of a given geometrical problem.

5. Professor Folley discussed generalizations of theorems proved in Sierpinski's book, *The Hypothesis of the Continuum*. By considering a space of a

transfinite number of dimensions, it was shown that the generalized hypothesis of the continuum is equivalent to decompositions of this space in various ways. In particular, one such decomposition is into two sets A and B, A having at most $\aleph_\alpha$ points on every line parallel to any one axis of the space and B having at most $\aleph_\alpha$ points on every line parallel to any other axis of the space.

6. Dr. Rainville discussed elementary algebraic properties of certain linear operators. One of these operators has the formal properties of the Laplace integral transformation in its application to linear differential expressions. Another operator transforms each linear differential operator into its adjoint. Application was made to the determination of differential equations for the solution of which the Laplace integral transformation is useful.

7. Mr. Holtz discussed polynomials whose coefficients are in a division algebra. The main body of the paper dealt with the application of the general theory of non-commutative polynomials, as presented in a previous paper by M. H. Ingraham and M. C. Wolf, to polynomials with coefficients in a quaternion algebra or in a division algebra of order nine.

8. Dr. Sugar's paper was principally concerned with some of the famous and less famous unsolved problems of number theory. Among these were the problems of prime pairs, prime triples, representation of primes by linear and quadratic forms, the Goldbach problem, Fermat's last theorem, and the Euler conjecture.

9. Dr. Coral first pointed out that a non-singular surface S, which is represented parametrically by coördinate functions having continuous partial derivatives of the first order, may be called *minimal* if along S the first variation of the area integral vanishes. In order to prove that such a surface is minimal in the usual sense, *i.e.*, has mean curvature zero, it is necessary to find a parametric representation of S in which the coördinate functions have continuous second derivatives. Such a proof was given in 1926 by T. Radó, who found a representation of S which is locally isothermic, so that the coördinate functions of S are harmonic functions and therefore analytic. Dr. Coral gave a modification of Professor Radó's proof which simplified the details materially.

10. Professor Anning studied the typical quartic polynomial and its Sturm functions from a graphical point of view, and pointed out the pictorial significance of the conditions for four real zeros.

P. S. DWYER, *Secretary*

REPORT OF THE COMMITTEE TO REVIEW THE ACTIVITIES OF THE MATHEMATICAL ASSOCIATION OF AMERICA

The creation of this committee, let it be said at once, is in no way to be attributed to the imminence of anything resembling a crisis in the Association's affairs, or of any danger of impairment of the organization's usefulness. On the contrary, the Association approaches the completion of the first twenty-five years of its existence strong in the possession of the loyalty and confidence of its members, endorsed by an increasing membership, and with its material affairs in perfect order. The inspiration for a review of its activities is to be referred to a different source. The Association, having now passed through that period of years in which its continued existence could have been regarded as problematical, looks to a future which in at least some of its aspects differs from that which was envisaged by its founders. In the presence of the many and fundamental changes in social adjustments peculiar to the time, it is all too possible for individual or organization to find at some crucial moment that it has inadvertently allowed itself to fall into the rôle of a foolish virgin while awaiting the appearance of the bridegroom of opportunity. This the Trustees and officers of the Association—mindful of their charge—have wished to forestall, or to do so at least insofar as such may be possible by the raising and throwing into relief of such matters as in the minds of a chosen group of individuals may seem worthy of the consideration and the renewed appraisal and judgment of the membership of the organization. A committee to undertake this was, therefore, authorized by the Trustees at the Summer Meeting of 1938, and pursuant thereto the undersigned committee was appointed by Professor A. J. Kempner, then President of the Association. We, of this committee, now beg to submit herewith the following report.

In accepting the commission of the Association to review its activities we have not been unmindful of the honor which in the implication of confidence has been done us. Nor were we to remain long unmindful of the magnitude, and more especially of the unspecific character and the vagueness of delimitation of our assignment. In the face of this it seemed to us a wise precept of Descartes "to divide the difficulties into as many parts as possible, and by commencing with the simplest and easiest to ascend step by step to the more complex." This has been our method—may the result speak for itself.

The phases of any review are without violence subsumable under the designations of retrospect, aspect, and prospect. The first of these has in the present instance concerned us but little. The history of the Association is sufficiently familiar to require no recapitulation from us, and the gamut of its accomplishments has happily been such as stands in no need of embellishment or inspired praise. Our review, therefore, has in the main concerned itself with the Association's present, and with the outlook to its future.

As to aspect, the Association is, in brief, at present an organization of something over twenty-one hundred individual and one hundred institutional mem-

bers. It is of national, or perhaps better said, of continental scope, and proclaims in its by-laws as its object: "to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings for the presentation and discussion of mathematical papers, by publishing mathematical books, monographs and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a library and by coöperating with other organizations whenever that may be desirable for attaining these or similar objects." It is governed by a staff of eight officers and twelve additional Trustees, publishes as its official organ THE MATHEMATICAL MONTHLY, and holds two meetings each year. By authority of its by-laws the Association may grant to any group of not less than ten members the permission to organize as a "Section of the Association" for the purpose of holding local meetings, and there have been organized and are in existence twenty-two such Sections. The organizations of national scope which have objectives bordering most directly upon those of the Association are on the one hand The National Council of Teachers of Mathematics, which is devoted to "the advancement of mathematics teaching in the elementary and secondary schools," and on the other hand The American Mathematical Society which is devoted to mathematical research.

Crude though the outline which we thus present may be, it nevertheless serves in some manner as a basis for the considerations which follow. To begin with, it places into some relief the great breadth of objectives which the founders of the Association preëmpted for it. Thus it has been said concisely and yet not inaccurately that "Everything that is worth doing for mathematics, other than research, is a function of the Association" (J. W. Young, MONTHLY, Jan. 1932). In any such extension of its realm of purposes an organization cannot fail to find sources both of strength and of weakness. While a diversity of interests calls forth inspiration from a multiplicity of origins and permits of an enriched program of activities, it must almost inevitably defer to a singleness of purpose in the matters of determinate strength and efficiency of execution. The broader organization must attract membership from individuals with different preferences, and its guidance requires the coöperation of manifold talents. Its problems will be of many kinds, and while some will be of a national significance, others again will be of a regional application and importance, many distinctions being no less than inevitable in a country of such greatly varying physical aspects, populational density, and traditions as ours.

Considerations such as these have led us to the first matter upon which we have recommendations to submit. The administration in a truly statesmanly manner of an organization with many objectives such as the Association is, seems to us to call for a peculiarly large amount of conscious planning, of judgments on the importance of projects which offer opportunities for participation, and of the appraisal of relative merit in activities which invite financial support. While in such matters the leadership of the Association in the past has been able and even distinguished, we hope and even expect that the Association will

in the future enjoy such growth that its effective management will become a task of ever increasing magnitude. In the death of Professor Slaught, who held the offices of Manager and Honorary President, the guidance of the Association lost, along with much else, an appreciable pillar of stability. With the single exception of the Secretary-Treasurer the officers of the Association are in a position to give their leadership only during relatively brief terms. Unfortunately, moreover, very little responsibility for the care of the Association's affairs seems by custom to have attached itself to some of its offices, notably the vice-presidencies. We believe that there would be definite advantage in so modifying the present adjustments as to cloak the principal officers more equitably with responsibility for the organization's direction, and by aggregating these officers into an Executive Committee charged explicitly with the conscious planning of the Association's program, to assure greater stability and continuity for that program.

The general need for organizations of a local compass with the purposes of the Association as their own has palpably disclosed itself in the calls for the formation of Sections in increasing number. Sections now exist in almost all parts of the country, and in them one may see, at least in theory, a decentralization of the organization's functioning. We believe that such decentralization embraces many desirable features, and that in it the Association may well find the potential means for carrying out its future work with a high degree of effectiveness. As to their practical aspect, however, we believe that the arrangements which at present maintain could be very materially improved.

The Sections are in themselves local associations of which each has its own officers and committees. A goodly number of them not only hold meetings with regularity, but are also otherwise vital and active and have a conscious and definite purpose. In some instances they enjoy the distinction of being the only organizations within their respective territories that concern themselves with the welfare of mathematics, and that afford the occasions and means for group discussions of matters of interest or moment in that connection. Despite this, the inter-relation between the Sections and the Association as a central body is tenuous almost to the point of non-existence. In its by-laws the Association specifically disclaims all obligation to pay from its treasury any of the expenses of the Sections, and although, to be sure, it has not in practice held itself entirely aloof in this matter of extending some financial aid, nevertheless such assistance as has been given has been sporadic and not in accordance with any definitely established program. There is no provision in the way of regular channels through which information regarding situations which locally affect the welfare of the Sections or the good of mathematics as a whole may be communicated from the Sections to the Association, and in turn none through which the Sections may be kept officially informed of, or made responsive to, any program to which the national organization devotes itself. There is no organized service of advice to the Sections, and in turn no consciousness on their part that they either fill any rôle whatsoever in the formation of the policies of the parent body,

or that they have any responsibility toward carrying such policies into effect.

We believe, and regard it as of considerable importance, that the Association should establish means of more direct contact with its local units, and that it should, insofar as may be feasible, give to the local units material assistance and the opportunity to take part in its councils. In extending assistance, especially to such Sections as carry on their activities under the handicaps of sparsity of population, great distances and remoteness from centers of intellectual initiative, the Association places itself into the position of an intermediary through which those of its members in the more favorably situated parts of the country may assist in the work of those who in this respect are less fortunate. To do this is entirely consistent with the Association's objectives. In giving some direct voice to the membership in local units the Association may improve its responsiveness both to opportunities and dangers which locally arise, and may develop means by which the weight of its influence and prestige may be brought to bear effectively upon local situations. There is perhaps no part of the country in which the Association's membership, or potential membership, is not faced with some local manifestation of the situation in which mathematical instruction in the schools generally finds itself. The effect of this situation upon the colleges shows itself indirectly in the quality of the mathematical preparation of the students and directly in assaults upon the college entrance requirements. In their efforts to support and maintain high mathematical standards in such conditions the local groups individually find themselves almost inevitably foredoomed to ineffectiveness. With assistance and advice at hand, and the resources and prestige of a national organization behind them, they may perhaps in many instances find themselves less unequal to their task.

Although it might almost be inferred from the foregoing statements that we advocate a replacement of the Association by some federation of Sections, that would be entirely incorrect. The effectiveness of the Association in the future will undoubtedly depend upon its maintaining a clear-cut identity as an organization of national scope. It must maintain itself always in the position to draw its leadership from men of outstanding accomplishment who stand high in the esteem of the mathematical world. The means through which the Association may make the guidance of such men available to itself should be held free from all local sanction and independent of regional interests.

Guided by the considerations which have thus been set forth, we have drawn up a plan for some modifications in the Association's scheme of organization which we hereby propose as a recommendation to the following effect:

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary-Treasurer, and an Associate Secretary.
2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected

members (hereinafter called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, and to administer and control these activities, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision.

3. There shall be an Executive Committee advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor, and the Secretary-Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or interest to the Association. This Committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. There shall be a Finance Committee to conduct the business affairs of the Association, *i.e.*, to receive and administer its funds, to control its properties and investments, to make its contracts, *etc.* This Committee shall consist of three members, of which the Secretary-Treasurer shall be one. It shall report its actions at the annual business meeting of the Association and in the official journal, and its general policies shall be subject to the approval of the Board of Governors.

5(a). The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by this membership in constituencies (hereinafter called "Regions") established by the Board.

5(b). The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

5(c). The membership in each Region shall elect biennially a Governor for a term of two years.

5(d). The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary-Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary-Treasurer) for four years.

5(e). Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled, and the Board may in any case reject all nominations made and call for a new list.

5(f). The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined as follows:

5(g). For elections at large, by a Nominating Committee to be appointed annually for that purpose by the Board. This committee shall proceed as is at present prescribed in Sec. 7 of Article III of the by-laws.

5(h). For each regional election, by the Section or Sections of the Association existing within the Region, or, in the absence of such Sections, by a committee appointed for that purpose by the Governor representing the Region.

5(i). The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

Although the Regions referred to in this recommendation are to be fixed by the Board and are to be subject to revision by the Board, the following is suggested as an initial delimitation of them:

1. New England
2. New York and Eastern Canada (east of Manitoba)
3. New Jersey, Pennsylvania, West Virginia and Delaware
4. District of Columbia, Virginia and Maryland
5. North Carolina, South Carolina, Georgia, Alabama and Florida
6. Mississippi, Louisiana and Arkansas
7. Tennessee, Kentucky and Ohio
8. Indiana, Illinois and Michigan
9. Wisconsin, Minnesota and Iowa
10. Nebraska, Kansas and Missouri
11. Oklahoma and Texas
12. New Mexico, Arizona, Utah, Colorado and Wyoming
13. South Dakota, North Dakota, Montana, Idaho, Washington, Oregon and Western Canada (west of Ontario)
14. California and Nevada

By way of final remarks in direct connection with the matter of the Association's organization we would say that if our recommendation is adopted the transition from the existing order to that proposed can be made without any serious discomposure. We suggest as a procedure to this end the following:

That the membership proceed with its present schedule of Presidential elections. That it elect a First Vice-President at the end of 1940 for the partial term through 1941, and begin regular elections to this office at the end of 1941. That it begin its regular elections of Governors at large at the end of 1941. That the Board elect an Associate Secretary at or before the summer meeting of 1940 for the partial term through 1944. That the Board elect a Second Vice-President at the end of 1940, an Editor at the end of 1941, and a Secretary-Treasurer at the end of 1942. That it elect two members of the Finance Committee at the summer meeting of 1940, one for the partial term through 1941, and one for the partial term through 1943. That all present Trustees whose terms do not expire with the year 1939, and the Trustees to be elected at the annual meeting of 1939 become members of the Board of Governors as follows:

Trustee Curtiss as ex-president, through 1942;

Trustee Kempner as ex-president, through 1944;

Trustees Betz, Coble, Dresden and Weaver as Governors at large, through 1940;

Trustees Bennett, Carmichael and Evans as Governors at large, through 1941;

Trustees to be elected in 1939 as Governors at large, through 1942.

That the Regions 5, 7, 10, and 12 elect a Governor at the summer meeting of 1940 for the partial term through 1941, and that these and the Regions 1, 2, and 13 begin their regular elections of a Governor at the annual meeting of 1941. That the Regions 3, 4, 6, 8, 9, 11, and 14 begin their regular elections of a Governor at the annual meeting of 1940. That in the interim while any Region remains without representation by a Governor, the Executive Committee fulfill all such duties as shall ordinarily devolve upon such a representative.

The inclusion in our recommendation of such features as the creation of a Finance Committee, the addition of an Associate Secretary, the omission from the staff of officers of the members of the present "Committee on Official Journal" except for the Editor, *etc.*, hardly require comment. Under existing conditions they seem to us to be desirable adjustments. The omission of the office of Librarian will also be noted, and is to be explained on the ground that we believe the care of the library to be more properly a matter to be entrusted to a standing committee than to an officer. The whole matter of the Association's library is to be discussed by us below. Some further comments in connection with our recommendation, which will, indeed, lead us to another proposal are the following:

Our belief in the desirability of regional representation on the Association's governing Board has already been elaborated. The form in which it has been embodied in the recommendation made has been dictated largely by the exigencies which workability of the plan seemed to impose. In the event that our recommendation is adopted it is our hope that the Sections, as they may be grouped within any Region, will avail themselves to the utmost of the opportunities we wished to extend to them, and that they will coöperate by returning as their representatives those persons who are best qualified for that position and who are genuinely interested in furthering the Association's objectives.

The extent to which a representative body will bring to bear upon its deliberations those advantages which are peculiarly inherent to it, depend very materially upon a full attendance at its meetings. Such attendance, especially in the instance of members from the more remote corners of the country, is necessarily bound up with considerable financial expenditure. It is the Association as a whole which is the primary beneficiary of this expenditure, and since, as will be obvious, the latter must in some cases reach proportions which would be excessive for many members of the organization personally to assume, we shall propose that the Association adopt the sharing of this burden as a part of its established policy. The Association as an altruistic organization may without apology ask of its members any reasonable expenditure of their time and talent.

This right should not, however, be regarded as extending to their pecuniary resources. The organization has regularly established dues through which the load of financial obligation inseparable from its administration and the conduct of its activities is intended to be placed equitably upon the members. We do not, of course, believe that the Association would be the sole beneficiary from the attendance of representatives at its meetings. Such attendance at the meetings of a national organization have for any alert individual personal advantages which it is certainly not necessary for us to recount here. If attendance calls for sacrifice it seems but fair that this should be shared. We recommend, therefore, the following:

(a) *That the Association undertake, whenever, in the judgment of the Board its financial status permits and the annual meeting is reasonably central, to reimburse each regional Governor for one-third of his first-class railroad fare to and from this meeting.*

(b) *That when this appropriation remains unused by any such Governor in any year it be held to the credit of his Region to be placed at the disposal of its Governor in the following year, in addition to such appropriation as may be made for that year.*

(c) *That the Association similarly undertake to reimburse each member of its Executive Committee for the amount of his first-class railroad fare to and from each of two meetings of this Committee in any one year.*

We believe, on the basis of some specimen calculations which we have made, that the expenditures which the Association would be called upon to make in connection with this recommendation are not excessive. The further discussion of this we defer, however, to an appropriate place below.

Second in the order of our review, though in importance it is secondary to none, is the question of the adjustment of the Association's activities to its professed objectives. An organization's charter to the right of existence lies in its purposes. Through its proclamation of these it calls for the support of individuals who subscribe to them, and the individual in turn by holding membership proclaims his endorsement of them. In the course of time a revision of an organization's purposes may become desirable or necessary, and the members will endorse such changes as may be made, or failing to endorse them will discontinue their membership. In the case of the Mathematical Association we do not believe that the objectives need revision. There seems to be every reason to believe that they have the unconditional endorsement of the membership, and it is our opinion that they are as vital or more vital in the presently existing order of school and society as they have been at any time past. The range of these objectives is large. From contact and some interpenetration with the objectives of those organizations which are devoted to the teaching of mathematics in the secondary schools, they extend over the collegiate field and over that of the graduate school, to make contact with that of mathematical research.

The organizations devoted primarily to the secondary school field are con-

cerned in the main with questions which center upon the pupil, and their problems are accordingly almost entirely of a pedagogical nature. The Association regards these problems as essentially outside its field, although it has not in the past, and cannot afford in the future, to hold itself aloof from interest in them. The mathematical instruction in the schools very directly affects that in the colleges, and what is detrimental to the former is almost certain to have a similar effect upon the latter. In the teachers of the secondary schools the Association finds a personnel in which it is directly interested. Ideally these teachers should all be college trained. In practice they all do have some training of a post-secondary grade, and it is distinctly within the range of the Association's objectives to concern itself with the problems presented by projects for the betterment of their training and for the enlargement of their intellectual scope and of their appreciation of their subject.

At the collegiate level the concern of the Association focuses itself both upon the student and the teacher. In the early, or so-called junior college years the enrollments in mathematics are generally large, and the great majority of the students involved finish their mathematical training at this stage. The teaching which they receive must be looked upon as not merely supplying them with facts, but as being of immediate influence upon their subsequent reactions to things mathematical and the understanding of mathematics which they carry on through life. The fostering of the best standards of teaching at this level, and the improvement of the teachers, are objectives of first rate importance. In the upper college years and throughout the graduate school the students of mathematics are predominantly engaged in training themselves for the profession of the teacher. The character and quality of this training, the extent to which it should be carried, the certification of successful candidates, the definition of the various degrees and many other matters in this field are legitimately the Association's concern. Aside from these purposes, the Association is dedicated to the promotion of mathematical scholarship by any and all means. It accordingly has among its objectives the sponsorship of expository mathematical writing and the dissemination of the results of mathematical research, the stimulation of an *esprit de corps* among mathematicians and the facilitation of the exchange of ideas at meetings, the support of travelling lecturers, the dignified scientific popularization of mathematics and the publication of books, papers and pamphlets designed to broaden the understanding of the cultural significance of mathematics, the historical importance and vitality of its theories and practice, and its many bearings upon science, philosophy and technology.

It will be evident even to the casual reviewer that the present activities of the Association are unequally distributed over the many reaches of the domain which has thus been laid out. He will observe a distinct concentration of these activities and the associated expenditure of resources upon the field which borders closely upon that of mathematical research. The tendency, moreover, has been toward such concentration rather than away from it, and this despite the fact that research is not one of the Association's objectives and is the single

objective of the American Mathematical Society, a singularly well directed, strong, and efficient organization.

We can lay small claim to originality in the matter of drawing these facts to the attention of the Association's members, for among many of the members they form a well worn subject of discussion, and on many occasions they have been publicly discussed by the Association's officers. In 1932 President W. B. Carver, then the Editor of the MONTHLY, wrote concerning that journal: "it is devoted to the interests of collegiate mathematics if one accepts the statement on the cover. But if one looks further than the cover page he may find reason to doubt the whole-heartedness of this devotion." Now there is, of course, nothing reprehensible in any encroachment of the Association upon the field of mathematical research. At present this is not one of the Association's stated objectives, although it could without any ado whatever be made one if such were found to be desirable by the membership. We believe, however, basing our judgment upon the fullest contact with the membership which we have been able to make, that only a very small minority would advocate this, and that the great majority would not. It is certainly undeniable that a very large proportion of the Association's present membership is vitally interested in mathematical research. It may be assumed, however, that they give primary utterance to this interest by their membership in the American Mathematical Society, and this being so it is reasonable to assume that their membership in the Association is articulate, not of their research interests, but of their interests in those other activities for which the Association more particularly stands.

The matter which is here at issue is one which current events seem to be pressing toward the state of urgency. During the lifetime of the Association the student enrollment in the colleges has been tripled, and the junior college as an institution has risen to a position of importance. There seems to be much reason to believe that in the future the separation between the work of the junior college years and the senior college years will undergo further differentiation regardless of whether they are carried out in the same or in separate institutions. The number of mathematics teachers engaged in the work of the junior years of college has thus grown to the point where they have become conscious of their numbers and importance. Their affiliations with the senior personnel of the secondary schools, the most influential teachers, the principals and the superintendents, may well be considered natural. The welfare of mathematics lies to no mean extent in the hands of this large group of teachers. The Association has the fostering of this welfare as one of its primary objectives, but—let it be said—at present it exerts very little influence in this group. There is already a stirring in several parts of the country—in some cases already a movement—for the formation of an independent organization of the junior college mathematics personnel. In the light of this it seems clear that the Association is confronted by the necessity of making without delay a decision of great importance. Shall it reassert all of its present objectives and, acknowledging the fact that it has thus far neglected some of them in the institution and prosecution of its activities,

undertake at once to enter actively into their cultivation and support, or shall it allow itself to be essentially displaced in a large part of its domain by another organization?

We believe that the best welfare of mathematics calls for a decision in favor of the first of these alternatives, and our soundings of the sentiment of the membership makes us believe that this decision would obtain an overwhelming support. To some members it apparently seems that only by making this decision can the Association expect to maintain in the future its status as an organization of truly primary importance. Pursuant to the beliefs which we have already fully expressed, we advocate, therefore, that the Association give thought at once to the diversification of its activities, and to the attraction into its membership of a greatly increased number of those persons who, though not directly concerned with research activity, have other mathematical interests. This need not, and does not, mean that the Association should abandon any of its present activities. It means rather, to use a figure not our own, that the roots and trunk of the tree should be brought under cultivation as well as the branches, this being quite as important as the care of the blossom and fruit.

To merely invite into membership in the Association individuals engaged in the junior college field, will, we believe, promise but small return. The Association does not seem to this group to be concerned with many matters that are of moment to it. It should not be regarded in this connection as a question of the existing membership of the Association assuming to supply to the junior college group those things they need to fill their want, but rather of the Association's assuring that group of the fundamental identity of interests and objectives, and with the invitation to participate in membership *en masse* offering it the opportunity and the facilities which the organization has at hand for it to serve its own needs within the Association and with the coöperation of the Association's entire membership. We are convinced that a consummation of this would yield great advantages to all concerned, and accordingly make the following recommendation:

That the Association appoint a small committee to seek out the leaders of the Junior College mathematics teaching personnel and ascertain from them what they regard as their essential needs, and under what conditions they would, as a group, seek to carry out their objectives under membership in the Association. That this committee be empowered to assure the Junior College personnel of a cordial welcome to membership in the Association and participation in its counsels, and to discuss such matters as:

(i) *the remission of initiation fees for some specific period,*

(ii) *space in the* MONTHLY *for papers dealing with their problems, and editorial representation on the* MONTHLY,

(iii) *facilities for the presentation of their programs and the discussion of problems at the Association's meetings,*

(iv) *representation on the Association's Executive Committee.**

* Action on this recommendation as it appeared in our preliminary report was taken by the Trustees at the summer meeting of 1939.

The financial status and transactions of the Association have come under our review essentially as they have been reported by the Secretary-Treasurer during the years 1934 to 1938 inclusive. In brief and somewhat rough recapitulation the figures involved appear when reduced to annual averages to be the following:

ANNUAL RECEIPTS

1. From dues, subscriptions, *etc.*	$9,700	
2. From initiation fees	250	
3. From advertising	450	
4. From interest on endowment, *etc*	760	
Total		$11,160

ANNUAL DISBURSEMENTS

5. For the MONTHLY, the *Register*, *etc.*	$6,020	
6. For the Secretariat	3,235	
7. For assistance (speakers, *etc.*) to Sections	225	
8. For meetings	190	
9. For committees and commissions	150	
10. For the library (binding)	85	
11. Membership in the American Mathematical Society	100	
12. Subvention to *Annals of Mathematics*	200	
13. Subvention to *Duke Mathematical Journal*	200	
Total		$10,405
14. Special appropriation for *Mathematical Reviews* made in 1939		$1,000

At the close of the year 1938 the wealth actually in the possession of the Association, as represented by its balance on hand and its endowment, was $22,700. In addition to this the Association has in its control special funds which at that time amounted to the following:

The Carus Monograph Fund	$6,946
The Arnold Buffum Chace Fund	7,784
The Chauvenet Prize Fund	688

These figures, their size and their diversity, may be looked upon as a measure of the scope and vitality of the Association's enterprise. In their steady increase the Association's growth has been reflected, and the membership will hope for the continuation of such increase and growth. The receipts of the Association have come, and it must be assumed will continue to come, in the main from item 1 as listed. The revenue from this source can be increased only by increasing the membership. The membership total has not materially changed during the last decade, and stands at present in the neighborhood of twenty-one

hundred. This, it must be observed, is only a fraction of what can justifiably be regarded as the membership potentiality. On the basis of data collected in 1935 (R. G. D. Richardson, the MONTHLY, vol. 43, 1936, p. 205) it appeared that of teachers of mathematics alone in the colleges, universities, junior colleges, and degree-granting normal schools of the United States and Canada there were approximately 4,444. Of these only 1,333 were noted to be members of the Association. There is no reason to suppose that the proportion is different at the present time. We have already discussed above the possibility of enlisting an entire group to increase the membership, in connection with the teachers of the elementary college subjects. Aside from this we believe that any increase of material proportions will result only from a diversification of the Association's activities. The appeal of the Association is at present strongest for the personnel of the universities and the larger colleges, and dwindles to small proportions for those whose interests center upon the work of the small colleges, the secondary schools and intermediate institutions.

Item 2, the initiation fee, is a matter which we have discussed but upon which we have come to no conclusion. There seems to be a considerable diversity of opinion as to whether this fee is or is not a ponderable deterrent to membership, and if it is whether it is or is not compensated for by the income it supplies. The question might merit the Association's study and perhaps some experimentation.

Of the regular disbursements of the Association by far the greatest portion goes to the publication of the MONTHLY and the expenses of the Secretariat. This is entirely as it should be, since these are the media through which the organization functions and makes itself articulate. We believe that the MONTHLY must necessarily play an important rôle in conjunction with any diversification of the Association's activities. At present, as its own Editors have been very ready to acknowledge, it reflects the concentration of the Association's interest in the sub-research field. The recent institution of a Department of Mathematical Education seems to us a step in the right direction, and one that could well be followed further. The editors, it must be said, have on many occasions earnestly solicited articles on subjects directly in the line of the Association's objectives, and the response has not been overly large. The responsibility for the production of suitable material lies, after all, with the membership, and this has not been too generously forthcoming.

It has been suggested that from the standpoints of typography, design, format and legibility the MONTHLY might well be overhauled with a view to its modernization and the enhancement of its appeal and attractiveness as a journal. We feel technically incompetent properly to appraise this suggestion, but believe that the Association may perhaps do well to refer the matter to the study of some qualified committee.

The item of disbursement numbered 10 places into relief the whole problem of the Association's library. Although this enterprise is theoretically in the charge of the Librarian, it has for many years in fact been cared for by the

Secretariat. A catalogue was issued by the Association in 1931. It shows the library to consist of some eighty-three journals which the Association receives by exchange arrangements, and some books. The latter represent mainly the residue of volumes not otherwise disposed of from the influx which is intended for review. In its intent the library duplicates completely that of the American Mathematical Society—in completeness it is in no way comparable with it. Since the membership presently avails itself of the library's loan service to the extent of no more than a dozen instances per year, it will be clear that as an enterprise the library is at present essentially negligible.

In the light of this we regard the average annual expense of $85 connected with it as one which the Association cannot afford to continue. Its discontinuance, however, in the absence of other suitable dispositions will mean that the many journals that are currently received will remain unbound, a matter which will involve certain obvious loss and waste. There seem to be two alternatives between which the Association must choose. The one is to abandon the enterprise of a library and to dispose of the present one in some way, recognizing the fact that duplication with the American Mathematical Society's activity in this respect is bound to be bootless. The other alternative is to seek to build up the present library mainly with books and journals covering subjects more definitely identifiable with the Association's field. This might involve a considerable expenditure of funds to be effective.

We believe that it will require more study than we can give this matter to determine which alternative the Association should take. The ultimate decision could reasonably be made to depend largely upon the outcome of an investigation as to whether or not the Association could find some large and centrally located institution with which it could consummate an agreement concerning its library somewhat along the lines of the agreement between the American Mathematical Society and Columbia University.

There are many matters which can be drawn into a discussion of the annual disbursements numbered 11, 12 and 13 above, and of a proposed annual subvention to *Mathematical Reviews*. The amount of spendable money which remains to the Association after it has cared for its own organ and secretariat is not large. The items in question have been taking the largest share of it. They belong in the category of the Association's activities which appeals most directly to that part of the membership which is vitally interested in mathematical research, and therein lies the crux of the matter. The Association's resources being limited, it is clear that its ability to diversify its activities will be contingent upon its not concentrating them too heavily in any one field. By advocating that it do the former we have already made our position clear.

The Association's membership in the American Mathematical Society is generally recognized for what it is, namely, a testimonial of the sincere interest which the Association takes in the work of the Society, and a token of the satisfaction which it feels in the cordiality that has always characterized the relations between the two organizations. The reasons for the subventions to the *Annals*

of Mathematics and to the *Duke Mathematical Journal* can, we believe, be reappraised with entire propriety. The original motivation in at least one case was that of providing publication space for work of an expository character. This purpose has never been materially served. The journals concerned are well established and are in no way really dependent upon the subventions. On the other hand the membership of the Association enjoys in return for them a favorable rate of subscription to the journals and in each case about 150 members avail themselves of this advantage.

In the case of the new journal, *Mathematical Reviews*, the Association has already proclaimed its interest by the grant of $1,000 which was made by the Trustees during the present year. The venture appeals because it is new and also because it earnestly professes itself to be dedicated to the objective of being of use to a very wide range of interests in the mathematical field. We believe that as a project it deserves the good wishes and support of all the country's mathematicians. In the matter of a subvention from the Association, however, we have not been able to reach the unanimity of opinion which we have on all other matters, and our very considerable soundings of the sentiments of the Association's membership have helped us little. There are members who are enthusiastically in favor of it and many others that are quite as earnestly opposed to it on the ground that other activities of the Association have a more legitimate and pressing claim upon its funds.

In the list of the Association's average annual receipts and disbursements as it has been given above, there is an obvious discrepancy between the two totals, the receipts fortunately being the larger. While in some small part this is to be accounted for by minor unlisted items, the larger portion, an average of $600 annually, was a genuine excess, and was allowed to remain as an increment of the Association's funds. These funds accordingly grew from the figure of $19,700 at the end of 1933, to the $22,700 noted above at the end of 1938. While the accumulation of an endowment has advantages which need no discussion, we believe that the Association in its present condition can afford to enter into activities to the entire extent of its income. The further swelling of its reserve seems to us to be presently of less moment than the broadening of its services and the extension of its sphere of influence. The amount by which the receipts have in the last years been in the excess may be looked upon as a fund from which new expenditures can be made.

We propose in the following to make recommendations for such new expenditures, and recall that we have already made such a recommendation above, namely to the effect that the Association undertake to partially reimburse its regionally elected Governors and the members of its Executive Committee for expenses incurred by them in their conduct of the Association's administration. The disbursements which will be called for by an adoption of this recommendation will, of course, depend upon the places of meeting and the domiciles of the persons involved. The following, however, will show by way of sample the general order of the amounts which must be thought of in this connection. For a meeting

in Chicago the proposed assistance to regional Governors coming from Albuquerque, New Mexico; Los Angeles; Lincoln, Nebraska; Dallas, Texas; Urbana, Illinois; Cincinnati; Washington, D. C.; Pittsburgh; Boston; Minneapolis; Memphis, Tennessee; Atlanta, Georgia; New York; and New Orleans amounts to about \$200. We think the sample will not be regarded as a particularly favorable one. For two meetings in Chicago and members resident in New York; Cincinnati; Urbana; Minneapolis; and Lincoln, the proposed assistance to the Executive Committee amounts to about \$150 above what the Association now spends annually for the expenses of some of its officers.

In the Carus Monograph and the Arnold Buffum Chace Funds the Association has two special assets which are of great potential value. The publications associated with these funds have been well received and are highly regarded by the Association's members. It is rather widely deplored, however, that the last six year period has seen no further Carus Monograph. We have noted with interest the recent actions of the Association enlarging the committee in charge of the monographs and appropriating the income of the Chace Fund during the next five years to the proposed new mathematical historical journal. The Chauvenet Prize Fund is directed to an end which we thoroughly endorse. We do not believe, however, that the present manner in which the fund is used is the best possible one. It is to be feared that the influence of the prize upon the amount or character of expository writing has been very small indeed.

The encouragement and sponsorship of expository and critical writing is one of the objectives of the Association which enjoys the unanimous support of the members. There is a ready welcome and a general demand for more readable scholarly papers on all kinds of mathematical subjects from the classical to the modern, from the elementary to the advanced, on theory, on applications, on history, or on philosophy. In the past there have, of course, been the Carus Monographs, and from time to time excellent papers in the MONTHLY. There seems, however, to be at the present little or no means for the ready publication of writings which in length are intermediate between the relatively few pages of a journal paper, and the relatively many pages of a complete monograph. Such papers, say in length between twenty and a hundred pages, could be profitably written on subjects in many categories, including among others, elementary introductory expositions of theories and their applications, more advanced expositions and interpretations of modern viewpoints and theories, philosophical essays and criticisms, broad historical accounts of important schools, or biographical accounts of individuals. The Association could perhaps well undertake to publish such papers in the form of a series of pamphlets to be made available to the members at cost. And we believe that these costs could well be made to include small token honoraria to the authors. It is not our intention in this connection to suggest that papers be bought and paid for, but rather that the Association recognize that this branch of authorship calls for talent and involves much work, while yielding but little in return. We believe that a material token will do much toward enlisting the ablest authors, and that this same end will be

furthered if the series is properly dignified in form and name. We recommend, therefore,

That the Association create a standing "Committee on Expository Writing," to be charged with receiving, refereeing, and at its discretion inviting papers of intermediate length. That such papers be published in the form of a series of pamphlets, which could appropriately be designated the Herbert Ellsworth Slaught Memorial Publications, and that the Association undertake to award a token honorarium to each author (from $25 to $75 is suggested) in appreciation of his authorship and of his donation of the paper to the Association.

The instructional programs in mathematics in the institutions of collegiate grade throughout the country manifest in many respects a diversity which is no less than wide and lamentable. Accepted norms are lacking at all stages—in the entrance requirements, in the content of courses, in the requirements for graduation and the attainment of teachers' certificates or licenses to teach, and in the requirements and routines for advanced degrees. In some schools the curricula remain fixed over inordinate periods of time, while in others they are in almost continual flux being subject to perpetual experimentations and adjustment. While activity in this, as in any matter, may perhaps be overdone, it seems clear that obsolescence of material or of teaching procedures in mathematics will not fail to be harmful to the position of the subject in the future educational programs. In the face of all this, we find very little means by which the facts of instructional plans and procedures currently in operation may be disseminated, and we believe that such dissemination is actually very scant. Except for the pedagogical journals, whose viewpoint is, generally speaking, far from what we have in mind, there is no agency which announces to others the pioneering or experimentation which is being carried on at one institution, and none by which the experience of one is made available to all. We are convinced that there is opportunity for the Association here. It can supply the means which may hasten and facilitate the modernization of courses and teaching practices, the removal of inadequacies and deficiencies, and the abandonment of outworn elements. It can help to minimize the disappointments and discouragement of students who must transfer from one institution to another, by publishing the specifications and prerequisites for advanced study at typical larger universities. By reviewing and perhaps codifying laudable standards and practice in the training of teachers, it may go far toward perpetuating the good of the recent report of its committee on this subject. We recommend, therefore,

That the Association create a standing "Committee on Collegiate Curricula," to be charged with the collection, the review and the collation of facts pertinent to mathematical instruction in the colleges—including entrance requirements, course contents, teaching procedures, requirements for degrees with mathematics as a major or as a minor subject, requirements for teachers' certificates or teaching licenses, etc., and that this committee be instructed to set forth its findings from time to time as it sees fit either for publication in the MONTHLY *or, at the discretion of the Board*

of Governors, in the form of special bulletins to be circulated among the teaching staffs of the colleges throughout the country.

The domain of activity of the Association abuts or overlaps that of many other organizations in various parts of the country or in the country as a whole. Of these organizations some are of national scope such as The American Mathematical Society, The National Council of Teachers of Mathematics, The Society for the Promotion of Engineering Education, the Pi Mu Epsilon fraternity, *etc.*, while others are of a regional character, such as the New England Association of Teachers of Mathematics, The Central Association of Science and Mathematics Teachers, the various education associations of the States, regional schoolmasters' clubs, the State Academies, *etc.* There are also organizations primarily devoted to other subjects such as physics, chemistry, statistics, engineering, *etc.*, with which the Association will have some affinity of objectives. We believe that the effectiveness and influence of the Association in its national aspect will depend increasingly upon the degree to which it establishes and maintains cordial and even close relations with many of these groups, and upon the degree to which it places itself into a position of readiness to grasp opportunities for coöperation and joint meeting with them. It is important to this end that the Association provide itself with some regular and dependable means for the discovery of the current programs and proposed activities of related groups. We recommend, therefore,

That the Association create a standing "Committee on the Activities of Other Organizations," to establish communication with the secretaries of such organizations with the purpose of discovering their activities and programs and remaining currently informed thereof. That such committee be instructed to report items of apparent interest or significance to the Association or its Sections, and transmit appropriate items to the MONTHLY *for publication.*

We believe that something can be done in the matter of strengthening the programs of the meetings of the Association's Sections. In many instances prevalent weakness in the Section meetings unquestionably militates against the appeal of the Section to wider interest and membership. The weaknesses are often and in large part due to absence of objective and lack of coherence in the program, and to material unimportance of the papers presented. The programs are generally arranged by local committees which are appointed for that purpose and are renewed from year to year. These committees are thus frequently inexperienced and are at the same time compelled to build their programs from such material as may be randomly and voluntarily offered. While the Association should leave the initiative, the structure, and the conduct of the Section meetings strictly to the local committees, it nevertheless seems that it could be greatly helpful in an advisory capacity. A central committee reviewing the local programs and receiving reports concerning them, might glean from them elements of strength which are of more than local interest. By making its material available to local committees it could perhaps often inspire the invitation of

papers on worthwhile subjects, or suggest the discussion of more vital matters. From time to time it might discover matters of moment to the Association as a whole, the discussion of which in all of the Sections could only be desirable.

In the case of the less centrally located Sections the Association has in the past, and from time to time, supplied a speaker for specific meetings. We not only endorse this past practice, but believe it to be one which the Association should undertake to expand. It is a means through which the central organization can greatly help its local units in the less favored parts of the country. With judicially chosen speakers who are informed of and active in the Association's work, it is a means through which the bonds between the Association, its Sections and its Regions can be drawn tighter, and the interest of the several units in each other improved. We recommend, therefore,

That the Association create a standing "Committee on Sectional Meetings," to act in an advisory capacity to local committees, and to supervise such support as the Association may extend to its Sections in the matter of lecturers or speakers for their programs.

In drawing this report to a close we remark that if the Association adopts our recommendation relative to its reorganization, it will find itself to have at hand an Executive Committee which is specifically charged with the prosecution of precisely the planning and appraisal of the Association's activity in which we have been engaged. The *raison d'être* for our committee will cease, therefore, to maintain, and we accordingly beg to be discharged from our assignment. In doing so it is but meet that we should record for the possible use of the Executive Committee such other matters as we have somewhat discussed and upon which in the course of time we might have come to some conclusions.

The programs of the national meetings have concerned us to some extent. Has there been too much of a tendency to follow methods and construct programs along lines determined by the American Mathematical Society without due regard to the Association's different needs and objectives? Could the prevalent plan of unrelated papers be sometimes or often replaced by symposia either on mathematical subjects or on other matters in which the Association is vitally interested? Are the present summer meetings ordinarily scheduled so closely upon the opening of the school year that attendance at them by the teaching personnel of the small colleges and secondary schools is precluded? If the junior college group can be drawn into the Association in important numbers, facilities for their programs must be provided. It seems that these could well be scheduled simultaneously with meetings of the Mathematical Society, as the conflict of interests would be small.

The method by which the Sections at present finance themselves, namely, by the collection of a small fee, usually twenty-five cents, does not seem to be above improvement. It lacks dignity and emphasizes a dissociation between the Sections and the organization as a whole. Can some scheme be contrived whereby a single dues will take care of all obligations to the Association?

The Association has during the last several years supported a committee on

the testing of collegiate mathematics. This committee reported at the summer meeting of 1939, and the chairman, Professor E. W. Chittenden, is of the opinion that the work should be continued. We transmit without comment the following letter which we have from him:

"The study of written examinations has become an important part of current educational research. It has been carried on until a variety of techniques have been developed, and there is now an extensive literature on the subject. The Mathematical Association of America is in a position to assume leadership in the adaptation of these techniques to the teaching of college mathematics as well as to exert influence otherwise. A number of college teachers believe that it is in the interest of the teaching of mathematics for the Association to take an active part in this movement.

"I suggest that a permanent committee of nine be appointed, six members to serve in an advisory capacity with regional representation, and two active members to serve with the chairman in conducting the work of the committee. The term of membership on the committee should be three years, although it may be desirable for the chairman to serve indeterminately.

"The appropriations for this committee should cover the cost of an initial meeting and include an annual allowance of two hundred dollars for current expenses.

"The committee would be expected to keep in touch with current progress in the field of mathematics testing, to coöperate and advise with other organizations engaged in the preparation of mathematics tests, devise methods, compile statistical information, make regular reports, and engage in actual test construction as far as practical. The committee should be concerned with the preservation of the freedom of mathematics teaching from dominance by any testing program.

"The past committee on tests has accumulated a file of several hundred tested questions on Junior College Mathematics. This file should be extended and provision made for further use by colleges of its contents.

"The proposed committee might undertake to prepare examinations for colleges willing to coöperate with the committee with the understanding that the results become a part of the committee's records.

"Coöperating colleges and universities should be changed frequently because of the considerable effort involved. Such changes will increase the breadth of the committee's sampling and enrich its experience.

"A well organized and efficient committee might secure assistance from other organizations. For example, the tests of the past committee were published and distributed by the Coöperative Test Service."

Undergraduate mathematics clubs exist in many institutions and are generally regarded as not being as successful as they might be. In many instances the directors of such clubs find themselves hard pressed for suitable material within the range of the members. Can the Association make itself of use at this point?

The Society for the Promotion of Engineering Education has had some considerable success with its ventures into summer schools of short duration devoted to specific subjects. Are there any possibilities for the Association in some similar enterprise?

It will be evident, even to those who run as they read, that our work of review remains incomplete. We do not apologize for this, since it seems to us at once inevitable and by no means wholly undesirable that it should be so. We have in some cases raised for the Association questions which we evaded answering, be the reason what it may. Where we have come to definite conclusions we have invariably obligated ourselves to the membership of the Association from which we sought and drew advice. It would be difficult for us to list all those who have been verbally helpful to us. It is often difficult to trace the source of an idea. We do, however, desire to make public acknowledgment to those who were of assistance to us through correspondence. They were in many instances put to considerable inconvenience by our requests, and in the press of affairs it is to be feared that personal thanks for their assistance was often passed over. We are greatly indebted to Secretary-Treasurer W. D. Cairns for his unfailing readiness to meet our every request, and our thanks are due, among others, to the following of the Association's members:

R. C. Archibald, Ethelwynn R. Beckwith, E. T. Bell, A. A. Bennett, R. W. Brink, H. E. Buchanan, W. H. Bussey, President W. B. Carver, P. A. Caris, R. D. Carmichael, E. W. Chittenden, H. H. Conwell, Rev. L. A. V. DeCleene, L. L. Dines, Arnold Dresden, P. S. Dwyer.

Cornelius Gouwens, W. C. Graustein, W. L. Hart, E. R. Hedrick, R. C. Huffer, E. V. Huntington, M. H. Ingraham, Dunham Jackson, A. J. Kempner, H. W. Kuhn, W. R. Longley, David Moskovitz.

G. A. Parkinson, T. A. Pierce, W. D. Reeve, H. L. Rietz, J. Riordan, H. A. Robinson, Sister Mary Felice, R. P. Stephens, J. M. Thomas, J. L. Walsh, J. H. Weaver, K. P. Williams, W. M. Whyburn.

Respectfully submitted,

D. R. Curtiss	C. V. Newsom
T. C. Fry	G. B. Price
F. L. Griffin	Marie J. Weiss
Martha Hildebrandt	Rudolph E. Langer, *Chairman*

THE FIBONACCI SEQUENCE AND ALLIED TRIGONOMETRIC IDENTITIES*

L. S. JOHNSTON, University of Detroit

The Fibonacci Sequence $[u_i]$, where

$$u_1 = u_2 = 1, \qquad u_{n+2} = u_{n+1} + u_n,$$

has been famous for some seven centuries and has appeared in very large volume in the literature of botany, zoology, and psychology in addition to that of mathematics.† The connection between the sequence and some of the standard identities of trigonometry seems to be less well known, however, than even some of these non-mathematical connections mentioned. It is the purpose of this note to show some of these allied trigonometric identities, and in particular to show convenient methods of setting up problem exercises.

In various places in the literature of trigonometry appear such problems and formulas as

(1) $\pi/4 = \text{arc tan } 1/2 + \text{arc tan } 1/5 + \text{arc tan } 1/8$;‡

(2) $\sum_{n=0}^{3} \text{arc cot } (3n - 1) = 0$;§

(3) $\text{arc cot } 1 = \text{arc cot } 2 + \text{arc cot } 5 + \text{arc cot } 13 + \text{arc cot } 34 + \cdots,$

where the numbers 1, 2, 5, 13, 34, $\cdots$ are every other term of the Fibonacci Sequence.‖

It is seen at once that (1) and (2) are identical, and that their form suggests in their first two or three elements some connection with (3) and therefore with the Fibonacci Sequence. We shall set up a general theory which includes all of these as special cases.

Consider any sequence $[t_i]$ where t_1 and t_2 are any two numbers not both

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* An extension of remarks on the same subject before the Michigan Section of the Association, March 19, 1938.

† For some discussion of these applications and for an extensive bibliography of the literature on the sequence itself, see Dynamic Symmetry—the Greek Vase, by Jay Hambidge, Yale University Press, 1920, Chapter 1, and Appendix of the same volume, pages 146–157. See also College Algebra, Revised Edition, 1919, by Rietz and Crathorne, page 85, exercise 48.

‡ Carmichael and Smith, Mathematical Tables and Formulas, page 268, formula 36. The writer has not found this formula in any classroom text, though the somewhat similar formula

$$\pi/4 = \text{arc tan } 1/3 + \text{arc tan } 1/5 + \text{arc tan } 1/7 + \text{arc tan } 1/8$$

appears in some texts, *e.g.*, Palmer and Leigh, Plane and Spherical Trigonometry, 1934, page 126, exercise 36.

§ Proposed as a problem by A. A. Bennett, this MONTHLY, March 1936.

‖ Proposed as a problem by D. H. Lehmer, this MONTHLY, November 1936.

See also: On the Derivation of Arctangent Equalities, J. W. Wrench, Jr., this MONTHLY, February 1938; and Lewis Carroll and a Geometrical Paradox, Warren Weaver, this MONTHLY, April 1938.

zero, the recurrent relation of the sequence being $t_{n+2}=t_{n+1}+t_n$. If we can find a function $f(t)$ such that $f(t_n)=f(t_{n+1})+f(t_{n+2})$ for every integral value of n, we can set up either the finite sum

$$f(t_n) = \sum_{k=1}^{r} f(t_{n-1+2k}) + f(t_{n+2r}), \tag{4}$$

or, subject to convergency conditions, the infinite sum

$$f(t_n) = \sum_{k=1}^{\infty} f(t_{n-1+2k}). \tag{5}$$

One such function is $f(t_n)=(-1)^{n-1}t_n$, that is, the sequence formed by changing the sign of every other term of $[t_i]$ beginning with the second. This sequence admits the finite expansion (4) but not the infinite expansion (5). We shall exhibit a function $f(u)$, however, which admits both expansions.

Consider the identity

$$\begin{aligned} u_{n+1} &= \frac{u_{n+2}u_n + u_{n+1}^2 - u_{n+2}u_n}{u_{n+1}} \\ &= \frac{u_{n+2}u_n + u_{n+1}^2 - u_{n+2}u_n}{u_{n+2} - u_n}. \end{aligned}$$

It is well known (or quite easily proved by one who may not already be familiar with the fact) that $u_{n+1}^2-u_{n+2}u_n=(-1)^n$. Hence the last equation above can be written

$$u_{n+1} = \frac{u_{n+2}u_n + (-1)^n}{u_{n+2} - u_n}. \tag{6}$$

For $n=2p$, p being integral, we have

$$u_{2p+1} = \frac{u_{2p+2}u_{2p} + 1}{u_{2p+2} - u_{2p}}.$$

The right-hand member of this equation is the familiar formula for cot (arc cot u_{2p} − arc cot u_{2p+2}), whence we have the equation

$$\text{arc cot } u_{2p+1} = \text{arc cot } u_{2p} - \text{arc cot } u_{2p+2},$$

or

$$\text{arc cot } u_{2p} = \text{arc cot } u_{2p+1} + \text{arc cot } u_{2p+2},$$

which is a solution of the functional equation

$$f(u_{2p}) = f(u_{2p+1}) + f(u_{2p+2}).$$

In view of (4) we have the finite sum

$$\text{arc cot } u_{2p} = \sum_{k=1}^{r} \text{arc cot } u_{2p-1+2k} + \text{arc cot } u_{2p+2r}, \tag{7}$$

and, since the remainder term in (7) approaches zero as r increases, we have the infinite sum

$$\text{arc cot } u_{2p} = \sum_{k=1}^{\infty} \text{arc cot } u_{2p-1+2k}. \tag{8}$$

Now consider the Lehmer problem. In that problem it is stated that the numbers 1, 2, 5, 13, 34, $\cdots$ are every other term of the sequence $[u_i]$, that is, that the element 1 in that problem is u_1 of the sequence. But if we consider 1 as u_2 instead of u_1, we see that the problem is merely a special case of our (8) in which $p=1$. If we set $p=1$, $r=2$ in (7) we have precisely the Bennett problem.

Referring again to (6), if we set $n=2p-1$, we have

$$u_{2p} = \frac{u_{2p+1}u_{2p-1} - 1}{u_{2p+1} - u_{2p-1}}.$$

The right-hand member of this equation is the familiar formula for

$$\coth\,(\text{arg coth } u_{2p-1} - \text{arg coth } u_{2p+1}),$$

where we have used the notation arg coth instead of the more conventional $\coth^{-1}$ for typographical convenience.* We then have the equation

$$\text{arg coth } u_{2p} = \text{arg coth } u_{2p-1} - \text{arg coth } u_{2p+1},$$

or

$$\text{arg coth } u_{2p-1} = \text{arg coth } u_{2p} + \text{arg coth } u_{2p+1},$$

which exhibits a solution of the functional equation

$$f(u_{2p-1}) = f(u_{2p}) + f(u_{2p+1}).$$

In view of (4) and (5) we can expand arg coth u_{2p-1} in either the finite sum or the infinite sum just as we expanded arc cot u_{2p}.

For any sequence $[t_i]$ we can always choose the first two terms so that $t_2^2 - t_3t_1$ is negative, that is so that $t_2^2 - t_3t_1 = -b^2$. For such a sequence $[t_i]$ it is easily shown that

$$t_{n+1}^2 - t_{n+2}t_n = (-1)^{n-1}(t_2^2 - t_3t_1) = (-1)^n b^2.$$

Now if a new sequence $[t_i']$ be formed such that $t_n' = t_n/b$, the terms of this sequence will fit into (7) and (8) and their corresponding formulas involving arg coth just as the terms of $[u_i]$ fit into these formulas.

Other identities based upon $[u_i]$ can be derived in almost unlimited number.

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* This notation was suggested by Professor Norman Anning of the University of Michigan in remarks following the reading of the paper at the Michigan meeting.

We exhibit some without proof; their proofs may interest the reader as exercises, and may supply him with good problems for his students.

$$\text{(i)} \qquad \operatorname{arc\,cot} \frac{u_n}{u_{n+1}} - \operatorname{arc\,cot} \frac{u_{n-1}}{u_n} = (-1)^n \operatorname{arc\,cot} u_{2n},$$

$$\text{(ii)} \qquad \operatorname{arc\,cot} u_{2p-1} + \operatorname{arc\,cot} u_{2p+2} = \operatorname{arc\,cot} (u_{2p}/u_3),$$

$$\text{(iii)} \qquad \operatorname{arc\,cot} u_{2p-1} + \operatorname{arc\,cot} u_{2p+3} = \operatorname{arc\,cot} (u_{2p+1}/u_4),$$

$$\text{(iv)} \qquad \operatorname{arg\,coth} u_{2p} + \operatorname{arg\,coth} u_{2p+3} = \operatorname{arg\,coth} (u_{2p+1}/u_3),$$

$$\text{(v)} \qquad \operatorname{arg\,coth} u_{2p} + \operatorname{arg\,coth} u_{2p+4} = \operatorname{arg\,coth} (u_{2p+2}/u_4).$$

Equations (ii) and (iii) furnish a good example of the danger of assuming that a law is formulated by citing two examples (or any finite number) of apparent applications of the alleged law. Thus, in view of these two equations, the rash student might infer that

$$\operatorname{arc\,cot} u_{2p-1} + \operatorname{arc\,cot} u_{2p-1+q} = \operatorname{arc\,cot} (u_{2p-3+q}/u_q),$$

but a little inspection will show that this alleged law does not hold for $q=5$, nor for any value of q greater than 4. A similar remark applies to (iv) and (v). These might be taken as instances showing the necessity for the step in mathematical induction which so many freshmen fail to appreciate.

Certain functional equations show remarkable analogies between $[u_i]$ and trigonometric identities. Thus the functional equation $[f(x)]^2-[f(y)]^2 = [f(x+y)][f(x-y)]$ has the solutions $f(x)=x$, $f(x)=\sin x$, and, subject to the condition $x=y+2p$, the solution $f(x)=u_x$. Some other analogies of the sort are*

$$u_n(u_{n+2p} - u_{n-2p}) = u_{2n}u_{2p},$$

analogous to

$$[\sin nx][\sin (n+p)x - \sin (n-p)x] = \sin 2nx \sin px;$$

and

$$u_{p+2k}u_{q+2k} - u_p u_q = u_{p+q+2k}u_{2k},$$

analogous to

$$\sin (p+k)x \sin (q+k)x - \sin px \sin qx = \sin (p+q+k)x \sin kx.$$

Some "semi-analogies" are

$$u_{n+2p+1}^2 + u_n^2 = u_{2p+1}u_{2n+2p+1},$$

almost analogous to

$$\cos^2 (n+2p+1)x + \cos^2 nx = 1 + \cos (2p+1)x \cos (2n+2p+1)x;$$

* The referee to whom this paper was first submitted suggests that these analogies and semi-analogies remind one of the well known fact that the algebraic theory of recurring series of the second order is simply isomorphic with the theory of circular and hyperbolic functions.

and

$$u_n(u_{n+2p+1} + u_{n-2p-1}) = u_{2p+1}u_{2n},$$

almost analogous to

$$[\cos nx][\cos (n + 2p + 1)x + \cos (n - 2p - 1)x] = [\cos (2p + 1)x][1 + \cos 2nx];$$

and

$$u_{p+2k+1}u_{q+2k+1} + u_p u_q = u_{p+q+2k+1}u_{2k+1},$$

almost analogous to

$$\begin{aligned}\cos (p + 2k + 1)x \cos (q + 2k + 1)x \\ = \cos (p + q + 2k + 1)x \cos (2k + 1)x + \cos (p - q)x.\end{aligned}$$

THE PASCAL CONFIGURATION IN A FINITE PROJECTIVE PLANE*

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1. Properties of the configuration. *If a simple hexagon is inscribed in a conic, the three points determined by pairs of opposite sides are collinear* [1]. This is Pascal's Theorem. The three points are known as Pascal points and the line as the Pascal line of the hexagon. The figure obtained by considering all of the Pascal points and Pascal lines associated with six given points of a conic by taking them in different ways as vertices of simple hexagons is called a Pascal Configuration or a Mystic Hexagram.

It will be assumed throughout this paper that the six points are distinct and that the conic is non-degenerate. The six points will be called fundamental points and the fifteen lines determined by them fundamental lines. The conic will be referred to as the fundamental conic.

We list for purposes of reference some of the well known properties of the configuration. The proofs may be found in [3], [4], or [5].

There are 45 *Pascal points P and* 60 *Pascal lines h. The points lie by threes on the lines and the lines by fours on the points.*

The 60 *h-lines lie by threes on* 20 *points G, known as Steiner points. The* 20 *G-points may be divided into* 10 *pairs of conjugate points relative to the fundamental conic.*

The G-points lie by fours on 15 *lines i, called Steiner-Plücker lines.*

The h-lines also intersect by threes in 60 *Kirkman points H.* An H-point is determined by three h-lines associated with the three hexagons that can be

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* Presented before the Kansas section of the Association at Pittsburg, Kansas, April 2, 1938.

† Most of the results of this paper constitute a portion of a Master's Thesis written at the University of Kansas. I wish to acknowledge my indebtedness to Professor U. G. Mitchell under whose direction the work was done.

formed from the nine fundamental lines which remain when the sides of one particular hexagon are omitted [6].

There are 20 *Cayley-Salmon lines g, each of which contains one G- and three H-points.*

The 20 *lines g lie by fours on* 15 *Salmon points I.*

There exists a definite (1, 1) *dual correspondence between the above indicated points and lines* [7]:

60 h-lines correspond to 60 H-points,
20 g-lines correspond to 20 G-points,
15 i-lines correspond to 15 I-points.

An H-point corresponds to the h-line of the hexagon which is omitted in obtaining the point.

A g-line corresponds to the G-point which is conjugate to the G-point contained by the given g-line.

An i-line corresponds to the I-point which contains the four g-lines which themselves correspond to the four G-points contained by the given i-line.

The purpose of this paper is to obtain a few of the more elementary properties of the configuration in the simplest of the finite projective planes; *viz.* the modular planes. The following section contains a brief discussion of a modular plane.

2. The modular plane PG (2, p).* Let p be a prime positive integer and let m be any given integer; then there exist integers q, r where $r \geqq 0$ such that $m = qp + r$. The number r is called the *residue of m modulo p*, and is said to be the result of *reducing m modulo p*. We also write $m \equiv r \bmod p$, which reads, *m is congruent to r modulo p*. It is clear that the only required values for r are $0, 1, 2, \cdots, p-1$.

It is not difficult to show that the above set of residues constitutes a *field* F_p under the ordinary operations of addition and multiplication (reducing, of course, all results modulo p). In other words they can be combined as ordinary numbers under addition, subtraction, multiplication, and division.†

Now consider the set of all "*points*" given by the homogeneous triples (x_1, x_2, x_3), where x_1, x_2, x_3, belong to F_p. The triples (mx_1, mx_2, mx_3) and (x_1, x_2, x_3) are understood to represent the same point provided $m \not\equiv 0 \bmod p$. The triple (0, 0, 0) is excluded. Similarly, consider the set of all "*lines*" defined by linear congruences

$$a_1x_1 + a_2x_2 + a_3x_3 \equiv 0 \bmod p,$$

where a_1, a_2, a_3, x_1, x_2, x_3 are again elements of F_p. The coefficients a_1, a_2, a_3 determine the line and the triples (x_1, x_2, x_3) which satisfy the congruence are

* For a detailed discussion of finite projective geometries, see [8].

† Division here means the inverse of multiplication; *i.e.* the result of dividing b by a is an element x such that $ax \equiv b \bmod p$. Division by 0 is of course excluded.

points which lie on the line. This finite set of points and lines can be shown to satisfy the postulates for a finite projective geometry of two dimensions, called a modular plane and denoted by $PG(2, p)$ [1]. The number of points (or lines) in the geometry is p^2+p+1. Each line contains exactly $p+1$ points and each point is on exactly $p+1$ lines. A conic is represented by any quadratic form

$$\sum_{i,j=1}^{3} a_{ij}x_ix_j \equiv 0 \bmod p,$$

where, as before, the coefficients and variables are elements of F_p. Each non-degenerate conic contains exactly $p+1$ points.

3. The configuration in $PG\ (2,\ p)$. The number of points on a conic in $PG(2, p)$ is $p+1$; therefore, the convention that the fundamental points be distinct necessitates the restriction $p \geqq 5$. It is not difficult to prove that in every Pascal configuration there exist three fundamental lines which are not copunctual and which do not intersect on the conic. Three such fundamental lines may be chosen as sides of the triangle of reference for a system of homogeneous coördinates. Denote the six fundamental points by 1, 2, 3, 4, 5, 6 and the sides of the triangle of reference by $\overline{12}$, $\overline{34}$, $\overline{56}$.

The equation of the conic may now be written in the form

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 + \left(r + \frac{1}{r}\right)x_2x_3 + \left(s + \frac{1}{s}\right)x_1x_3 \\ + \left(t + \frac{1}{t}\right)x_1x_2 \equiv 0 \bmod p, \end{aligned} \tag{1}$$

and coördinates of the six fundamental points become*

$$\begin{array}{lll} 1:(0,\ r,\ -\ 1), & 3:(-\ 1,\ 0,\ s), & 5:(t,\ -\ 1,\ 0), \\ 2:(0,\ -\ 1,\ r), & 4:(s,\ 0,\ -\ 1), & 6:(-\ 1,\ t,\ 0), \end{array} \tag{2}$$

where r, s, t denote parameters which are arbitrary except for the following restrictions necessitated by the assumption of distinct fundamental points and a non-degenerate conic:

$$\begin{array}{lll} r \not\equiv 0,\ \pm\ 1, & rs \not\equiv t, & \\ s \not\equiv 0,\ \pm\ 1, & st \not\equiv r, & rst \not\equiv 1 \bmod p. \\ t \not\equiv 0,\ \pm\ 1, & rt \not\equiv s, & \end{array} \tag{3}$$

These results hold in the ordinary plane if the congruences are replaced by equalities [9], [10].

4. Coincidences among the Pascal points. We consider in this section the

* In what follows we will use interchangeably integers which are congruent modulo p; *e.g.*, -1 will often be used in place of its residue $p-1$.

maximum number of coincidences that can occur among the Pascal points and the dependence of the maximum on the geometry in which the configuration is immersed.

It is not difficult to see that the only way the Pascal points can coincide is by threes, and that, when a particular Pascal point is given, the two other points are thereby determined [11]. If we agree to call a point of the configuration at which three of the Pascal points coincide a *triple point*, then this property may be stated as follows: *The* 45 *Pascal points can be divided into* 15 *separate and unique groups of three each, where each group represents a possible triple point.* Evidently a triple point occurs when three of the fundamental lines are copunctual (*i.e.* at a point not on the conic). Corresponding to each of the 15 groups will be an expression in r, s, t whose vanishing gives the condition that the associated triple point shall occur. We shall denote a triple point determined, for example, by the lines $\overline{13}$, $\overline{24}$, $\overline{56}$ by the symbol (13, 24, 56). The 15 expressions are given in the following table. Observe that the choice of coördinates prohibits occurrence of the triple point (12, 34, 56). Each expression and associated triple point will be noted by the corresponding symbol at the right of the table.

TABLE I

Triple point	Expression	Symbol
(12, 34, 56)	1	—
(14, 23, 56)	r^2-s^2	A_1
(13, 24, 56)	r^2s^2-1	A_2
(12, 36, 45)	s^2-t^2	B_1
(12, 35, 46)	s^2t^2-1	B_2
(16, 25, 34)	t^2-r^2	C_1
(15, 26, 34)	r^2t^2-1	C_2
(14, 25, 36)	$r^2s^2t^2+2rst-r^2s^2-s^2t^2-r^2t^2$	T_1
(16, 23, 45)	$r^2+s^2+t^2-2rst-1$	T_2
(15, 24, 36)	$r^2s^2t^2-2rst-s^2t^2+s^2+t^2$	U_1
(13, 26, 45)	$r^2s^2+r^2t^2-r^2-2rst+1$	U_2
(14, 26, 35)	$r^2s^2t^2-2rst-r^2s^2+r^2+s^2$	V_1
(15, 23, 46)	$r^2t^2+s^2t^2-t^2-2rst+1$	V_2
(13, 25, 46)	$r^2s^2t^2-2rst-r^2t^2+r^2+t^2$	W_1
(16, 24, 35)	$r^2s^2+s^2t^2-s^2-2rst+1$	W_2

Among the last eight of the expressions we have the relations,

$$
\begin{aligned}
T_1 + T_2 &= (r^2 - 1)(s^2 - 1)(t^2 - 1), \\
U_1 - U_2 &= (r^2 - 1)(s^2 - 1)(t^2 - 1), \\
V_1 - V_2 &= (r^2 - 1)(s^2 - 1)(t^2 - 1), \\
W_1 - W_2 &= (r^2 - 1)(s^2 - 1)(t^2 - 1).
\end{aligned} \tag{4}
$$

It is clear from these relations that both expressions in any one of the four

pairs (T_1, T_2), (U_1, U_2), (V_1, V_2), (W_1, W_2), can not vanish simultaneously without violation of the conditions (3). Therefore, *no Pascal configuration can have more than* 10 *triple points.*

Now suppose that a configuration in $PG(2, p)$ contain 10 triple points; then the first six expressions of Table I must vanish. Adding the first two gives

$$A_1 + A_2 = (r^2 - 1)(s^2 + 1).$$

Therefore, since $r^2 \not\equiv 1 \mod p$, we have $s^2 \equiv -1 \mod p$. In a similar manner we obtain $r^2 \equiv -1$ and $t^2 \equiv -1 \mod p$. The values of the remaining eight expressions then reduce to

$$(5)\quad \begin{aligned} T_1 &\equiv 2rst - 4, & U_1 &\equiv -2rst - 4, & V_1 &\equiv -2rst - 4, & W_1 &\equiv -2rst - 4,\\ T_2 &\equiv -2rst - 4, & U_2 &\equiv -2rst + 4, & V_2 &\equiv -2rst + 4, & W_2 &\equiv -2rst + 4. \end{aligned}$$

Four of these vanish only when $rst \equiv \pm 2$, *i.e.* $r^2s^2t^2 \equiv 4 \mod p$. But we know that $r^2s^2t^2 \equiv -1 \mod p$; therefore, $4 \equiv -1 \mod p$, or $4 = p - 1$ and $p = 5$. Hence, if a Pascal configuration in $PG(2, p)$ contains the maximum of 10 triple points, then $p = 5$. On the other hand, consider the configuration in $PG(2, 5)$; then the only values which r, s, t may assume are 0, 1, -1, 2, -2, and conditions (3) reduce these to 2 and -2. Therefore, $r^2 \equiv s^2 \equiv t^2 \equiv -1 \mod 5$, and it is immediate that the first six expressions vanish. The remaining ones have the values

$$\begin{aligned} T_1 &\equiv 1 \mp 1, & U_1 &\equiv 1 \pm 1, & V_1 &\equiv 1 \pm 1, & W_1 &\equiv 1 \pm 1\\ T_2 &\equiv 1 \pm 1, & U_2 &\equiv -1 \pm 1, & V_2 &\equiv -1 \pm 1, & W_2 &\equiv -1 \pm 1 \end{aligned} \quad \mod 5,$$

where the ambiguous sign is determined according as $rst \equiv +2$ or $rst \equiv -2 \mod 5$. In either case, four of these expressions vanish. Therefore, *a necessary and sufficient condition that a Pascal configuration in $PG(2, p)$ shall have the maximum of* 10 *triple points is that $p = 5$.*

The relations (5) in the above proof show that if the first six expressions vanish, then the vanishing of just one more implies $p = 5$. This suggests that, if $p = 5$ is excluded, the maximum number of triple points may be six. In order to show that this is actually the case, we need the following relations:

$$(6)\quad \begin{aligned} U_1 - W_1 &= (t^2 - 1)A_1, & T_1 + V_2 &= (t^2 - 1)A_2,\\ V_1 - W_1 &= (1 - r^2)B_1, & T_1 + U_2 &= (r^2 - 1)B_2,\\ U_1 - V_1 &= (1 - s^2)C_1, & T_1 + W_2 &= (s^2 - 1)C_2,\\ U_2 - W_2 &= (t^2 - 1)A_1, & T_2 - V_1 &= (1 - t^2)A_2,\\ V_2 - W_2 &= (1 - r^2)B_1, & T_2 - U_1 &= (1 - r^2)B_2,\\ U_2 - V_2 &= (1 - s^2)C_1, & T_2 - W_1 &= (1 - s^2)C_2. \end{aligned}$$

It is obvious that if more than three of the first six expressions of Table I vanish then all six must vanish. This case has already been considered. Moreover, if more than six of the expressions of Table I vanish, then at least three

out of the first six must vanish. Therefore, we need only consider those cases in which *exactly* three out of the first six vanish. The only such triples that need be considered are

$$(7) \qquad A_1B_1C_1, \qquad A_1B_2C_2, \qquad B_1A_2C_2, \qquad C_1A_2B_2,$$

for these are the only ones that can vanish without implying the vanishing of all six.

Assume first that $A_1 \equiv B_1 \equiv C_1 \equiv 0 \bmod p$. This implies, by relations (6), that $U_1 \equiv V_1 \equiv W_1$ and $U_2 \equiv V_2 \equiv W_2 \bmod p$. In order for more than six of the expressions to vanish in this case, we must have either $U_1 \equiv 0$ or $U_2 \equiv 0 \bmod p$. Suppose first that $U_1 \equiv 0 \bmod p$. From $A_1 \equiv B_1 \equiv C_1 \equiv 0$ we get $r^2 \equiv s^2 \equiv t^2 \bmod p$; therefore, U_1 reduces to $r^2(r \mp 1)^2(r^2 \pm 2r + 2)$, where the ambiguous sign is determined according as $rst \equiv +r^3$ or $-r^3 \bmod p$. Relations (6) show that the only other expression that can vanish under our assumptions is T_1, which reduces to $r^3(r \mp 1)^2(r \pm 2)$. It follows that in order for T_1 and U_1 to vanish, we must have both

$$r \pm 2 \equiv 0, \quad \text{and} \quad r^2 \pm 2r + 1 \equiv 0 \bmod p.$$

However, these conditions on r are inconsistent for $p > 5$. Assuming $U_2 \equiv 0$ and $T_2 \equiv 0 \bmod p$ leads to the conditions

$$2r \pm 1 \equiv 0, \quad \text{and} \quad 2r^2 \pm 2r + 1 \equiv 0 \bmod p,$$

which are also inconsistent for $p > 5$.

Exactly these same results are obtained if any of the other triples in (7) are assumed to vanish. Therefore, when $p > 5$, a configuration can have at most six triple points. It will be shown immediately that a configuration can actually have six triple points when $p > 5$. Hence, *the maximum number of triple points possible for a configuration in $PG(2, p)$, where $p > 5$, is six.*

It has already been observed that, if the first six expressions of Table I vanish, then $r^2 \equiv s^2 \equiv t^2 \equiv -1 \bmod p$. In order for values of r, s, t to exist so that this condition is satisfied, the quadratic congruence $x^2 \equiv -1 \bmod p$ must have a solution; in other words, -1 must be a quadratic residue modulo p. The condition that such be the case is given by the following theorem, known as Euler's Criterion [**12**].

If p is an odd prime and a is an integer not divisible by p, then a is a quadratic residue or a quadratic non-residue modulo p according as $a^{(p-1)/2} \equiv +1$, or $a^{(p-1)/2} \equiv -1 \bmod p$.

Evidently, if $(p-1)/2$ is even, we can always find a configuration in $PG(2, p)$ which contains the maximum of six triple points; because r, s, t can be chosen so that $r^2 \equiv s^2 \equiv t^2 \equiv -1 \bmod p$, in which case the first six expressions vanish. We wish now to show that this restriction on p is also necessary; or, more precisely, that a configuration in $PG(2, p)$, for $(p-1)/2$ odd, cannot have more than four triple points.

There are associated with each of the last eight expressions of Table I three quadratic discriminants. They are given in the following table.

TABLE II

Discriminants

Expression	r	s	t
T_1	$s^2t^2(s^2-1)(t^2-1)$	$r^2t^2(r^2-1)(t^2-1)$	$r^2s^2(r^2-1)(s^2-1)$
T_2	$(s^2-1)(t^2-1)$	$(r^2-1)(t^2-1)$	$(r^2-1)(s^2-1)$
U_1	$s^2t^2(s^2-1)(t^2-1)$	$t^2(r^2-1)(1-t^2)$	$s^2(r^2-1)(1-s^2)$
U_2	$(s^2-1)(t^2-1)$	$r^2(r^2-1)(1-t^2)$	$r^2(r^2-1)(1-s^2)$
V_1	$s^2(s^2-1)(1-t^2)$	$r^2(r^2-1)(1-t^2)$	$r^2s^2(r^2-1)(s^2-1)$
V_2	$t^2(s^2-1)(1-t^2)$	$t^2(r^2-1)(1-t^2)$	$(r^2-1)(s^2-1)$
W_1	$t^2(s^2-1)(1-t^2)$	$r^2t^2(r^2-1)(t^2-1)$	$r^2(r^2-1)(1-s^2)$
W_2	$s^2(s^2-1)(1-t^2)$	$(r^2-1)(t^2-1)$	$s^2(r^2-1)(1-s^2)$

It is evident that in order for a given one of these eight expressions to vanish, each of its discriminants must be a quadratic residue modulo p. Euler's Criterion shows that if a number a is a quadratic residue modulo p, then $-a$ is a quadratic residue modulo p only in case $(p-1)/2$ is even. An examination of Table II shows that, if $(p-1)/2$ is odd, then at most *one* of the eight expressions can vanish. Moreover, under the same condition at most three of the first six expressions of Table I can vanish. It follows that, when $(p-1)/2$ is odd, a configuration can have at most four triple points. However, under this condition it is possible to have $r\equiv s\equiv 2$ and $t\equiv -2 \bmod p$, in which case A_1, B_1, C_1, and T_1 all vanish. Therefore, *if $(p-1)/2$ is odd, the maximum number of triple points that a Pascal configuration in $PG(2, p)$ can have is four.*

This last result also holds in the ordinary *real* plane. An example of a configuration in this case which has four triple points is given by the configuration determined by the vertices of a regular hexagon inscribed in a circle. Three of the triple points are infinite and the fourth is the center of the circle.

5. The configuration in $PG(2, 5)$. The modular plane $PG(2, 5)$ is particularly interesting relative to the Pascal configuration, because each of its conics contains exactly six points. As a result, the configuration in this plane has very special properties, one of which has already been mentioned, *viz. every configuration in $PG(2, 5)$ has exactly* 10 *triple points.*

The proofs of the following properties are not difficult but, due to lack of space, will be omitted.

Each fundamental line of a configuration in $PG(2, 5)$ contains exactly two triple points, which, moreover, are a conjugate pair relative to the fundamental conic.

The 60 *h-lines coincide by sixes on the ten polars of the triple points. These lines, with the triple points, constitute a Desargues configuration, which is self-polar relative to the fundamental conic.*

The following is an alignment table for the configuration in $PG(2, 5)$. The lower case letters $a_1, a_2, b_1, b_2, \cdots, w_2$ denote respectively the polars of the corresponding triple points. The capital letters $D, E, F, \cdots, R$ denote the 15 Pascal points which are not involved in the triple points. The table is obtained for the

case $rst \equiv +2 \bmod 5$. Two perspective triangles of the Desargues configuration are $B_1C_2W_2$ and $C_1B_2U_2$. They are perspective from A_1.

TABLE III

	A_1	A_2	B_1	B_2	C_1	C_2	T_1	U_2	V_2	W_2	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
a_1		x					x		x		x									x				x	
a_2	x							x		x	x							x					x		
b_1				x			x	x									x		x						x
b_2			x						x	x			x		x										x
c_1						x	x			x		x		x							x				
c_2					x			x	x			x				x						x			
t_1	x		x		x									x			x							x	
u_2		x	x			x										x			x				x		
v_2	x			x		x							x							x		x			
w_2		x		x	x										x			x			x				

In $PG(2, 5)$ the 20 G-points, the 60 H-points, and the 15 I-points all reduce to the 10 triple points of the configuration, while all of the corresponding lines reduce to the polars of the triple points. The reduction takes place in exactly the same manner for corresponding points and lines. Moreover, *the (1, 1) correspondence becomes a polar correspondence relative to the fundamental conic.*

The 60 H-points coincide by sixes on the 10 triple points. The reduction of the G-points takes place through failure of one of the points in each conjugate pair to occur; *i.e.* the three h-lines which would ordinarily determine the point are coincident. The I-points reduce through failure of five of them to occur; *i.e.* the four g-lines which should determine a point fail themselves to occur.

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8. E. R. Ott, this MONTHLY, vol. 44, 1937, p. 86.
9. Cayley, Collected Mathematical Papers, vol. 1, p. 322.
10. F. Briochi, Nouvelles Annales de Mathématiques, vol. 16, 1875, p. 269.
11. F. Levi, Geometrische Konfigurationen, p. 184.
12. R. D. Carmichael, Theory of Numbers, p. 59.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NOTE ON STIRLING'S FORMULA

P. M. HUMMEL, University of Alabama

The usual developments of Stirling's formula* for $n!$ uses, directly or indirectly, Wallis' formula for π, namely

$$\frac{\pi}{2} = \frac{2^2}{1\cdot 2}\cdot\frac{4^2}{3\cdot 5}\cdot\frac{6^2}{5\cdot 7}\cdots .$$

The average college undergraduate, with nothing beyond an elementary calculus course, knows little about infinite series and still less about infinite products, hence gets very little out of one of the usual developments of Stirling's formula.

In the writer's probability course, which is open to students with a knowledge of calculus, the following inequalities†

$$e^{11/12}\sqrt{n}\, n^n e^{-n} < n! < e\sqrt{n}\, n^n e^{-n}, \qquad (n > 1), \tag{1}$$

are established by using only elementary calculus and simple algebra, after which Stirling's formula is accepted without further proof. Actually, the lower limit could be used as a good approximation for $n!$ as it differs from Stirling's formula only in having $\sqrt{2\pi}$ replaced by its approximate equal, $e^{11/12}$.

To establish these inequalities we start with the well known function $y=\log x$. Clearly, if $k\geqq 2$,

$$\int_{k-1}^{k} \log x\, dx > \tfrac{1}{2}[\log (k-1) + \log k], \tag{2}$$

the right member being the trapezoidal area formed by the chord. Hence we may write

$$\int_{k-1}^{k} \log x\, dx = \tfrac{1}{2}[\log (k-1) + \log k] + a_k, \tag{3}$$

where a_k is the area between the curve and chord.

In (3), let $k=2, 3, \cdots, n$ and add, getting

$$\begin{aligned}\int_{1}^{n} \log x\, dx = {} & \tfrac{1}{2}(\log 1 + \log 2) + \tfrac{1}{2}(\log 2 + \log 3) + \cdots \\ & + \tfrac{1}{2}[\log (n-1) + \log n] + (a_2 + \cdots + a_n).\end{aligned} \tag{4}$$

* Stirling's formula states that for large values of n, $n!$ is approximately equal to $\sqrt{2\pi n}\, n^n e^{-n}$.

† The upper limit of (1) is found in Elements of Probability, by Levy and Roth. The lower limit is believed to be new.

Integrating the left member and simplifying the right, gives

$$n \log n - n + 1 = \log n! - \tfrac{1}{2} \log n + (a_2 + a_3 + \cdots + a_n),$$

which may be written

$$\log n! = (n + \tfrac{1}{2}) \log n - n + 1 - (a_2 + a_3 + \cdots + a_n). \tag{5}$$

Since every $a_k > 0$, we have

$$\log n! < (n + \tfrac{1}{2}) \log n - n + 1,$$

and hence

$$n! < e\sqrt{n}\, n^n e^{-n}.$$

To get the lower part of (1), integrate the left member of (3) and solve for a_k, getting

$$a_k = -1 + (k - \tfrac{1}{2}) \log \frac{k}{k-1} \cdot \tag{6}$$

From elementary calculus, we have for any real $f(x) \not\equiv 0$,

$$\int_{k-1}^{k} [f(x)]^2 dx > 0,$$

and hence,

$$\int_{k-1}^{k} \left[\frac{1}{x} - \frac{1}{k}\right]^2 dx > 0. \tag{7}$$

Evaluating the left member of (7), one easily obtains

$$\log \frac{k}{k-1} < \frac{2k-1}{2k(k-1)},$$

and using this result in (6), gives

$$a_k < \frac{1}{4k(k-1)} = \frac{1}{4}\left[\frac{1}{k-1} - \frac{1}{k}\right].$$

Hence

$$a_2 + a_3 + \cdots + a_n < \frac{1}{4}\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right)\right]$$
$$= \frac{1}{4}\left(1 - \frac{1}{n}\right) < \frac{1}{4};$$

and using this result in (5), we obtain

$$\log n! > (n + \tfrac{1}{2}) \log n - n + 1 - \tfrac{1}{4},$$

whence

$$n! > e^{3/4}\sqrt{n}\, n^n e^{-n}.$$

This is not as good a lower limit as is given in (1), but is easily obtained and is the one the writer develops in class. The class is then asked, as a problem, to establish the lower part of (1) by starting with the integral

$$\int_{k-1}^{k}\left[\frac{1}{x} - \frac{2k-1-x}{k(k-1)}\right]^2 dx > 0, \tag{8}$$

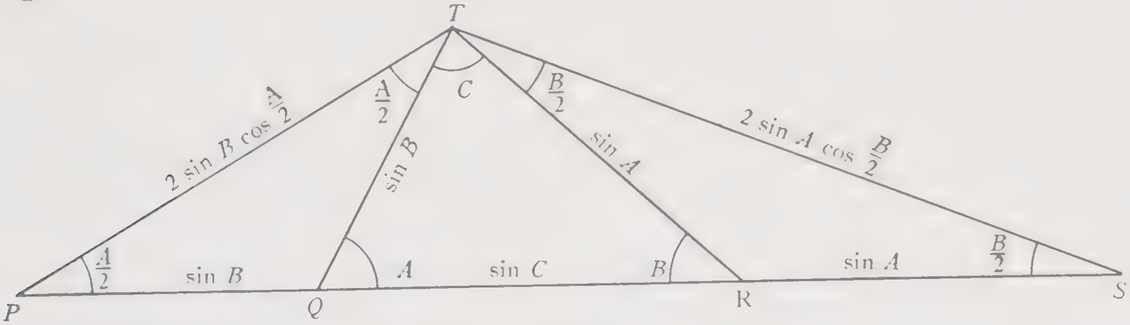
in place of (7). The procedure is essentially the same.

To those who like to accompany their proofs with geometric interpretations we might add that the integrand function of (7) is the square of the difference of the function $y=1/x$ and the horizontal line $y=1/k$. The integrand function of (8) is the square of the difference of the function $y=1/x$ and the chord.

ON THE DISCOVERY OF CERTAIN TRIGONOMETRIC IDENTITIES

R. E. GREENWOOD, JR., University of Texas

The proof of certain trigonometric identities such as the double angle and half angle formulas (for limited range of the angle) by geometrical means is well known.* Certain geometrical methods can also be used to "discover" certain identities where the angles concerned are the angles of a triangle. Let A, B, and C be the angles of a triangle, A and B being acute, and let $\sin A$, $\sin B$, and $\sin C$ be the sides of the triangle. We then construct the following figure:



In $\triangle TQR$, applying the law of cosines, we have

$$\sin^2 B + \sin^2 A - 2\sin B\sin A\cos C = \sin^2 C,$$

while in $\triangle PTS$ we have

$$\begin{aligned}4\sin^2 B\cos^2 A/2 + 4\sin^2 A\cos^2 B/2 \\ - 8\sin A\sin B\cos A/2\cos B/2\cos(C + A/2 + B/2) \\ = \sin^2 A + \sin^2 B + \sin^2 C + 2\sin A\sin B + 2\sin B\sin C + 2\sin C\sin A,\end{aligned}$$

* Hobson, A Treatise on Plane Trigonometry, third edition, pages 55–57, gives several geometrical methods. See also Roscoe Woods, The trigonometric functions of half or double an angle, this MONTHLY, vol. 43, 1936, page 174.

and similar identities for the other triangles involved. The process can be carried still further, as is readily seen.

Furthermore, if C is also acute, the center of the circumscribed circle will be in the interior of the triangle (if C is greater than 90° the same final results can be obtained by a slight modification of the diagram), and the radius of the circumscribed circle will be 1/2. An easy proof of one of the double angle formulas is afforded by applying the law of cosines to $\triangle PQS$:

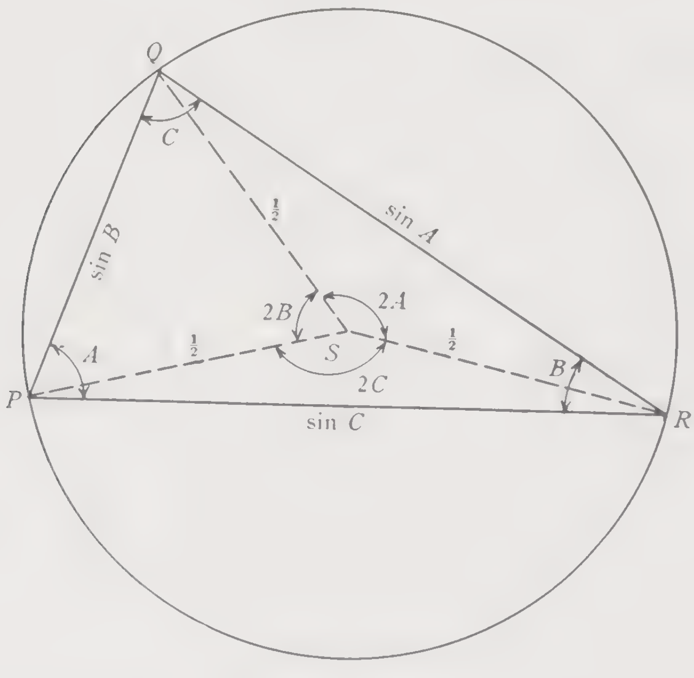
$$\tfrac{1}{4}+\tfrac{1}{4}-\tfrac{1}{2}\cos 2B=\sin^2 B,$$

or

$$1-\cos 2B=2\sin^2 B.$$

From area considerations we have

$$\text{Area } \triangle PQR = \text{Area } \triangle QRS + \text{Area } \triangle RSP + \text{Area } \triangle QSP.$$



Using the fact that the area of a triangle is one-half the product of two sides multiplied by the sine of the included angle, we have

$$\frac{\sin A\sin B\sin C}{2}=\frac{\sin 2A}{8}+\frac{\sin 2C}{8}+\frac{\sin 2B}{8},$$

or

$$4\sin A\sin B\sin C=\sin 2A+\sin 2B+\sin 2C.$$

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department at the Mathematical Association of America, 531 West 116th St., New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Tests of Significance. What they mean and how to use them. By John E. Smith. Chicago, The University of Chicago Press, 1939. 9+90 pages. $1.00.

Mathematics for Actuarial Students. By Harry Freeman. Part II: Finite Differences, Probability, and Elementary Statistics. Cambridge, published for the Institute of Actuaries at the University Press, 1939. 13+339 pages. 25s.

Modern Elementary Theory of Numbers. By L. E. Dickson. Chicago, The University of Chicago Press, 1939. 7+309 pages. $3.00.

Development of the Minkowski Geometry of Numbers. By Harris Hancock. New York, The Macmillan Company, 1939. 24+839 pages. $12.00.

James Gregory Tercentenary Memorial Volume containing his correspondence with John Collins and his hitherto unpublished mathematical manuscripts, together with addresses and essays communicated to the Royal Society of Edinburgh July 4, 1938. Edited by H. W. Turnbull. Royal Society of Edinburgh, 1939. 12+524 pages. 25s.

Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Band I: Algebra und Zahlentheorie. 1. Teil. A. Grundlagen; B. Algebra. Heft 2. Zweite völlig neugearbeitete Auflage. By H. Hasse and E. Hecke. Leipzig and Berlin, B. G. Teubner, 1939. 28+30+56 pages.

Reelle Funktionen. Band I: Zahlen, Punktmengen, Funktionen. By Constantin Carathéodory. Leipzig and Berlin, B. G. Teubner, 1939. 6+184 pages. RM 11.20.

Punktreihengeometrie. By E. A. Weiss. Leipzig and Berlin, B. G. Teubner, 1939. 8+232 pages. RM 14.00.

Elementary Theory of Equations. By W. V. Lovitt. New York, Prentice-Hall, 1939. 11+237 pages. $2.50.

REVIEWS

First Course in the Theory of Numbers. By H. N. Wright. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 7+108 pages. $2.00.

An excellent text for a first, one semester, course in the theory of numbers. The material is vividly presented; the proofs are easy to follow and are amply illustrated with numerical examples. There are 260 exercises for the student, arranged in 31 sets.

Some of the development of the theory is left to the student in these exercises. For the convenience of any teacher who may use this book, a list is given here:

page 3	number 6
page 9	number 6
page 46	number 3
page 65	number 8
page 66	number 12
page 69	number 7
page 76	number 5

The student is presupposed to have some mathematical maturity, about that of two courses in calculus, but little else is assumed. This little includes an acquaintance with proofs by induction and a knowledge of division with a remainder. On pages 39 and 101, where the argument might be too deep for immature students, rigor is obtained by means of references to Perron and Dickson, respectively. In only one place is the discussion unmotivated, and that is on page 58, where the notion of x as an indeterminate symbol is introduced without sufficient explanation.

A great deal of emphasis (27 out of the 104 pages) is placed on the theory of continued fractions, which is applied in solving Diophantine equations, and in factoring large numbers. Also a little unusual in a first course is the inclusion of the Jacobi as well as the Legendre symbols, and the solution of the equation $\phi(x)=n$.

The format of the book is excellent, the type is large and clear, and the text is remarkably free from typographical errors. A table of the first 400 primes is an appendix.

H. H. Campaigne

Plane Trigonometry. With Tables. By W. T. Stratton and R. D. Daugherty. New York, Prentice-Hall, Inc., 1939. 7+88+118 pages. $2.25.

The authors present in 80 pages the essential material for the usual course in plane trigonometry. It is recommended by the reviewer to those teachers who are weary of wordy expositions and who can overlook a few lapses of rigor for the sake of brevity in a book of this type.

The functions of the acute angle are discussed in the first chapter, those of the general angle in chapter five. There is no separate chapter on trigonometric equations since their solution is made a part of several different chapters. A table of important trigonometric formulas is compiled at the end of the text and answers are given for certain problems at the discretion of the authors.

The tables include 5- and 7-place logs, 5-place log functions, 5-place natural functions, and conversion tables from radians to degrees.

Because of the brevity of the text, some teachers may find it necessary to amplify and justify certain statements—notably those involving the symbol ∞. For example, on page 25, last line, we find:

$$1/a^{\infty} = 0, \quad \text{or} \quad a^{-\infty} = 0. \qquad \therefore \quad \log_a 0 = -\infty.$$

The method of interpolation is given without the customary explanation that it is based on an assumption regarding the behavior of the functions under consideration.

No serious typographical errors were noted and the make-up of the book is entirely satisfactory.

HARRIET F. MONTAGUE

Tables for Converting Rectangular to Polar Coördinates. By J. C. P. Miller. London, Scientific Computing Service Limited, 1939. 16 pages.

The introduction consists of a description of the two sets of tables which follow it. The explanation is lucid and detailed. It can be understood by anyone who is familiar with the algebraic relations $r^2=x^2+y^2$, $\tan^{-1}\theta=y/x$ which connect rectangular with polar coördinates and the reduction formulas of elementary trigonometry. Although the tables are prepared for the maximum efficiency of a person working with either a computing machine or the slide rule, they are in no way dependent on the use of either.

The first set of tables can be used to change from rectangular to polar coördinates. The table is constructed for values of $k=l/s$, where l is the smaller and s the larger of $|x|$, $|y|$. For each value of k from 0 to 1 inclusive at intervals of .001 there are listed the corresponding values for $\sqrt{1+k^2}$, $\tan^{-1}k$ (in degrees and radians) and $\cot^{-1}k$ (in degrees only). By use of a subsidiary table, given at the bottom of each page, r and θ can be evaluated from the functions of k.

The second set of tables provides a scheme whereby the sine, cosine, and tangent of all angles up to 27 revolutions in degrees and 36 revolutions in radians can be expressed as functions of an angle in the first quadrant. The reviewer believes that the efficiency of this table would have been enhanced by the addition of a third table giving the values of the sine, cosine, and tangent of angles in the first quadrant.

The tables in the pamphlet ought to be particularly useful for those problems in astronomical or physical research where a large amount of data is obtained in the form of rectangular components and must be expressed in vector components.

ROBERTA F. JOHNSON

College General Mathematics for Prospective Secondary School Teachers. By L. E. Boyer. State College, Pennsylvania, School of Education, The Pennsylvania State College, 1939. 106 pages. $1.00.

This is a thesis submitted at the Pennsylvania State College in partial fulfillment of the requirements for the degree of Doctor of Education. It should be of particular interest to all who are preparing secondary school teachers in mathematics. The author has made a study of the existing situation regarding the mathematical preparation of such teachers. In spite of the widespread demand in teacher-training circles that our secondary school teachers be given a broad

general education which will acquaint them with the major fields of learning, he found that of 700 recently certified non-mathematics teachers 397 had no college mathematics credit. He examined the type of mathematics course given in the Pennsylvania State Teachers Colleges and found they were emphasizing the formal aspects of the subject and "little attempt was being made to bring out the subject's social history, its significance in our lives, and the universal dependence of mankind upon it."

In his thesis Dr. Boyer has set about to build a course which can be profitably given to all college students preparing to be teachers. He made a record of the mathematical topics reported as necessary in research papers in various fields, including the Social Sciences, Biology, Agriculture, Physics, and Chemistry; a study of mathematical terms and topics occurring in general reading; and an analysis of 12 recent text-books related to general mathematics. Then a questionnaire was sent to a large group of in-service teachers and 325 college professors in various fields asking their opinion as to the worthwhileness of 25 topics. The consensus is that there is a keenly felt need for a course in college general mathematics based on the majority of the 25 suggested topics.

After giving a full tabulation of the people participating, the material, and the results, Dr. Boyer closes by choosing the following as suggestive of the main topics a course in general mathematics should contain:

1. Meaning and historical development of number system.
2. Mathematics as a compact, precise, and exact language of numbers.
3. Approximate numbers, origin, nature, and correct manipulation in fundamental processes.
4. Historic development of algebra as a generalization of arithmetic.
5. Significance of mathematics in early history.
6. Nature of mathematics.
7. Connection between mathematics and reasoning in non-mathematical situations.
8. Uses and relationships of formulas, equations, functions, and graphs with manipulative techniques thereof.
9. An abbreviated and unified treatment of algebra, geometry, and trigonometry with respect to their utility to the race.
10. Uses of geometric principles and designs in art, nature, and the works of man.
11. Meaning and the use of statistics.
12. Theory of investment.
13. Significant relation of mathematics to modern science and philosophy.
14. Relation of mathematics to the culture of a civilization.
15. Miscellaneous—to include: Babylonian, Egyptian, and Greek contributions; elementary calculus; fractions, decimals, percentage, ratio; proportion and variation; group theory; permutations, combinations, and probability; three famous geometry problems.

C. A. Lester

The Nature of Proof. The Thirteenth Yearbook of the National Council of Teachers of Mathematics. By Harold P. Fawcett. New York, Bureau of Publications, Teachers College, Columbia University, 1938. 146 pages. $1.75.

Any teacher of mathematics may read this book with profit, and every teacher of high school geometry should read it.

The author sets up these criteria to distinguish the pupil who understands the nature of proof:

1. He will select the significant words and phrases in any statement that is important to him and ask that they be carefully defined.
2. He will require evidence in support of any conclusion he is pressed to accept.
3. He will analyze that evidence and distinguish fact from assumption.
4. He will recognize stated and unstated assumptions essential to the conclusion.
5. He will evaluate these assumptions, accepting some and rejecting others.
6. He will evaluate the argument, accepting or rejecting the conclusion.
7. He will constantly re-examine the assumptions which are behind his beliefs and which guide his actions.

The procedure used to insure this behavior in the members of a geometry class of twenty-five pupils in the Ohio State University School is then described. The first few weeks in class were spent in discussions, the results of which were summarized by the pupils as follows:

1. Definition is helpful in all cases where precise thinking is to be done.
2. Conclusions seem to depend on assumptions but often the assumptions are not recognized.
3. It is difficult to agree on definitions and assumptions in situations which cause one to become excited.

Following this introduction, the pupils were guided to a consideration of space where "the ideas studied are devoid of strong emotional content and the pupils native ability to think is not stifled by prejudice or bias."

After some time was spent on geometry, it was suggested that now the pupils might think about the problems of democracy in the same calm and objective manner they had been using in studying geometry. They then examined the logic in various newspaper articles, advertisements, and court decisions to see if the conclusions reached were justified.

The author presents evidence to show that his course increased the ability of pupils to analyze non-mathematical material more than did the traditional course in geometry, and that it yielded a control of the subject-matter of geometry which was not inferior to that obtained from the usual course.

Some teachers will consider that the procedure outlined is wasteful of time, some may feel that they could not successfully conduct a course of this kind, and some may be subject to certain course requirements that they could not meet under this procedure, but every teacher will find that reading this book will improve his teaching of geometry.

R. A. BEAVER

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.

CLUB TOPICS

In the following bibliographies accompanying the titles, the publications follow with their abbreviations: American Mathematical Monthly, AMM; *Isis*, I; *Mathematics Teacher*, MT; *National Mathematics Magazine*, NMM; *Science*, S; *Scripta Mathematica*, SM; *School Science and Mathematics*, SSM. The numbering of the topics is a continuation of that in use last year in this Department (see vol. 46, pp. 233–234, for example).

24. *Calculating Machines and Devices*, by Lao G. Simons.

Cajori, F. A notable case of finger reckoning in America. I: May 1926.
Locke, L. L. The ancient Peruvian abacus. SM: 1932.
Locke, L. L. The contribution of Leibniz to the art of mechanical calculation. SM: June 1933.
Simons, Lao G. Two reckoning tables. SM: June 1933.

28. *Magic Squares and Circles*, by Lao G. Simons.

Bragdon, C. The Franklin 16×16 magic square. SM: April 1936.
Heath, R. V. A panelled magic square. SM: April 1936.
Heath, R. V. A curious magic square. SM: July 1935.
Heath, R. V. A magic circle. SM: October 1935.

29. *Nomographs*, by Dale Leipper, Marie M. Johnson.

Alcock, H. J. and Jones, J. R. The Nomogram. Sir Isaac Pitman and Sons, Ltd., London, 1932.
Brodetsky, S. A First Course in Nomography. G. Bell and Sons, Ltd., London, 1920. Also Open Court Publishing Company, LaSalle, Illinois.
Brown, O. E. An unusual nomogram. AMM 42: 227–231.
Friauf, J. B. Nomographic solution of a problem in spherical trigonometry. AMM 42: 232–235.
Georges, J. S. and Gorsline, W. W. Nomography. SSM 36: 267–272.
Hewes, L. I. and Seward, H. L. The Design of Diagrams for Engineering Formulas and the Theory of Nomography. McGraw-Hill, New York, 1926.
Hezlet, R. K. Nomographs or the Graphic Representation of Formulae. The Royal Artillery Institution, Woolwich, England, 1913.
Lipka, Joseph. Graphical and Mechanical Computation. Wiley, New York, 1918.
Mackey, C. O. Graphical Solutions. Wiley, New York, 1936. (See pages 50–90 for nomography.)
Marshall, W. C. Graphical Methods. McGraw-Hill, New York, 1921. (Contains a list of 92 formulas and tells where to find their developed nomograms; pages 167–185 concern nomography.)
Mavis, F. T. The Construction of Nomographic Charts. International Textbook Company, Scranton, Pa., 1939.
D'Ocagne, Maurice. Nomography. Encyclopedia Britannica, 14th ed., vol. 16: 483–484. (Includes additional references.)
Rose, W. N. Line Charts for Engineers. Chapman and Hall, London, 1923.
Swett, G. W. Construction of Alignment Charts. Wiley, New York, 1928.
Van Voorhis, M. G. How to Make Alignment Charts. McGraw-Hill, New York, 1937. (This text is concerned only with the actual construction and contains little theory.)
Wood, F. M. A Short Monograph on Nomography. McGill University Library. Also the Engineering Journal: June and August, 1930.

33. *Constructions with Compasses Alone*, by Marie M. Johnson.

Cajori, F. A forerunner of Mascheroni. AMM 36: 364–365.
Carnahan, W. H. Compass geometry. SSM 32: 384–390.
de Lanascol, A. Q. Geómétrie du Compas. Librairie Scientifique, Albert Blanchard, Paris, 1925.
Mascheroni, Lorenzo. La Geometrie del Compasso, Nuova edizione. Palermo, Era Nova, 1901.
Mohr, Georg. Euclides Danicus, 1672. Royal Danish Scientific Society at Kopenhagen, A. F. Host and Son, 1928.
Shively, L. S. An Introduction to Modern Geometry. Wiley, New York, 1939. (See pages 132–135.)

35. *The History of American Mathematics*, by Lao G. Simons.

Bradley, A. D. Pennsylvania German arithmetical books. SM: Jan. 1938.
Cajori, F. The earliest arithmetic published in America. I: Dec. 1927.
Hellman, C. D. Jefferson's efforts towards the decimalization of United States weights and measures. I: Nov. 1931.
Karpinski, L. C. The elusive George Fisher "Accomptant"—writer or editor of three popular arithmetics. SM: Oct. 1935.
Simons, Lao G. The German-American algebra of 1837. SM: Sept. 1932.

36. *The Origin of Various Mathematical Terms and Symbols*, by Lao G. Simons.

Cajori, F. Note on our sign of equality. I: 1924, No. 19.
Cajori, F. Leibniz, the master-builder of mathematical notations. I: 1925, No. 23.
Cajori, F. Empirical generalizations on the growth of mathematical notations. I: 1924, No. 18.

39. *Numerals and Number Systems*, by Lao G. Simons.

Archibald, R. C. Mersenne's numbers. SM: April 1935.
Gandz, S. The origin of the ghubar numerals, or the Arabian abacus and the articuli. I: Nov. 1931.
Lehmer, D. N. Hunting big game in the theory of numbers. SM: Mar. 1933.
Shaw, A. A. Note on Roman numerals. NMM: Dec. 1938.

41. *History of Algebra*, by Lao G. Simons.

Nordgaard, M. A. Sidelights on the Cardan-Tartaglia controversy. NMM: April 1938.

43. *American Mathematicians*, by C. B. Read.

Archibald, R. C. A catalogue of a special exhibition of manuscripts, books, portraits, and personal relics of Nathaniel Bowditch. Peabody Museum, Salem, Mass.
Archibald, R. C. Florian Cajori. I: April 1932.
Archibald, R. C. Simon Newcomb, bibliography of his life and work. Memoirs of the National Academy of Sciences, XVII, first memoir, part II.
Archibald, R. C. Simon Newcomb. SM: Jan. 1936.
Archibald, R. C. Benjamin Peirce. AMM 32: 8–30.
Archibald, R. C. Unpublished letters of James Joseph Sylvester and other new information concerning his life and work. Osiris: Jan. 1936.
Cajori, F. George Bruce Halsted. AMM 29: 338–340.
Coolidge, J. L. Robert Adrian, and the beginnings of American mathematics. AMM 33: 61–76.
Lane, E. P. Ernest Julius Wilczynski. AMM 39: 567–569.
Schreiber, E. W. Florian Cajori—a tribute. SSM 32: 117–134.
Slaught, H. E. Eliakim Hastings Moore. AMM 40: 191–195.
Smith, D. E. Thomas Jefferson and mathematics. SM: Sept. 1932.
Smith, D. E. Eliakim Hastings Moore. MT 26: 109–110.
Yates, R. C. Sylvester at the University of Virginia. AMM 44: 194–201.

44a. *Mathematics in Certain Countries*, by C. B. Read.

Bompiani, E. Italian contributions to modern mathematics. AMM 38: 83–95.

Cheng, D. C. On the mathematical significance of the Chinese Ho T'u and Lo Shu. AMM 32: 499–504.
Karpinski, L. C. The mathematics of the Orient. SSM 34: 467–472.
Radó, T. On mathematical life in Hungary. AMM 39: 85–90.
Vetter, Q. The development of mathematics in Bohemia. AMM 30: 47–58.

44b. *Mathematics in Certain Countries*, by Lao G. Simons.

Archibald, R. C. Mathematics before the Greeks. S: Jan. 1930.
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Bruce, R. E. Sicily, and the march of ancient mathematics and science to the modern world. SM: April 1938, July 1938.
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Neugebauer, O. Babylonian mathematics. SM: Aug. 1934.
Simons, Lao G. The influence of French mathematicians at the end of the eighteenth century upon the teaching of mathematics in American colleges. I: Feb. 1931.
Smith, D. E. Hindu mathematics—the geometry of the Hindus. I: Aug. 1913.
Struik, D. J. Mathematics in the Netherlands during the first half of the sixteenth century. I: May 1936.

45. *First Printed Mathematical Books*, by C. B. Read.

Benedict, Suzan R. The algebra of Francesco Ghaligai. AMM 36: 275–278.
Cowley, Elizabeth B. An English text on mathematics written about 1810. AMM 30: 189–193.
Ebert, Emiline R. A few observations on Robert Recorde and his "ground of arts." MT 30: 110–121.
Kunkel, P. V. A study of Davies' University arithmetic. MT 26: 471–476.
McClenon, R. B. Leonardo of Pisa and his *Liber Quadratorum*. AMM 26: 1–8.
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46. *Mathematics and Art*, by Lao G. Simons.

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Bowes, Julian. Dynamic symmetry. SM: March 1933, June 1933.
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47. *Mathematics and Music*, by Lao G. Simons.

Archibald, R. C. Mathematics and music. I: 1925, No. 22.
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48. *Scholars in Other Fields—Interest in Mathematics.*

Collard, A. Goethe et Quetelet. Leurs relations de 1829 à 1832. I: Jan. 1934.
Locher, L. Goethe's attitude toward mathematics. NMM: Dec. 1936.
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49. *Notes on Famous Mathematicians.*

Archibald, R. C. A rare pamphlet on de Moivre and some of his discoveries. I: Oct. 1926.
Bacon, H. M. The young Pascal. MT 30: 180–185.
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ADDITIONAL NOTES

The following excerpts from club reports may prove useful to other clubs:

Pi Mu Epsilon of *Duke University* writes: "At our social meeting this year for entertainment we had mathematical topics written on slips of paper and mixed in a hat. One member would draw a slip, give a three minute talk on the topic he found, and then designate who was to be the next person to draw from the hat."

Pi Mu Epsilon of the *University of Colorado* sponsors a "Math Teaser" in each issue of the college paper, with the answer published in the following issue.

Square Circle members at *Women's College* of the *University of North Carolina* were given a list of books at the first meeting of the year, from which each was to read at least one and report on it when her name was drawn. To this might be added the suggestion of having reports at the monthly meetings on the best articles appearing in the current mathematics magazines.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 406. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that

$$\sum_{k=0}^{[n/2]} \binom{2n+1}{4k} = 2^{n-1}(2^n \pm 1).$$

E 407. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Let A', B', C' be the feet of the altitudes of a triangle ABC, and H the orthocenter. Let the parallels through H to $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in D, E, F, respectively; and let the parallels through H to BC, CA, AB meet $B'C'$, $C'A'$, $A'B'$ in D', E', F'. Prove that the six points D, D', E, E', F, F' lie on one line, perpendicular to the Euler line (which joins H to the circumcenter).

E 408. *Proposed by W. C. Rufus, Observatory of the University of Michigan.*

From front to rear of an advancing army detachment was ten miles. A rear-guard messenger, dispatched to a guard-house directly behind his position in the line of march, returned without loss of time and then proceeded immediately to the van-guard and again returned. He then noted that he had overtaken his guard ten miles from the starting point, and that the time spent on each errand had been the same. How far was the guard-house from the starting point, and how far did the messenger travel altogether?

E 409. *Proposed by V. Thêbault, Le Mans, France.*

Consider a hexagon whose vertices are the ends of three diameters of a circle. Show that the sum of the products of the distances of a variable point on the circle from pairs of opposite sides of the hexagon is constant.

E 410. *Proposed by C. H. Hardingham, Harpenden, England.*

What are the smallest positive integers a, b, c which are the sides of a triangle whose medians also are integers?

SOLUTIONS

E 366 [1939, 106]. *Proposed by C. O. Oakley, Haverford College.*

Two ferry boats ply back and forth across a river with constant speeds, turning at the banks without loss of time. They leave opposite shores at the same instant, meet for the first time 700 feet from one shore, continue on their way to the banks, return and meet for the second time 400 feet from the opposite shore. As an oral exercise, determine the width of the river.

Solution by W. C. Rufus, Observatory of the University of Michigan.

Ferry boat A left one shore, travelled 700 feet and met B; together they had travelled the width of the river. A continued across the river to the opposite shore and back 400 feet, where he met B again. They had then travelled a total of three times the width of the river. As their speeds were constant, A travelled three times 700 feet, or 2100 feet. The width of the river was 400 feet less than the distance A travelled, that is, 1700 feet.

Also solved by W. B. Clarke, M. L. Constable, Wm. Douglas, E. K. Paxton, Phillips University Mathematics Club, C. W. Trigg, and the proposer.

E 367 [1939, 106]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

The point P moves on the circumcircle of triangle ABC, and the bisectors of angles APC and APB meet AC and AB at Q and R respectively. Show that QR passes through the center of the circle inscribed in the triangle ABC. Show also that if PS and PT are perpendicular to PQ and PR and cut AC and AB at S and T respectively, then ST passes through the center of the escribed circle which touches side BC between B and C.

Solution by L. M. Kelly, Boston University.

Bisect the arc AC at B' and B'', and the arc AB at C' and C'', so that B'' lies on the same side of AC as B, and C'' on the same side of AB as C. Let I be the incenter, and E the relevant excenter. An application of Pascal's Theorem to the hexagon $ACC'PB'BA$ shows that Q, I, R are collinear. Similarly, an application to the hexagon $ACC''PB''BA$ shows that S, E, T are collinear.

Also solved by J. H. Butchart, and Rufus Crane.

E 368 [1939, 107]. *Proposed by W. B. Campbell, Drexel Institute.*

A clock has its hour, minute, and second hands turning about the common center, and all together at noon. They will not be all three coincident again for another twelve hours, although there will be several earlier instants when two of the three hands are superimposed. At which of these instants will the remaining hand make the least angle with the two which are superimposed?

Solution by E. P. Starke, Rutgers University.

Once every minute and a fraction, the second hand overtakes both the hour and the minute hands. To find the time demanded by the problem, we put the minute and hour hands together (a familiar problem from high school algebra)

and move the second hand forward or backward until it overtakes the nearer of the other hands. For any time between noon and 6:00 p. m. there is another between six and midnight at which the hands are in the reflected positions. The times between noon and six at which the two slower hands are together are (in hours and minutes) $1{:}05\frac{5}{11}$, $2{:}10\frac{10}{11}$, $3{:}16\frac{4}{11}$, $4{:}21\frac{9}{11}$, and $5{:}27\frac{3}{11}$. At these times the second hand is at the positions (in seconds) $27\frac{3}{11}$, $54\frac{6}{11}$, $21\frac{9}{11}$, $49\frac{1}{11}$, and $16\frac{4}{11}$. Of these situations, the third finds the second hand nearest the others. The final solution is obtained by backing the second hand until it agrees with the hour hand. (In this process the minute and hour hands are backed too, the minute hand more than the hour hand.) Let x be the number of seconds through which the second hand must be backed. Meanwhile the hour hand will turn $\frac{1}{720}$ of that angular distance, and we have

$$21\tfrac{9}{11} - x = 16\tfrac{4}{11} - x/720,$$

or $x = 5\frac{3655}{7909}$. Thus their position is $3{:}16\frac{256}{719}$. During this time, the minute hand has been backed $x/60$ or $\frac{720}{7909}$, and its position is $3{:}16\frac{196}{719}$. The difference in position is $\frac{60}{719}$ or, as an angle, $360°/719$, which is about $30'\ 02\frac{1}{2}''$.

Also solved by the proposer.

E 369 [1939, 107]. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Quebec.*

Find the form of n if $1^4+2^4+3^4+\cdots+n^4$ is exactly divisible by $1^2+2^2+3^2+\cdots+n^2$.

Solution by E. P. Starke, Rutgers University.

By any of the usual methods of summation (see for example Witmer, *The sums of powers of integers*, this MONTHLY, November 1935, pp. 540–548) we find

$$1^4 + 2^4 + 3^4 + \cdots + n^4 = n(n+1)(2n+1)(3n^2+3n-1)/30,$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6.$$

The condition of divisibility reduces to the requirement that $3n^2+3n-1$ be a multiple of 5. We can put

$$3n^2 + 3n - 1 = 3(n-3)(n-1) + 5(3n-2),$$

which is a multiple of 5 if and only if n is of one of the forms $5k+1$, $5k+3$.

Also solved by Calvin Foreman, Wm. Forman, V. W. Graham, Elmer Latshaw, K. W. Miller, W. O. Pennell, Harry Siller, W. R. Talbot, C. W. Trigg, and the proposer.

E 370 [1939, 107]. *Proposed by V. Thébault, Le Mans, France.*

Locate the point P within the irregular tetrahedron $ABCD$ so that each of the six planes, each through P and an edge, will bisect the surface of the tetrahedron.

Solution by H. V. Lyons, University of Toronto.

Let a, b, c, d be the areas of the faces BCD, CDA, DAB, ABC of the tetrahedron, and let (x, y, z, t) be the barycentric coördinates of P. The plane ABP cuts the edge CD at the point Q or $(0, 0, z, t)$, which divides CD in the ratio $t:z$. The lines AQ and BQ divide the areas of the triangles ACD and BCD in this same ratio. By the conditions of the problem, we have the following relation concerning areas of triangles:

$$ABC + ACQ + BCQ = ABD + ADQ + BDQ,$$

i.e.,

$$d + \frac{tb}{z+t} + \frac{ta}{z+t} = c + \frac{zb}{z+t} + \frac{za}{z+t}\cdot$$

This is easily reduced to

$$\frac{z}{a+b-c+d} = \frac{t}{a+b+c-d},$$

and the other planes lead to similar relations. Thus the point P is

$$(-a+b+c+d,\ a-b+c+d,\ a+b-c+d,\ a+b+c-d).$$

E 371 [1939, 168]. *Proposed by H. T. R. Aude, Colgate University.*

A box contains a sufficient number of coins, which are pennies, dimes, and half-dollars. Find the smallest sum, also the largest sum, which can be counted out in two and only two ways, by using one hundred coins including at least one of each kind.

Solution by B. A. Hausmann, University of Detroit.

If a, b, c, primed or unprimed, represent the number of pennies, dimes, and half-dollars, respectively, we have

$$a + 10b + 50c = a' + 10b' + 50c', \tag{1}$$

$$a + b + c = a' + b' + c', \tag{2}$$

and therefore

$$9(b - b') + 49(c - c') = 0.$$

Since 9 and 49 are relatively prime, we deduce

$$b - b' = 49k, \qquad c - c' = -9k.$$

We may suppose $b > b'$. Then, since every kind of coin must be used, $k=1$. Applying (2) again to get a similar equation for a and a', we have

$$a = a' - 40, \qquad b = b' + 49, \qquad c = c' - 9.$$

The largest sum will occur when a, a', b, and b' are as small as possible.

Hence $a=1$, $a'=41$, $b=50$, $b'=1$, $c=49$, $c'=58$, and the sum is \$29.51.

Similarly, the smallest sum will occur when b, b', c, and c' are as small as possible. Hence $b=50$, $b'=1$, $c=1$, $c'=10$, $a=49$, $a'=89$, and the sum is \$5.99.

Also solved by W. B. Campbell, Wm. Douglas, Daniel Finkel, Wm. Forman, H. D. Larsen, Elmer Osborne, C. W. Trigg, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3941. *Proposed by N. A. Court, University of Oklahoma.*

The polar plane of a point common to three given spheres, with noncollinear centers, with respect to a variable sphere tangent externally to the three given spheres, describes a coaxal pencil.

3942. *Proposed by H. E. Tester, Isleworth, Middlesex, England.*

A man is standing at the junction of two perpendicular cross-roads, and his dog, at a distance a from the junction along one of the roads, is watching him. At a given instant the man starts to walk with speed v along the other road, and the dog to run directly towards his master with speed $2v$. Determine the curve of pursuit.

3943. *Proposed by H. E. Tester, Isleworth, Middlesex, England.*

Express the integral

$$I = \int_0^3 dx \int_0^{(5-5x^2/9)^{1/2}} dy$$

by the use of the variables λ and μ defined by the equations

$$\lambda + \mu = [(x+2)^2 + y^2]^{1/2}$$
$$\lambda - \mu = [(x-2)^2 + y^2]^{1/2}$$

and thus verify that the value of the integral is $3\pi\sqrt{5}/4$.

3944. *Proposed by V. Thébault, Le Mans, France.*

The positive integral point-masses x, y, z are placed respectively at the vertices A, B, C of an equilateral triangle with sides of length a. Determine x, y, z, a so that the distances of the centroid of the three masses to the three vertices shall be integers.

3945. *Proposed by V. Thébault, Le Mans, France.*

If in a triangle the distances d_1, d_2, d_3 of the midpoints of the sides of the triangle to a tangent to the nine-point circle satisfy a relation $\sqrt{d_1} \pm \sqrt{d_2} \pm \sqrt{d_3} = 0$, the point of contact of that tangent is one of the Feuerbach points.

3946. *Proposed by V. Thébault, Le Mans, France.*

Given a tetrahedron $ABCD$ and an arbitrarily chosen point P: (1) Show that the radical planes of an arbitrary sphere through P with each of the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, circumscribing the tetrahedrons in the parentheses, meet respectively the planes of the faces BCD, CDA, DAB, ABC in four straight lines lying in a plane. (2) If an arbitrary plane cuts the plane of the faces BCD, CDA, DAB, ABC in the straight lines Δ_a, Δ_b, Δ_c, Δ_d, show that the planes (P, Δ_a), (P, Δ_b), (P, Δ_c), (P, Δ_d) cut the spheres $(PBCD)$, $(PCDA)$, $(PDAB)$, $(PABC)$, respectively in four circles of a single sphere through P.

3905 [1939, 112]. *Correction.* In the fifth line after $\cdots$ *two determined directions* insert *if M lies on (O).*

SOLUTIONS

3836 [1937, 394]. *Proposed by H. P. Thielman, College of St. Thomas, St. Paul, Minn.*

Given Kelvin's function

$$\operatorname{bei} x = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x/2)^{4n-2}}{[(2n-1)!]^2},$$

evaluate

$$\int_0^{\infty} \frac{\operatorname{bei} x}{x}\, dx.$$

A supplementary note by E. S. Pondiczery, Princeton, N. J.

In his discussion in this MONTHLY [1939, 517], Hsü shows that the integral

$$\int_0^{\infty} \frac{\operatorname{bei}(x)}{x}\, dx \tag{1}$$

does not converge, and that (1) would have the value $\pi/4$ "if it existed." That is, formal computations with Laplace integrals assign the value $\pi/4$ to the divergent integral (1). Such a formal evaluation of an integral indicates that it should be summable to the formal value; I shall show that (1) is in fact summable to the value $\pi/4$ by a well known variant of the Abel process. More precisely, I shall show that

$$\lim_{\delta \to 0+} \int_0^{\infty} \frac{\operatorname{bei}(u)}{u} e^{-\delta u^2} du = \frac{\pi}{4}; \tag{2}$$

the integral exists for each positive δ because

$$\operatorname{bei}(x) = O(x^2), \quad x \to 0; \qquad \operatorname{bei}(x) = o(e^{x/\sqrt{2}}), \quad x \to \infty.$$

Making a change of variable, I have to show that

$$\lim_{\epsilon \to 0+} \int_0^\infty \frac{\operatorname{bei}(2t^{1/2})}{t} e^{-\epsilon t} dt = \frac{\pi}{2}. \tag{3}$$

Now

$$\begin{aligned}
\int_0^\infty \frac{\operatorname{bei}(2t^{1/2})}{t} e^{-\epsilon t} dt &= \int_0^\infty \operatorname{bei}(2t^{1/2}) dt \int_\epsilon^\infty e^{-st} ds \\
&= \int_\epsilon^\infty ds \int_0^\infty \operatorname{bei}(2t^{1/2}) e^{-st} dt \\
&= \int_\epsilon^\infty \frac{1}{s} \sin \frac{1}{s} ds = \int_0^{1/\epsilon} \frac{\sin u}{u} du \\
&\to \frac{\pi}{2}, \qquad (\epsilon \to 0+),
\end{aligned}$$

provided that the interchange of integrations is legitimate. I have used the fact, established by Hsü, that

$$\int_0^\infty \operatorname{bei}(2t^{1/2}) e^{-st} dt = \frac{1}{s} \sin \frac{1}{s}, \qquad s > 0.$$

To complete the proof, I have to show that if $\epsilon > 0$

$$\int_\epsilon^\infty ds \int_0^\infty \operatorname{bei}(2t^{1/2}) e^{-st} dt = \int_0^\infty \operatorname{bei}(2t^{1/2}) dt \int_\epsilon^\infty e^{-st} ds.$$

I break the integral on the left into two parts:

$$\int_\epsilon^\infty ds \int_0^\infty \operatorname{bei}(2t^{1/2}) e^{-st} dt = \int_\epsilon^\infty \int_0^1 + \int_\epsilon^\infty \int_1^\infty = I_1 + I_2.$$

From the power series expansion of $\operatorname{bei}(x)$, we see that there is a constant A such that

$$|\operatorname{bei}(2t^{1/2})| < At, \qquad 0 \leqq t \leqq 1.$$

Consequently I_1 is dominated by the convergent integral

$$\int_\epsilon^\infty ds \int_0^1 te^{-st} dt \leqq \int_\epsilon^\infty ds \int_0^\infty te^{-st} dt = \int_\epsilon^\infty \frac{ds}{s^2} = \frac{1}{\epsilon};$$

therefore the integrations in I_1 can be interchanged.*

* E. W. Hobson, The Theory of Functions of a Real Variable, vol. 2, 1926, p. 347.

As for I_2, from Hsü's work we see that there is a constant B such that

$$|\operatorname{bei}(2t^{1/2})| < B exp(2t)^{1/2}, \qquad t \geqq 1;$$

hence I_2 is dominated by

$$B\int_\epsilon^\infty ds\int_1^\infty e^{-st}exp(2t)^{1/2}dt,$$

and this integral converges. In fact, we have, since $s \geqq \epsilon$ in I_2,

$$\begin{aligned}\int_1^\infty e^{-st}exp(2t)^{1/2}dt &= \int_1^\infty e^{-st/2}e^{-st/2}exp(2t)^{1/2}dt\\ &< e^{-s/2}\int_0^\infty e^{-\epsilon t/2}exp(2t)^{1/2}dt\\ &= A(\epsilon)e^{-s/2},\end{aligned}$$

where $A(\epsilon)$ is independent of s. Consequently I_2 is dominated by the convergent integral

$$BA(\epsilon)\int_\epsilon^\infty e^{-s/2}ds,$$

so that the interchange of integrations is legitimate. This completes the proof of (3).

3855 [1938, 53]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Solve in positive integers

$$(x^{1/2}+y^{1/2})^4+(x^{1/2}-y^{1/2})^4=z^4.$$

Solution by Elijah Swift, University of Vermont.

If we set $z=2w$, and simplify, the equation becomes

$$2x^2+12xy+2y^2=16w^4; \tag{1}$$

dividing by $2w^4$ and letting $X=x/w^2$ and $Y=y/w^2$, we get

$$X^2+6XY+Y^2=8. \tag{2}$$

If we cut this conic by any line through (1, 1) with rational slope, we obtain a rational solution of (2), and conversely we can obtain all rational solutions in this way. Taking the slope as m/l where m and l are integers, we find X and Y and hence x, y and w^2, and then x, y and z^2, in terms of the integers l and m and an arbitrary factor k. These solutions are

$$\begin{aligned}x &= k(m^2-2lm-7l^2),\\ y &= k(-7m^2-2lm+l^2),\\ z^2 &= 4k(m^2+6lm+l^2).\end{aligned} \tag{3}$$

From the derivation it is clear that equations (3) give all possible solutions of the original equation.

Not every pair of values (m, l) gives a solution in positive integers, since the three parentheses on the right of equations (3) must have the same sign. Setting these equal to 0, and graphing the degenerate conics thus obtained, we find that all three parentheses will have the same sign (negative) for two sectors, one in the second and one in the fourth quadrant, lying between the lines $m=l(1-\sqrt{8})$ and $l=m(1-\sqrt{8})$. Considering the sector in the fourth quadrant (for m, l and $-m$, $-l$ give the same values for x, y, z) we find that we must take m as a positive integer and l as a negative integer so that

$$1.828\cdot m > -l > 0.547\cdot m.$$

If we now choose m and $-l$ as any two integers satisfying this relation, then choose k as a negative integer such that z^2 is a perfect square, we shall obtain solutions of the given equation. We find that $-l=m$ yields the obvious solutions $(a^2, a^2, 2a)$. Also if (x, y, z) form a solution, so will (a^2x, a^2y, az). Apart from these we find the following solutions:

$$\begin{array}{lllll} m=2, & l=-3; & x=47\cdot 23, & y=7\cdot 23, & z=46; \\ m=3, & l=-4; & x=47\cdot 79, & y=47\cdot 23, & z=94; \\ m=3, & l=-5; & x=136\cdot 14, & y=8\cdot 14, & z=56; \end{array}$$

or, reduced by suitable division, this last solution becomes

$$x=17\cdot 7, \qquad y=7, \qquad z=14.$$

In this way we can find as many solutions as may be desired.

Solved also by Daniel Finkel, and E. P. Starke.

3861 [1938, 122]. *Proposed by V. Thébault, Le Mans, France.*

A convex polygon $A_1A_2 \cdots A_{2n}$, with its opposite sides equal and parallel, is inscribed in a conic. If from a point of the conic parallels are drawn to the directions conjugate to those of the sides, the intersections of these parallels with the conjugate sides form a polygon of constant area, and its opposite sides have conjugate directions.

Editorial Note. The proposer stated that it is always possible to determine a plane upon which the ellipse projects into a circle (O); the polygon (A) with the vertices A_i inscribed in the ellipse projects into another polygon (P) inscribed in (O) with pairs of opposite sides equal and parallel; and its diagonals joining opposite vertices are diameters of the circle. From this it is easily deduced that the pedal polygon (B) with respect to (P) of a point M on the circle (O) has its opposite sides perpendicular. Also it is easy to show that the locus of a point such that the pedal polygon of this point with respect to (P) has a constant area is a circle concentric with (O). This suffices to prove the theorem of the problem.

From the above it appears that the proposer means that the conic is an ellipse. Since no other details of the proof were given, we shall give proofs of the properties of the figure for the circle (O) and show its relation to a certain limaçon.

The convex polygon (A) with vertices $A_1, A_2, \cdots, A_{2n}$ is inscribed in a circle (O) with center O and radius R so that each diagonal A_iA_{n+i} is a diameter. Any chosen point M of its plane is projected orthogonally upon the sides of (A) giving the points B_i, vertices of the polygon (B), on the sides A_iA_{i+1}, respectively. A necessary and sufficient condition, $n>2$, that the directed area (vector area) of (B) remains constant as M varies is that $r=OM$ remains constant.

We shall suppose that the vertices of (A) are in positive order of rotation. Set $\angle MOA_i=\theta_i$ and $\angle A_iO A_{i+1}=2\phi_i$, where $\phi_1+\phi_2+ \cdots +\phi_n=\pi/2$. Then

$$MB_i = R \cos \phi_i - r \cos (\theta_i + \phi_i),$$
$$MB_{i+1} = R \cos \phi_{i+1} - r \cos (\theta_i + 2\phi_i + \phi_{i+1}),$$

and we obtain MB_{n+i} and MB_{n+i+1} from the above by replacing the minus sign before the r by the positive sign. Thus

$$\begin{aligned} &MB_i\cdot MB_{i+1} + MB_{n+i}\cdot MB_{n+i+1} \\ &\qquad = 2R^2 \cos \phi_i \cos \phi_{i+1} + 2r^2 \cos (\theta_i + \phi_i) \cos (\theta_i + 2\phi_i + \phi_{i+1}). \end{aligned} \tag{2}$$

Multiplying each member of (2) by $\sin (\phi_i+\phi_{i+1})/2$ we have, after reductions,

$$\begin{aligned} &\text{area } MB_iB_{i+1} + \text{area } MB_{n+i}B_{n+i+1} \\ &\qquad = \frac{R^2}{4} [\sin (2\phi_i + 2\phi_{i+1}) + \sin 2\phi_i + \sin 2\phi_{i+1}] + \frac{r^2}{4} \sin (2\phi_i + 2\phi_{i+1}) \\ &\qquad + \frac{r^2}{4} [\sin (2\theta_i + 4\phi_i + 2\phi_{i+1}) - \sin (2\theta_i + 2\phi_i)]. \end{aligned} \tag{3}$$

The sum of the terms containing θ_i from $i=1$ to $i=n$ gives

$$\frac{r^2}{4} [\sin (2\theta_n + 4\phi_n + 2\phi_{n+1}) - \sin (2\theta_1 + 2\phi_1)] \tag{4}$$

after cancellations by using $\theta_{i+1}=\theta_i+2\phi_i$. Also, since $\phi_{n+1}=\phi_1$ and $\theta_n+2\phi_n-\theta_1 =\pi$, the expression (4) reduces to $-(r^2/2) \cos (2\theta_1+2\phi_1) \sin \pi=0$. Hence

$$\text{area } (B) = \frac{R^2 + r^2}{4} \sum_{i=1}^{n} \sin (2\phi_i + 2\phi_{i+1}) + \frac{R^2}{2} \sum_{i=1}^{n} \sin 2\phi_i. \tag{5}$$

The last term is one-half of the area of (A), and the first term is easily interpreted as the sum of two areas. There are two cases according to the parity of n. This completes the proof of the necessary and sufficient condition.

It will be seen from the derivation that (5) is true when some of the ϕ_i's are negative and (A) is not convex. If $n=2$, (A) is an inscribed rectangle; and,

since $2\phi_1+2\phi_2=\pi$, the area of (B) is one-half of the area of (A) for all values of r. This is easily verified by inspection of a figure, and the result is trivial. If $n=3$, where $2\phi_1+2\phi_2+2\phi_3=\pi$,

$$\text{area } (B) = \tfrac{3}{4} \text{ area } (A) + \tfrac{1}{4} \text{ area } (A'),$$

where (A') is similar to (A) and inscribed in a circle of radius r. Thus, if M is on (O), the areas of (B) and (A) are equal; this special result is useful for a later problem.

We now consider the relative position of (B). Draw a tangent to (O) at the midpoint of arc A_iA_{i+1} cutting MB_i produced in B_i', and a tangent at the midpoint of arc $A_{n+i}A_{n+i+1}$ cutting MB_{n+i} produced in B_{n+i}'. Construct a circle on OM as a diameter, and let M_i be the other intersection of $B_{n+i}'B_i'$ with this circle. Then M_i is the common midpoint of $B_{n+i}B_i$ and $B_{n+i}'B_i'$. It is clear that this is the construction of a pair of points B_i' and B_{n+i}' on a limaçon with the circle on the diameter $OM=r$ as a basis and the moving line through M as a generator, where $M_iB_i'=R$ and $B_{n+i}'M_i=R$. Hence the vertices of (B) lie within this limaçon which is fixed by (O) and M. The expression for area (B) in (5) gives as a limit for convex polygons (A), $R^2\pi+(r^2/2)\pi$ as the area of the limaçon. If M is on (O), the limaçon becomes the cardioid.

In what follows the point M is on (O). We shall show that if (A) is a hexagon of the type here considered, convex or not, the consecutive sides of the hexagon (B) are perpendicular. Consider the three consecutive sides of hexagon (A) given by A_1, A_2, A_3, A_4 which determine the vertices B_1, B_2, B_3. We shall prove that B_2B_3 and B_1B_2 are perpendicular. In order to take care of all possible positions of M, A_2, A_3 on (O) when the points A_1 and A_4 are fixed as ends of a diameter, we shall use the notion of directed angles in Johnson's *Modern Geometry*, pages 11–15. Since M, B_1, B_2, A_2 are concyclic, we have $\measuredangle MA_2B_1=\measuredangle MB_2B_1$. From the concyclic set M, B_2, A_3, B_3 we have $\measuredangle B_3B_2M=\measuredangle B_3A_3M$. Also A_2B_1 cuts (O) in A_1, and therefore $\measuredangle MA_2B_1=\measuredangle MA_2A_1=\measuredangle MA_3A_1=\measuredangle MB_2B_1$. We now have $\measuredangle B_3B_2B_1=\measuredangle B_3B_2M+\measuredangle MB_2B_1=\measuredangle B_3A_3M+\measuredangle MA_3A_1=\measuredangle B_3A_3A_1=\pi/2$. Hence the hexagon (B) has its consecutive sides perpendicular; and obviously its opposite sides are perpendicular, and both statements are true whether hexagon (A) is convex or not.

It will now follow that in all cases of $2n$ vertices the opposite sides of (B) are perpendicular. For, the two opposite sides B_iB_{i+1}, $B_{n+i}B_{n+i+1}$ result from the six vertices A_i, A_{i+1}, A_{i+2}, A_{n+i}, A_{n+i+1}, A_{n+i+2}, and they determine in this order a hexagon whose corresponding B hexagon has for two opposite sides B_iB_{i+1}, $B_{n+i}B_{n+i+1}$, and as shown these are perpendicular.

If A_i and A_{i+2} fall upon A_{i+1}, then B_i and B_{i+1} coincide in a point B_{i+1}' on the cardioid, and B_{n+i} and B_{n+i+1} coincide in a point B_{n+i+1}' on the cardioid so that B_{i+1}', M, B_{n+i+1}' are collinear. We would then suspect that the straight lines B_iB_{i+1} and $B_{n+i}B_{n+i+1}$ approach tangents at B_{i+1}' and B_{n+i+1}'; and, if this is true, the two tangents must be perpendicular. We shall prove this. The circle (MA_{i+1}) on MA_{i+1} as diameter passes through B_i and B_{i+1}; and, as A_i and A_{i+2} approach

A_{i+1}, B_i and B_{i+1} approach a point B'_{i+1} on this circle and also on the cardioid, and the line of the chord B_iB_{i+1} approaches the tangent at B'_{i+1} to (MA_{i+1}). Let (MA_{i+1}) cut OA_{i+1} in C_{i+1}; then $MB'_{i+1}A_{i+1}C_{i+1}$ is a rectangle and (OM), the circle on OM as diameter which is the basis circle for the cardioid, passes through C_{i+1}. Hence C_{i+1} is the instantaneous center of rotation for $B'_{i+1}B'_{n+i+1}$, and the circle (MA_{i+1}) is tangent to the cardioid at B'_{i+1}. Hence the limit position of B_iB_{i+1} is the tangent to the cardioid at B'_{i+1}. The proof for the limit position of $B_{n+i}B_{n+i+1}$ is similar.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Under a grant from one of the educational foundations, a mathematics teaching seminar on a post-doctoral level is being conducted this year at Reed College, with Professor F. L. Griffin in charge. The four Fellows now participating are H. E. Goheen, L. Louise Johnson, Henry Scheffé, and R. A. Rosenbaum.

For the summer quarter, 1940, Stanford University has appointed Professor J. D. Tamarkin of Brown University and Professor Emil Artin of the University of Indiana. Professor Tamarkin will lecture on "Modern Theory of Integration" and "Selected Topics from Elementary Mathematics," while Professor Artin will give a course on "Algebraic Numbers."

GRANT FOR STUDY OF JUNIOR COLLEGES

The following announcement, by Professor W. C. Eells, may be of interest to many readers of the MONTHLY:

The American Association of Junior Colleges has received a grant of $25,000 from the General Education Board, of New York City, to finance a series of exploratory studies in the general field of terminal education in the junior college. Approximately 500 accredited junior colleges are now found in the United States besides another hundred which are not yet thus recognized.

About two-thirds of the 175,000 students enrolled in these institutions do not continue their formal education after leaving the junior college. The new study will be concerned particularly with courses and curricula of a semi-professional and cultural character designed to give this increasing body of young people greater economic competence and civic responsibility. There is increasing evidence that existing four-year colleges and universities are not organized adequately to meet the needs of a large part of this significant group.

It is anticipated that the exploratory study will reveal the need and the opportunity for a series of additional studies and experimental investigations and demonstrations which may cover several years of continuous effort.

The new study will include a large proportion of the junior colleges in the United States. It will be sponsored by a nation-wide representative committee, consisting of the following:

Doak S. Campbell, Dean of the Graduate School, Peabody College; George F. Zook, President, American Council on Education; J. C. Wright, Assistant United States Commissioner of Education for Vocational Education; Leonard V. Koos, Professor of Secondary Education, University of Chicago; Aubrey A. Douglass, Chief of the Division of Secondary Education, State Department of Education, Sacramento, Calif.; Guy M. Winslow, President, Lasell Junior College,

Auburndale, Mass.; Byron S. Hollinshead, President, Scranton-Keystone Junior College, La Plume, Pa.; Leland L. Medsker, Department of Occupational Research, Chicago Junior Colleges; J. E. Burk, President, Ward-Belmont Junior College, Nashville, Tenn.; David L. Soltau, President, Lower Columbia Junior College, Longview, Wash.; Rosco C. Ingalls, Director, Los Angeles City College, Calif.

Immediate responsibility for the study will be vested in an executive committee consisting of Rosco C. Ingalls, Chairman, Doak S. Campbell, and Bryon S. Hollinshead. The Director of the study will be Walter Crosby Eells, Executive Secretary of the American Association of Junior Colleges, Washington, D. C.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N.H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

- ALLEGHENY MOUNTAIN, Morgantown, W. Va., April 20.
- ILLINOIS, Bloomington, May 3–4.
- INDIANA, Richmond.
- IOWA, Mt. Vernon, April 19–20.
- KANSAS, Wichita, March 30.
- KENTUCKY
- LOUISIANA-MISSISSIPPI, Oxford, Miss., March 8–9.
- MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
- MICHIGAN, Ann Arbor, April 26–27.
- MINNESOTA
- MISSOURI, Warrensburg, April 19–20.
- NEBRASKA, Omaha.
- NORTHERN CALIFORNIA, Berkeley, January 27.
- OHIO, Columbus, April 4 or 6.
- OKLAHOMA
- PHILADELPHIA
- ROCKY MOUNTAIN, Fort Collins, Colo., April 19.
- SOUTHEASTERN, Athens, Ga., March 29–30.
- SOUTHERN CALIFORNIA, Compton, March 2.
- SOUTHWESTERN, Tucson, Arizona.
- TEXAS, Dallas, March 29–30.
- WISCONSIN, Milwaukee.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE TWENTY-FOURTH ANNUAL MEETING OF THE ASSOCIATION

The twenty-fourth annual meeting of the Mathematical Association of America was held at Columbus, Ohio, Thursday to Saturday, December 28–30, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the American Mathematical Society, and the National Council of Teachers of Mathematics. Four hundred fifty-three were in attendance at the meetings, including the following two hundred fifty-two members of the Association:

O. S. Adams, U. S. Coast and Geodetic Survey
J. B. Adkins, Phillips Exeter Academy
V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
A. A. Albert, University of Chicago
G. E. Albert, Ohio State University
C. B. Allendoerfer, Haverford College
W. E. Anderson, Miami University
Emil Artin, Indiana University
Max Astrachan, Antioch College
C. S. Atchison, Washington and Jefferson College
H. T. R. Aude, Colgate University
H. G. Ayre, Western Illinois State Teachers College
Frank Ayres, Jr., Dickinson College
W. L. Ayres, University of Michigan

R. W. Babcock, Kansas State College
F. R. Bamforth, Ohio State University
Grace M. Bareis, Ohio State University
I. A. Barnett, University of Cincinnati
H. M. Beatty, Ohio State University
E. F. Beckenbach, Rice Institute
Ethelwynn R. Beckwith, Milwaukee-Downer College
May M. Beenken, State Teachers College, Oshkosh, Wis.
R. F. Bell, York College
H. A. Bender, University of Akron
A. A. Bennett, Brown University
Brother Bernard Alfred, Manhattan College
S. F. Bibb, Armour Institute of Technology
G. D. Birkhoff, Harvard University
H. L. Black, Westminster College
Archie Blake, U. S. Coast and Geodetic Survey
Henry Blumberg, Ohio State University
L. M. Blumenthal, University of Missouri
R. P. Boas, Jr., Duke University
Paul Boeder, American Optical Company
Julia W. Bower, Connecticut College
M. G. Boyce, Western Reserve University
J. W. Bradshaw, University of Michigan
J. B. Brandeberry, University of Toledo
W. C. Brenke, University of Nebraska
Foster Brooks, Kent State University
O. E. Brown, Case School of Applied Science
C. T. Bumer, Kenyon College
R. S. Burington, Case School of Applied Science
Jewell Hughes Bushey, Hunter College
W. E. Byrne, Virginia Military Institute

W. D. Cairns, Oberlin College
V. B. Caris, Ohio State University
F. E. Carr, Oberlin College
W. B. Carver, Cornell University
E. W. Chittenden, University of Iowa
Laura E. Christman, Senn High School, Chicago
H. C. Christofferson, Miami University
R. V. Churchill, University of Michigan
E. H. Clarke, Hiram College
L. M. Coffin, Coe College
L. W. Cohen, University of Kentucky
J. B. Coleman, University of South Carolina
J. A. Cooley, University of Tennessee
A. H. Copeland, University of Michigan
W. A. Cordrey, Southeastern Louisiana College
Rufus Crane, Ohio Wesleyan University
H. B. Curtis, Lake Forest College
J. H. Curtiss, Cornell University

Wayne Dancer, University of Toledo
A. H. Diamond, Oklahoma A. and M. College
L. L. Dines, Carnegie Institute of Technology
H. S. DoBell, New York State College for Teachers
H. H. Downing, University of Kentucky
Olive M. Draper, Taylor University
Arnold Dresden, Swarthmore College
O. L. Dustheimer, Baldwin-Wallace College
P. S. Dwyer, University of Michigan

E. D. Eaves, University of Tennessee
P. D. Edwards, Ball State Teachers College

G. C. Evans, University of California
G. W. Evans, Swampscott, Mass.
H. P. Evans, University of Wisconsin

B. F. Finkel, Drury College.
F. A. Foraker, University of Pittsburgh
L. R. Ford, Armour Institute of Technology
W. B. Ford, University of Michigan
Orrin Frink, Jr., Pennsylvania State College
Thornton C. Fry, Bell Telephone Laboratories

Sister Mary Cleophas Garvin, Notre Dame College
B. E. Gatewood, Louisiana Polytechnic Institute
H. M. Gehman, University of Buffalo
J. S. Georges, Wright Junior College
F. J. Gerst, Loyola University
B. C. Getchell, Butler University
B. C. Glover, Otterbein College
L. M. Graves, University of Chicago
V. G. Grove, Michigan State College

D. W. Hall, Brown University
W. W. Hart, Winter Haven, Florida
M. C. Hartley, University of Illinois High School
M. L. Hartung, University of Chicago
G. G. Harvey, Massachusetts Institute of Technology
C. T. Hazard, Purdue University
Olive C. Hazlett, University of Illinois
E. R. Hedrick, University of California at Los Angeles
Gertrude Hendrix, Eastern Illinois State Teachers College
M. R. Hestenes, University of Chicago
H. C. Hicks, Carnegie Institute of Technology
Elizabeth J. Hines, Montgomery, West Virginia, High School
T. R. Hollcroft, Wells College
B. P. Hoover, Carnegie Institute of Technology
H. K. Hughes, Purdue University
P. M. Hummel, University of Alabama
E. V. Huntington, Harvard University
W. A. Hurwitz, Cornell University

Dunham Jackson, University of Minnesota
R. D. James, University of Saskatchewan
E. D. Jenkins, Eastern Kentucky State Teachers College
Fritz John, University of Kentucky
R. P. Johnson, Carnegie Institute of Technology
F. E. Johnston, George Washington University
L. S. Johnston, University of Detroit
B. W. Jones, Cornell University
Margaret E. Jones, Ohio State University
E. M. Justin, Case School of Applied Science

Wilfred Kaplan, College of William and Mary
L. C. Karpinski, University of Michigan
D. K. Kazarinoff, University of Michigan
M. W. Keller, Purdue University
J. R. Kelley, University of Virginia
J. R. Kline, University of Pennsylvania
L. C. Knight, College of Wooster
L. A. Knowler, University of Iowa
H. W. Kuhn, Ohio State University

A. C. Ladner, Denison University
W. D. Lambert, U. S. Coast and Geodetic Survey
R. E. Langer, University of Wisconsin
Lincoln La Paz, Ohio State University
C. G. Latimer, University of Kentucky
Nathan Lazar, Bronx High School of Science
Solomon Lefschetz, Princeton University
D. H. Lehmer, Lehigh University
A. J. Lewis, University of Denver
G. H. Ling, University of Saskatchewan
Marie Litzinger, Mount Holyoke College
Mayme I. Logsdon, University of Chicago
C. I. Lubin, University of Cincinnati

R. H. MacCullough, Defiance College
C. C. MacDuffee, University of Wisconsin
Saunders Mac Lane, Harvard University
H. M. MacNeille, Kenyon College
P. H. McGrath, St. Peter's College
J. D. Mancill, University of Alabama
E. S. Manson, Ohio State University
Ruth G. Mason, Wright Junior College
W. L. Massey, University of Chattanooga
Karl Menger, University of Notre Dame
D. D. Miller, Ohio University
Gasperine Milo, New River State College
Virginia Modesitt, Wright Junior College
Max Morris, Case School of Applied Science
Richard Morris, Rutgers University
Sister Charles Mary Morrison, Nazareth College

MARSTON MORSE, Institute for Advanced Study
E. J. MOULTON, Northwestern University
J. R. MUSSELMAN, Western Reserve University
JACK NEELY, Kingston, West Virginia, High School

E. B. OGDEN, Union College, Lincoln, Nebraska
RUFUS OLDENBURGER, Armour Institute of Technology
E. G. OLDS, Carnegie Institute of Technology
EMMA J. OLSON, Kent State University
A. L. O'TOOLE, Mundelein College
E. R. OTT, University of Buffalo

W. V. PARKER, Louisiana State University
W. A. PATTERSON, Fenn College
O. J. PETERSON, State Teachers College, Emporia, Kans.
JESSE PIERCE, Heidelberg University
A. E. PITCHER, Lehigh University
G. B. PRICE, University of Kansas
D. W. PUGSLEY, Berea College
H. R. PYLE, Earlham College

TIBOR RADÓ, Ohio State University
J. F. RANDOLPH, Cornell University
S. E. RASOR, Ohio State University
F. W. REED, Ohio University
R. G. D. RICHARDSON, Brown University
C. C. RICHTMEYER, Central State Teachers College
P. R. RIDER, Washington University
R. F. RINEHART, Case School of Applied Science
FRED ROBERTSON, Iowa State College
W. H. ROEVER, Washington University
H. P. ROGERS, Kent State University
J. B. ROSENBACH, Carnegie Institute of Technology
S. A. ROWLAND, Ohio Wesleyan University

RUTH M. ST. CLAIR, Montgomery, West Virginia, High School
C. C. SAMS, Mars Hill College
M. A. SCHEIER, St. Bonaventure College
I. J. SCHOENBERG, Colby College
H. L. SCHUG, The Hoover Company
JOSEPH SEIDLIN, Alfred University
R. S. SHAW, College of the City of New York
L. S. SHIVELY, Ball State Teachers College
C. GRACE SHOVER, Carleton College
D. R. SHREVE, Purdue University

W. O. SHRINER, Indiana State Teachers College
D. T. SIGLEY, Kansas State College
MARY EMILY SINCLAIR, Oberlin College
S. A. SINGER, Capital University
E. R. SLEIGHT, Albion College
M. M. SLOTNICK, Humble Oil and Refining Company
E. R. SMITH, Iowa State College
F. C. SMITH, College of St. Francis
H. W. SMITH, Oklahoma A. and M. College
J. P. SMITH, Georgetown University
W. F. SMITH, New River State College
VIRGIL SNYDER, Cornell University
C. E. SPRINGER, University of Oklahoma
G. W. STARCHER, Ohio University
H. E. STELSON, Kent State University
RUTH W. STOKES, Winthrop College
W. T. STRATTON, Kansas State College
ELIZABETH C. STRAYHORN, Western Kentucky State Teachers College
Sister M. CLOTILDA SULLIVAN, Mercyhurst College
OTTO SZÁSZ, University of Cincinnati

E. H. TAYLOR, Eastern Illinois State Teachers College
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
H. P. THIELMAN, College of St. Thomas
J. M. THOMAS, Duke University
C. C. TORRANCE, Case School of Applied Science
H. C. TRIMBLE, University of Chicago
P. L. TRUMP, University of Wisconsin
A. W. TUCKER, Princeton University

F. W. URBAN, Central Missouri State Teachers College

J. I. VASS, University of Wisconsin Extension Division

A. D. WALLACE, University of Virginia
J. L. WALSH, Harvard University
J. H. WEAVER, Ohio State University
WARREN WEAVER, Rockefeller Foundation
M. S. WEBSTER, Purdue University
MARIE J. WEISS, Sophie Newcomb College
A. M. WELCHONS, Arsenal Tech. Schools
M. E. WESCOTT, Northwestern University
A. E. WHITE, Kansas State College
G. T. WHYBURN, University of Virginia
R. L. WILDER, University of Michigan

R. B. WILDERMUTH, Capital University
F. B. WILEY, Denison University
C. W. WILLIAMS, Armstrong Junior College
K. P. WILLIAMS, Indiana University
MARY E. WILLIAMS, Ashland, Kentucky, High School
F. L. WREN, George Peabody College
C. R. WYLIE, JR., Ohio State University
MABEL M. YOUNG, Wellesley College
R. T. ZOCH, U. S. Weather Bureau
MAX ZORN, University of California at Los Angeles

The meeting of the American Association for the Advancement of Science, including a great wealth of scientific programs, was begun officially Wednesday evening with the presidential retiring address by Dr. Wesley C. Mitchell on "The public relations of science"; this was followed by a public reception at the Deshler-Wallick Hotel. Other outstanding lectures of the week were the Sigma Xi Lecture by Dr. Kirtley F. Mather on "The future of man as an inhabitant of the earth," the fifth annual Phi Beta Kappa Lecture by Dr. Marjorie Hope Nicolson on "Science and literature," and an address by Dr. Julian S. Huxley of London, England, on "Science, war, and reconstruction" as the first address in America under the exchange arrangement between the American and British Associations. A large and interesting science exhibit was located for the week in the Civic Auditorium. An unusually pleasant social feature was the Yuletide Tea at the Governor's Mansion on Thursday afternoon. Many visited the Columbus Gallery of Fine Arts during the week. On Saturday morning Dr. A. F. Blakeslee of the Carnegie Institution of Washington was elected president of the Association for the year 1940. Professor A. B. Coble of the University of Illinois was elected vice-president and chairman of Section A for the year 1940, and Professor J. L. Walsh of Harvard University member of the Section Committee.

Those attending the mathematics meetings found suitable accommodations in Neil Hall, the Coöperative House, and in private homes near the campus of Ohio State University. The lobby and social rooms of Neil Hall afforded ample opportunity for social gatherings throughout the week. Tea was served by the ladies of the mathematics department on Tuesday, Thursday, and Friday afternoons. This afforded a pleasant relaxation in an otherwise strenuous week. A resolution expressing the thanks of the mathematicians to the administrative staff of Ohio State University for their generosity in furnishing fine facilities for the meetings, to the members of the Department of Mathematics for their effective aid in the arrangements for the meetings, and to the ladies of the Department of Mathematics for their highly successful contributions to the entertainment of the mathematicians, was offered by Dean G. H. Ling at the joint dinner and was adopted unanimously. Special praise is due to Professor S. E. Rasor and his fellow members on the committee on local arrangements.

Two hundred eighty-six attended the annual dinner of the three mathematical organizations on Thursday evening. This was held in the Faculty Club of the University with Professor Tibor Radó acting as toastmaster. Dean Stradley brought greetings to the mathematicians on behalf of the Ohio State University.

President Christofferson representing the National Council of Teachers of Mathematics told of the educational difficulties encountered in dealing with the greatly increased number of pupils in the schools and colleges of the United States. Professor Langer described the proposal under consideration to enlist the coöperation of all teachers of freshman and sophomore college mathematics. Professor Birkhoff gave an historical account of the establishment of the William Lowell Putnam Competition, and Professor Thomas described the present state of mathematical periodicals in America.

The American Mathematical Society held sessions for the reading of papers on Tuesday, Wednesday and Thursday afternoons, and Wednesday and Friday mornings. The fifteenth Josiah Willard Gibbs Lecture was delivered Wednesday afternoon on "The engineer grappling with non-linear problems" by Professor Theodore von Kármán. At the business meeting on Thursday morning the Frank Nelson Cole Prize in Algebra was awarded to Dr. A. A. Albert of the University of Chicago. Thursday afternoon Professor D. H. Lehmer spoke on "The application of Bernoulli polynomials to some problems in Diophantine analysis" by invitation of the program committee.

Aside from the joint session on Friday afternoon, the National Council held two sessions on Friday morning using the topics "Training teachers for relational thinking" with addresses by Dr. J. I. Johnson, Professor C. C. Richtmeyer, Dr. H. G. Ayre, and Dr. J. S. Georges; and "Relational thinking in secondary mathematics as viewed by the college teacher" with addresses by Professors F. L. Wren and A. A. Bennett, followed by discussion led by Charles Weidemann, Professor J. R. Overman, and Professor H. C. Christofferson. On Friday afternoon another session considered the topic "Teaching children to do relational thinking," the speakers being C. L. Thiele, Dale Wantling, and Marie S. Wilcox, while the discussion was led by Dorothy Wheeler, F. N. Marsh, A. Brown Miller, and Alma Wuest. These programs, together with that of the joint session with the Mathematical Association Friday afternoon, will be reported in the *Mathematics Teacher*.

The Mathematical Association held a joint session with Section A and the Society Thursday morning, a joint session with Sections A and E and the Society on Friday morning, and a joint session with the National Council Friday afternoon, and separate sessions Friday afternoon and Saturday morning. The Mathematical Association is greatly indebted to the program committee under the chairmanship of Professor R. P. Agnew for the preparation of a strong program. This follows, together with abstracts of some of the papers numbered in accordance with their place on the program:

Joint Session of the Association with Section A of the American Association and the American Mathematical Society

At this session Professor J. R. Kline gave his retiring vice-presidential address as Chairman of Section A on "The Jordan curve theorem." This address probably will be expanded and appear in monograph form.

Joint Session of the Association with Sections A and E of the American Association and the American Mathematical Society

1. "The beginnings of mathematical geophysics in Great Britain" by W. D. Lambert, U. S. Coast and Geodetic Survey.

2. "Use of mathematics in the delineation of magnetic and electric anomalies" by Professor Lachlan Gilchrist, University of Toronto.

3. "Gravimetric and seismic methods in exploratory geophysics" by Dr. M. M. Slotnick, Humble Oil and Refining Company.

4. "Mathematical problems in seismology" by Dr. Archie Blake, U. S. Coast and Geodetic Survey.

5. "A seismologist's difficulties with mathematical theory or the lack of it" by Professor Perry Byerly, University of California, read by Professor W. D. Cairns.

This session was a happy consequence of the joint session with Sections A and E a year ago at the Richmond meeting, and has furthered a very desirable rapprochement between geophysicists and mathematicians. Numerous comments were made on the strength of the papers, and it is hoped that arrangements will be made to publish these five papers as a symposium. Meetings of this sort cannot help being of great potential value by bringing the efforts of these two groups into a common pursuit of many outstanding problems of geophysics. To Professor H. A. Meyerhoff of Smith College is due the chief credit for the formation of this interesting program.

Joint Session of the Association with the National Council of Teachers of Mathematics

1. "The social challenge of teacher education" by Dr. Karl Bigelow, American Council on Education.

2. Discussion of implications of Dr. Bigelow's address:

a. From the viewpoint of the liberal arts colleges, by Professor B. W. Jones, Cornell University.

b. From the viewpoint of the state universities, by Professor H. H. Downing, University of Kentucky.

c. From the viewpoint of the teachers colleges, by Professor W. O. Shriner, Indiana State Teachers College.

d. From the viewpoint of the Association and Council, by Professor E. G. Olds, Carnegie Institute of Technology.

1. Dr. Bigelow gave a very general treatment of educational principles involved in teacher education, with no specific reference to the problems that face teachers of mathematics.

2, a. Professor Jones remarked that consideration of mathematical training in its place in the college course, together with the fact that it is not sufficient that a prospective teacher be prepared to teach mathematics alone, limited the requirements in mathematics to about five courses. This forced a reconsideration of the effectiveness of the present program. Secondly, individual differ-

ences should be provided for by making especially sure that the weaker students have a firm understanding of the arithmetic and algebra they are to teach. Finally, the impending expansion of requirements to a fifth year of collegiate training gives an opportunity for some preparation for a kind of mathematical investigation which the teacher can continue to pursue throughout his career.

2, b. Professor Downing called attention to the fact that state universities are not concerned alone with training prospective teachers, but must consider the needs of students preparing for life work in various fields. The university professor must, therefore, present his subject and conduct himself in such a manner that he will appeal to all types of students. The most effective way to do the job, it seems to this speaker, is for the professor himself to be thoroughly interested in his subject, to be interested in his students, and to have a good knowledge of his subject. No attempt is made to say which of these is the most important feature. It is believed that such a teacher will inspire the student to acquire an interest in the subject whatever may be his special interests.

Professor Downing also pointed out that there are many ways of trying to get the professor to do a good job. At one state university, at least, the arts and science college has had for a number of years a Committee on the Improvement of College Teaching. This committee has been studying the numerous problems involved and makes recommendations to the college. This same university finds some of its departments holding frequent meetings, during some of which are discussed questions relating to teaching with special consideration of the young instructor.

The speaker also emphasized the social phase of the teacher problem as implied in Dr. Bigelow's remarks. He gave several examples of attempts to meet the student personally on a common level, and thus to create in the student a desire and willingness to enjoy his course and to do creditable work in it. What appeals to one individual will not appeal to another, and the professor must have a sufficient knowledge of human nature to know how to meet the different students. Some professors tell too many jokes and relate too many personal experiences. Some do not tell enough jokes to break the monotony of the subject. A good teacher must truly be a gifted person.

2, d. Professor Olds distributed a mimeographed list of recent articles which he had consulted in order to get the viewpoint of the Association and the Council on the education of mathematics teachers. He expressed the hope that the audience would read these articles for the purpose of gauging the fairness of his appraisal.

Referring to the forthcoming Final Report of the Joint Commission on the Place of Mathematics in Secondary Education, he pointed out that "social and civic attributes" was listed as one of the "five major qualities that are to be considered in the mathematics teacher." He mentioned that the reports of the previous committees of 1923 and 1935 had not neglected to emphasize the importance of this phase of training, and he expressed the opinion that mathematics teachers exhibit intelligent interest in the problems of society and have a good reputation for citizenship and general culture.

Remarking that "general remarks on general education are inclined to be banal," he suggested that the three questions of paramount importance to both pre-service and in-service teachers are: (1) How much mathematics? (2) What mathematics? (3) How much Education? With respect to the first question, the committees of 1923 and 1935 have provided a generally satisfactory answer. With respect to the second, it is recognized that the prospective teacher needs broad general training in those zones which are the natural extension of the areas in which he expects to teach. Courses in history and in mathematical (not Educational) statistics should be included. Some of the suggestions regarding the type of mathematics desirable have been questioned because of lessened emphasis on physics, and of the recommendation of mathematics related to the social sciences. It was remarked that many of the applications provided by physics are outside the experience of the high school pupil, while the latter field seems more fruitful. The overburdened elementary teacher has little time to ferret out applications and, unless he has readily available sources, applications will probably go by the board. If some one of our authors would restrain his urge to write another elementary text and would direct his energy to the compilation of a dictionary of elementary applications, he would be doing a lasting service to the cause of mathematics.

As far as professional education courses are concerned, the average mathematician would probably be content to restrict requirements to a course in general psychology. Such courses as the teaching of secondary mathematics should be given by a mathematician. It is gratifying to hear from Dr. Bigelow that mathematicians and educationists are becoming more willing to coöperate, because, in the speaker's opinion, the educationist has much to contribute. Mutual mistrust is due perhaps to a time lag of ten to twenty years since the average mathematician has looked closely at education courses or the educationist has intelligently appraised the work in mathematics.

In closing, the speaker expressed the opinion that the members of the audience, being "toilers in the vineyard," knew more than he did about the best preparation for the teaching of elementary mathematics and that they therefore, in the light of their wide experience, should vocally evaluate the suggestions made.

First Separate Session of the Association

1. "Mathematical Reviews" by Dr. T. C. Fry, Bell Telephone Laboratories, and Professor W. B. Carver, Cornell University.

2. "On the technique of generalization" by Professor Henry Blumberg, Ohio State University.

1. Dr. Fry described the origin and development of plans for *Mathematical Reviews*. He spoke of the great value of the new journal for teachers of mathematics who are not primarily research mathematicians, saying that the American Mathematical Society is not indifferent to matters of mathematics outside of research, and the Mathematical Association of America cannot afford to be indifferent to matters of research. It is not the time to ask, "Is it the business

of the Mathematical Association of America to give a subsidy to *Mathematical Reviews* with the advantage that will come to its members through a reduction in the subscription price?" We should rather ask, "Is there any possible way in which the Mathematical Association can put its strong emphasis behind this new and valuable project?"

Professor Carver said that the establishment of *Mathematical Reviews* is a matter of the very greatest possible interest not merely to research workers but also to mathematics teachers who are not strongly interested in research. One vital function of the Mathematical Association activities is the improvement of mathematical teaching at the collegiate level, including what is commonly called the lower and upper college; we college teachers must keep in touch with current developments, and the new journal with its wide scope and low cost will serve this purpose admirably. In addition to this, the microfilm and photoprint service, as explained in the advertising material and in the initial number of *Mathematical Reviews*, will give the complete text of any article of which an abstract is printed, except books or copyrighted material, at a very low cost.

2. Although generalization is an activity of paramount importance for mathematicians, ways and means of securing generalizations have never been systematically considered in mathematical literature. Reasons for this lacuna are discussed. The technique of generalization relates to concepts, problems, theorems, and processes. Various principles of this technique are formulated and exemplified. In the matter of technique or strategy, it is important to bear in mind that the mere knowledge of a principle is of slight value; what counts is the degree to which the principle becomes consciously and instinctively operative. In conclusion, the bearing of mathematical strategy on other fields is considered, and *vice versa*. The address will be published in the MONTHLY.

SECOND SEPARATE SESSION OF THE ASSOCIATION

1. Annual business meeting and election of officers.

2. "Modular fields" by Professor SAUNDERS MAC LANE, Harvard University.

3. "Interior transformations" by Professor G. T. WHYBURN, University of Virginia.

4. "Space analogs of function-theoretic results" by Dr. E. F. BECKENBACH, The Rice Institute.

2. Professor Mac Lane's paper will appear in an early issue of the MONTHLY.

3. Professor Whyburn first gave an historical background, including a sketch of the work of Stoïlow who showed that the topological properties of analytic functions are identical with those of a light interior transformation from a special sort of two-dimensional manifold to a sphere. This was followed by a summary of some of the main results established to date concerning the general interior transformation from one compact metric set to another. First it was indicated how, by a certain factorization of the transformation, reduction in many cases could be made to the consideration of a monotone transformation

followed by a light interior one. Next the questions of existence, analysis, and invariance of properties were considered for the cases where the original set is a one- or two-dimensional compact manifold. The complete solutions to the questions in the one-dimensional case were indicated; and some of the major theorems on the two-dimensional case were quoted, notably the theorem involving the invariance of the two-dimensional-manifold property, the theorem analyzing the action of the transformation "in the small," and the theorem giving a simple numerical relationship between the Euler characteristics of the original and image surfaces which must hold in order for the transformation to exist.

4. This paper will appear in an early issue of the MONTHLY.

MEETING OF THE BOARD OF TRUSTEES

Eight members of the outgoing Board were present at the Columbus meeting on Friday evening.

The following thirty-one persons and two institutions were elected to membership on applications duly certified:

To Institutional Membership

EASTERN ILLINOIS STATE TEACHERS COLLEGE, Charleston, Illinois

NEW MEXICO COLLEGE OF AGRICULTURE AND MECHANIC ARTS, State College, New Mexico

To Individual Membership

C. B. ALLENDOERFER, Ph.D.(Princeton) Asst. Prof., Haverford Coll., Haverford, Pa.

H. A. ARNOLD, Ph.D.(Calif. Inst. of Tech.) Instr., Univ. of Minnesota, Minneapolis, Minn.

T. A. BANCROFT, A.M.(Michigan) Asst. Prof., Mercer Univ., Macon, Ga.

R. A. BAUMGARTNER, A.M.(Illinois) Instr., North Dakota Agric. Coll., Fargo, N. D.

C. V. BERTSCH, Ph.D.(Michigan) Asst. Prof., Southwestern Coll., Winfield, Kan.

R. G. BLAKE, A.B.(Florida) Teacher, Jr. High School, Brooksville, Fla.

B. E. BLAKEMAN, A.M.(Illinois) Instr., Albert Lea Jr. Coll., Albert Lea, Minn.

L. E. BOYER, Ed.D.(Pennsylvania State Coll.) Prof., State Teachers Coll., Millersville, Pa.

J. E. DARRAUGH, A.B. (Brooklyn Coll.) Clerk, Consol. Edison Co., New York, N. Y.

C. D. FIRESTONE. Student, class of '41, Univ. of New Mexico, Albuquerque, N. M.

EDWARD FISHER, B.S.(Mass. Inst. of Tech.) Real Estate, Boston, Mass.

B. E. GATEWOOD, Ph.D.(Wisconsin) Asst. Prof., Louisiana Poly. Inst., Ruston, La.

J. R. GORMAN, A.M.(U.C.L.A.) Instr., Compton Jr. Coll., Compton, Calif.

R. E. GREENWOOD, JR., Ph.D.(Princeton) Instr., Univ. of Texas, Austin, Texas

MARGARET H. HALL, A.B.(New River State Coll.) Teacher, Jr. High School, Montgomery, W. Va.

ELIZABETH J. HINES, A.M.(Kentucky) Teacher, High School, Montgomery, W. Va.

H. B. JACKSON, A.B.(Harvard) Head of Dept., Belmont Hill School, Belmont, Mass.

L. G. JOHNSON. Student, class of '40, Northern State Teachers Coll., Marquette, Mich.

B. C. KEELER, A.M.(Columbia) Prof., Webb Inst. of Naval Architecture, New York, N. Y.

BLAIR KINSMAN, B.S.(Chicago) Math. Master, Emerson School for Boys, Exeter, N. H.

C. J. KIRCHEN, A.M.(Wisconsin) Instr., Itasca Jr. Coll., Coleraine, Minn.

Z. L. LOFLIN, M.S.(Louisiana) Asst., Louisiana State Univ., Baton Rouge, La.

Rev. P. H. MCGRATH, A.M.(Woodstock Coll.) Asso. Prof., St. Peter's Coll., Jersey City, N. J.

IVAN NIVEN, Ph.D.(Chicago) Instr., Univ. of Illinois, Urbana, Ill.

G. B. OAKLAND, A.M.(Minnesota) Teacher, Scott Collegiate Inst., Regina, Sask.

J. C. POLLEY, Ph.D.(Cornell) Prof., Wabash Coll., Crawfordsville, Ind.
RUTH M. ST. CLAIR, A.B.(New River State Coll.) Teacher, High School, Montgomery, W. Va.
C. C. SAMS, A.M.(Michigan) Instr., Mars Hill Coll., Mars Hill, N. C.
G. J. STIGLER, Ph.D.(Chicago) Asst. Prof., Econ., Univ. of Minnesota, Minneapolis, Minn.
H. G. SWAIN, A.M.(Lehigh) Asso. Prof., Concord State Teachers Coll., Athens, W. Va.
GILBERT ULMER, Ph.D.(Kansas) Asst. Prof., Educ. and Math., Univ. of Kansas, Lawrence, Kan.

The financial report of the Secretary-Treasurer for the year 1939 was presented and accepted. It had been previously examined by President Carver for the Finance Committee and Professor Langer.

It was voted that the $750 which was given to C. C. Carter be paid from the general treasury and not from the Houck Fund.

It was voted to accept with thanks the invitation of Dartmouth College to meet there during the week of September 9, 1940, in conjunction with the summer meeting of the American Mathematical Society.

At an informal meeting Thursday afternoon of the Trustees and members of the Committee on Association Activities, Professor Langer as chairman had presented the full report of the committee; this was followed by an extended discussion. At the meeting Friday evening the Trustees voted to accept the report, to discharge the committee, and to have a suitable memorandum prepared expressing their appreciation of the committee's work. The report appeared in the February issue of the MONTHLY.

In connection with this report it was voted (1) to appoint a committee to study further the desirability of publishing expository monographs, intermediate in length between articles in the MONTHLY and the Carus Monographs; (2) to appoint a standing Committee on Section Meetings to bring about closer coördination and mutual relations between the Sections and the parent organization; and (3) to appoint a standing Committee on Collegiate Curricula which shall collate and publish information as to changes in curricula, major requirements in colleges and graduate schools, *etc.*

ANNUAL BUSINESS MEETING

The annual business meeting and election of officers was held Saturday morning, December 30, 1939. The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He also reported the deaths of the following members:

H. T. BURGESS, Milford, Connecticut. (August 16, 1939)
N. C. GRIMES, Professor of mathematics, Grove City College. (November 12, 1938)
S. C. HARRY, Head of department of mathematics, Baltimore City College. (September 19, 1939)
A. E. KENNELLY, Professor emeritus of mathematics, Harvard University. (June 18, 1939)
W. H. LYONS, Associate professor of mathematics, Kansas State College. (October 21, 1939)
T. E. MASON, Professor of mathematics, Purdue University. (May 26, 1939)
W. C. RISSELMAN, Professor of mathematics, Northern Arizona Teachers College. (June 26, 1939)
O. D. TYNER, Instructor in mathematics, Chicago Technical College. (October 18, 1939)

The amendments to the By-Laws, circulated to the membership in November, were adopted.

The results of the election of officers were as follows:

Vice-Presidents for 1940: R. W. BRINK, University of Minnesota; W. C. GRAUSTEIN, Harvard University.

Additional members of the Board of Governors (under the new By-Laws), to serve until January 1943: PHILIP FRANKLIN, Massachusetts Institute of Technology; F. L. GRIFFIN, Reed College; MAYME I. LOGSDON, University of Chicago; G. T. WHYBURN, University of Virginia.

MEETING OF THE INCOMING BOARD OF GOVERNORS

Six members of the Board met for a brief meeting Saturday noon and voted to appoint the following associate editors of the MONTHLY for the year 1940, as nominated by Professor Moulton:

W. B. Carver	M. R. Hestenes	D. E. Smith
H. S. M. Coxeter	E. H. C. Hildebrandt	Virgil Snyder
Otto Dunkel	C. A. Hutchinson	R. J. Walker
B. F. Finkel	J. R. Musselman	Marie J. Weiss
Orrin Frink, Jr.	R. G. Sanger	M. E. Wescott

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 12, 1939

RECEIPTS

Balance Dec. 9, 1938		$6,293.04
1938 indiv. dues	$ 396.37	
1938 subscriptions	12.60	
1939 indiv. dues	7,473.69	
1939 instit. dues	613.90	
1939 subscriptions	1,034.11	
Initiation fees	176.00	
Advertising	561.50	
Reprints	244.82	
Sale copies of MONTHLY	176.66	
Sale First Carus Mon.	16.25	
Sale Second Carus Mon.	12.50	
Sale Third Carus Mon.	23.75	
Sale Fourth Carus Mon.	20.00	
Sale Fifth Carus Mon.	38.43	
Sale Archibald's Outline of Hist. of Math.	213.71	
Annals subscriptions	7.50	
Duke Journal subscriptions	6.00	
Math. Reviews subscription	6.50	
Sale Rhind Papyrus	174.32	
Life membership fees	70.64	
Drury Coll. int. Hardy Fund	120.00	

EXPENDITURES

Publisher's bills (Oct.'38–Sept.'39)		$ 5,629.77
Reprints		264.05
President's office		8.54
Editor-in-chief's office		586.14
Register expense		277.01
Committee on tests		61.75
Committee on membership		81.33
Secretary-Treasurer's office		
Postage	$ 483.14	
Bond	11.26	
Office expense	159.78	
Express, tel., *etc.*	83.00	
Clerical work	2,277.25	
Printing	330.62	
Bank charge	29.32	3,374.37
Annals subvention		200.00
Duke Journal subvention		150.00
Expense of sections from init. fees		171.69
Virginia meetings		118.96
Madison meeting		55.00
Paid *Annals* subscriptions		15.00
Forwarded *Annals* subscription		2.50
Forwarded *Duke Journal* subscriptions		6.00
Sust. memb. in Amer. Math. Soc.		100.00

Receipts (continued)			Expenditures (continued)	
Contrib. to cost of printing Outline	75.00		Refund subscriptions	8.55
Sale Western United Bonds	2,606.25		Insurance back copies MONTHLY	6.40
Int. Genl. End. Fund	663.72		Storage back copies MONTHLY	30.00
Int. Carus Fund	131.25		Paid back copies MONTHLY	18.40
Int. Chace Fund	210.32		Paid B. F. Finkel int. Hardy Fund	120.00
Int. Chauvenet Fund	15.00		Library expense	71.77
Int. Houck Fund	127.50		Transfer to Carus Fund	207.83
Int. current funds	147.01		Transfer to Chace Fund	191.20
Payment from restricted Carus Fund	49.70		Expense Carus Fund	61.87
Payment from restricted Chace Fund	2.20	15,427.20	Printing Archibald's Outline 4th edition	286.67
Total 1939 receipts to date		$21,720.24	Paid to Houck Fund	750.00
			Legal advice, Houck Estate	10.00
			Award Chauvenet Prize	100.00
			Expense 1940 Congress	40.00
			Assn. for Symbolic Logic	10.00
			Paid on Shawinigan Bonds	610.16
			Paid for Montana Power and North American Bonds	2,101.11
Total expenditures		15,726.07	Total expenditures	$15,726.07
Balance to end of 1939 business		$ 5,994.17	Checking account	199.62
Received on 1940 business		868.48	Oberlin Savgs. Bk. acct. restricted	749.70
			Peoples Banking Co. savgs. acct.	1,764.15
			Cleveland Trust Co. savgs. acct.	2,149.18
			Youngstown Bonds	2,000.00
Book balance Dec. 12, 1939		$ 6,862.65	Bank balance Dec. 12, 1939	$ 6,862.65

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance December 9, 1938		$6,946.03
Receipts: Sales	$ 110.93	
Interest	186.45	297.38
		$7,243.41
Expense acct. Carus Fund		61.87
		$7,181.54
Certificate of deposit	$2,354.23	
C. & O. $3\frac{1}{2}\%$ Refunding Mortgage Bonds Series D, 1996	2,000.00	
U. S. Treasury $3\frac{1}{8}\%$ Bond of 1946–49	1,000.00	
HOLC 3% Bond 1944–52	1,000.00	
U. S. Savings Bonds	150.00	
Cash in bank, restricted, certificate of participation	447.30	
Cash in bank, unrestricted	230.01	
Balance December 12, 1939		$7,181.54

Arnold Buffum Chace Fund

Balance December 9, 1938		$7,784.36
Receipts: Sale Papyrus	$ 174.32	
Interest	227.22	401.54
		$8,185.90
Expenditures: Expense in purchase of Shawinigan, Montana Power and North American Bonds		125.45
		$8,060.45
U. S. Treasury $3\frac{1}{8}\%$ Bonds 1946–49	$2,000.00	
HOLC 3% Bond 1944–52	1,300.00	
U. S. Savings Bonds	1,125.00	
Montana Power Co. $3\frac{3}{4}\%$ First Mort. Bonds 1966	1,000.00	
North American Co. 4% Debenture Bond 1959	1,000.00	
One half Shawinigan W. & P. Co. $4\frac{1}{2}\%$ Bond	500.00	
Certificate of deposit	785.51	
Cash in bank, restricted, certificate of participation	19.80	
Cash in bank, unrestricted	330.14	
Balance December 12, 1939		$8,060.45

Jacob Houck Memorial Fund

Receipts: From executor of estate	$7,580.53	
Interest	175.53	
Profit in purchase of Gatineau Bond	17.50	$7,773.56
Expenditures: Expense in purchase of Gatineau and Shawinigan Bonds	77.46	
Legal advice	10.00	87.46
		$7,686.10
U. S. Treasury Bonds	$4,000.00	
$1\frac{1}{2}$ Shawinigan W. & P. Co. $4\frac{1}{2}\%$ Bonds	1,500.00	
Gatineau Power Co. $3\frac{3}{4}\%$ Bond	1,000.00	
Certificate of deposit	404.42	
Certificate of deposit	750.00	
Cash in bank	31.68	
Balance December 12, 1939		$7,686.10

Chauvenet Prize Fund

Balance December 9, 1938		$ 687.94
Interest		15.00
		$ 702.94
Award Chauvenet Prize December 31, 1938		100.00
		$ 602.94
HOLC 3% Bond 1944–52	$ 500.00	
Cash in bank	102.94	
Balance December 12, 1939		$ 602.94

LIFE MEMBERSHIP FUND

Liability on life memberships as of January 1, 1939	$ 782.77
Life membership payment	70.64
	$ 853.41
Liability on life memberships as of January 1, 1940, on new $3\frac{1}{2}\%$ basis	$ 892.10

GENERAL ENDOWMENT FUND

Balance December 9, 1938		$18,200.00
U. S. Treasury $3\frac{1}{4}\%$ Bonds 1944–46	$1,000.00	
U. S. Treasury $3\frac{1}{4}\%$ Bonds 1943–45	1,000.00	
HOLC 3% Bonds 1944–52	5,500.00	
Land Trust Certificate, Hotel Cleveland Site	700.00	
Montana Power Co. $3\frac{3}{4}\%$ First Mortgage Bonds	2,000.00	
Texas Power & Light Co. 5% First Morgtage Bond 1956	1,000.00	
C. & O. $3\frac{1}{2}\%$ Refunding Mortgage Bond Series C 1996	1,000.00	
Pennsylvania R. R. Co. $3\frac{3}{4}\%$ Bonds Series C 1970	2,000.00	
Bethlehem Steel Co. $3\frac{3}{4}\%$ Consol. Mortgage Bonds 1966	2,000.00	
Oberlin Savings Bank Savings Account	2.000.00	
Balance December 12, 1939		$18,200.00

Of the funds on hand, indicated in the first division of this financial report, $230.01 belongs to the Carus Monograph Fund, $330.14 to the Arnold Buffum Chace Fund, $31.68 to the Jacob Houck Memorial Fund, $102.94 to the Chauvenet Prize Fund, while $892.10 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date of January 1, 1940.

When the accounts were closed December 12, 1939, there remained on the total business for 1939 the following items:

BILLS RECEIVABLE

1939 individual dues	$200.00
Advertising	120.00
	$320.00

BILLS PAYABLE

Publisher's bills (Oct.–Dec. '39)	$1,800.00
Printing *Register*	525.00
Editor-in-chief's office	100.00
Committee on membership	100.00
Secretary-Treasurer's office	300.00
Subsidy *Duke Journal*	100.00
Carus Monograph Fund	230.01
Chace Fund	330.14
Houck Fund	31.68
Chauvenet Prize Fund	102.94
Life membership fund	892.10
Init. fees due to sections	880.00
	$5,391.87

If to the balance on 1939 business shown in this report, $5,994.17, there be added the estimated bills receivable, $320.00, and there be subtracted the estimated bills payable, $5,391.87, there results an estimated final balance on 1939 business of approximately $920, as compared with approximately the same amount one year ago.

W. D. CAIRNS, *Secretary-Treasurer*

THE FOURTEENTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The fourteenth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Lehigh University, Bethlehem, Pennsylvania, on Saturday, December 2, 1939, Professor J. B. Reynolds presiding.

The attendance was fifty-four, including the following thirty-five members of the Association: Laura M. Ashbaugh, J. A. Benner, A. H. Black, P. A. Caris, John Cawley, E. H. Cutler, J. E. Davis, Arnold Dresden, Tomlinson Fort, M. G. Galbraith, H. S. Grant, D. A. Hatch, Coleman Herpel, P. V. Kunkel, V. V. Latshaw, D. H. Lehmer, A. E. Meder, Jr., Richard Morris, C. A. Nelson, C. O. Oakley, F. W. Owens, Helen B. Owens, G. E. Raynor, J. B. Reynolds, J. A. Shohat, C. A. Shook, L. L. Smail, W. M. Smith, E. R. Stabler, E. P. Starke, A. W. Tucker, R. M. Walter, A. H. Wilson, C. R. Wilson, R. H. Wilson, Jr.

At the business meeting the following officers were elected for next year: Chairman, J. A. Shohat, University of Pennsylvania; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, Arnold Dresden, J. W. Clawson, C. B. Allendoerfer. It was voted to hold the 1940 meeting at the University of Pennsylvania, Philadelphia, Pa., on the Saturday after the official Thanksgiving Day of Pennsylvania.

The following papers were presented:

1. "Mechanical aids to the theory of numbers" by Professor D. H. Lehmer, Lehigh University.

2. "Equations of polygonal configurations" by Professor C. O. Oakley, Haverford College.

3. "Orthogonal polynomials in relation to Lagrangian and Hermitian interpolation" by Professor J. A. Shohat, University of Pennsylvania.

4. "Old mathematical books and instruments in the Schwenkfelder Library" by Dr. Elmer E. S. Johnson, Schwenkfelder Library, Pennsburg, Pa., introduced by Professor Fort.

5. "Mathematics clubs, old and new" by Dr. Helen B. Owens, Pennsylvania State College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Lehmer described and demonstrated the various mechanical devices available for experimental research in the theory of numbers. These included the commercial multiplying machines and their adaptations, punched card equipment, sieve and stencil devices including the new factor stencils as revised by J. D. Elder and the small model electric sieve constructed recently by the speaker.

2. Professor Oakley gave further developments in the theory of semilinear equations: *i.e.*, equations of the form $u_0+\Sigma m_i|u_i|=0$ where the u's are linear

forms and where the m's are constants. It is known that equations of this sort plot a variety of things such as broken lines, isolated points, areas, and combinations of these. In this note particular stress is laid on the problem of semilinear representation of areas. Also, by introducing the operator $A(L)=(2/\pi)\int_0^\infty(\sin Lx/x)dx$ (an operator used extensively by V. Alaci), the equations of some new configurations are written down.

3. Professor Shohat stated that it is well known that no matter how we may choose the abscissas in a Lagrangian interpolation formula, we cannot have convergence for *all* continuous functions (Bernstein, Faber). On the other hand, such convergence does hold in Hermitian interpolation formulas if the abscissas are properly chosen (Fejér). The paper discusses both formulas, from the point of view of convergence and mean-convergence, if the abscissas are the roots of orthogonal polynomials, in particular, of trigonometric polynomials.

4. Dr. Johnson exhibited numerous old mathematical books and old surveying instruments. He also spoke briefly of the history of some of the books and instruments, including, in some instances, interesting facts concerning their acquisition.

5. Dr. Helen B. Owens, after outlining the growth of undergraduate mathematics clubs in America, suggested greater use of problem contests, joint club meetings, mathematical exhibits, public lectures, and student mathematical publications as means of stimulating mathematical interest. She pointed out the great value of the *Department of Undergraduate Mathematics Clubs* in this MONTHLY in these and all other club matters. She urged increased attention of teachers to club activities and urged their coöperation, stressing the unusual opportunity afforded by these undergraduate groups for the development of logical approach to all questions, both within and outside the field of mathematics.

P. A. CARIS, *Secretary*

ON TRIANGLES HAVING A COMMON MEAN*

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Introduction. The term "mean" is used here in the sense defined by P. Delens [**1**]. C. E. Van Horn [**2**] has considered the same triangle and called it "the equilateral derivative" of a triangle. Van Horn gives the following construction for the mean triangle, or the equilateral derivative of a given triangle: Let ABC be any triangle and let O be its circumcenter. Let D, E, F, be the midpoints of the sides BC, CA, AB, respectively. Draw the side bisector DH of the side BC to cut the arc BAC of the circumcircle containing the opposite vertex A at the point H. The side bisectors EJ, FK are similarly drawn to cut the arcs ABC and BCA at the points J and K, respectively. Choose the point L on arc HA so that HL is one-third of the arc HA. Select the points M and N in a similar manner on the arcs JB and KC, respectively.

Employing conjugate coördinates we take the circumcircle of the fundamental triangle $A_1A_2A_3$ to be the unit or base circle and let the coördinates of the vertices A_i be the turns α_i, $(i=1, 2, 3)$. We consider α_i to be roots of the equation $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$. Then as Delens shows, the vertices of the mean triangle of $A_1A_2A_3$ are $\sigma_3^{1/3}$, $\sigma_3^{1/3}\omega$, $\sigma_3^{1/3}\omega^2$, where $1+\omega+\omega^2=0$. It is the purpose of this paper to discuss the relation of two triangles having a common mean triangle, and to apply some of the results to theorems discussed by Musselman in his article in this MONTHLY, *On the line of images* [**3**].

1. Mutually orthopolar triangles. Let s_i be the elementary symmetry functions of the vectors β_i of the vertices B_i of a second triangle inscribed in the base circle. From the definition it follows at once that the two triangles A_i and B_i have a common mean when $\sigma_3=s_3$. The vectors to the orthocenters of triangles A_i and B_i are σ_1 and s_1, respectively. The midpoint m of the segment joining the orthocenters is given by

$$m=\frac{\sigma_1+s_1}{2}=\frac{\sigma_1}{2}+\frac{1}{2}(\beta_1+\beta_2+\sigma_3\beta_1^{-1}\beta_2^{-1}), \tag{1.1}$$

when the triangles have a common mean. The right member of equation (1.1) identifies m as the orthopole of side B_1B_2 as to triangle A_i. The symmetry of the expression $(\sigma_1+s_1)/2$, however, leads to results given by Murnaghan [**4**] and Godeau [**5**]:

THEOREM I. *If two triangles inscribed in the same circle have the same mean, the midpoint of the segment joining their orthocenters is the orthopole of any side of one triangle with respect to the other. In other words, the two triangles are mutually orthopolar.*

We have at once the following:

* Presented at the meeting of the Maryland-District of Columbia-Virginia Section of the Association at Washington, December 9, 1939.

COROLLARY. *The center of the nine-point circle of a triangle is the orthopole of any side of its mean.*

2. The line of images. If we take any point T on the circumcircle of a triangle $A_1A_2A_3$ and reflect this point in the sides of the triangle we obtain three points lying on a line, the line of images of the point T. This follows at once from the property of the pedal line of T as to the triangle $A_1A_2A_3$, and since the pedal line bisects the segment joining the orthocenter to the pole T, it follows that the line of images passes through the orthocenter. Moreover, the line of images is the directrix of the parabola having T for focus and inscribed in the triangle $A_1A_2A_3$. Musselman [**3**] has given the equation of the line of images of T as to $A_1A_2A_3$ to be

$$Tx - \sigma_3\bar{x} = T\sigma_1 - \sigma_2. \tag{2.1}$$

Now consider a second triangle $B_1B_2B_3$ inscribed in the base circle, and let β_i be its vertices. Then s_1 is its orthocenter and the equation of the line of images of the same point T as to this triangle is

$$Tx - s_3\bar{x} = Ts_1 - s_2. \tag{2.2}$$

Lines (2.1) and (2.2) are coincident when $\sigma_3 = s_3$ and $T\sigma_1 - \sigma_2 = Ts_1 - s_2$, *i.e.*, $T = (\sigma_2 - s_2)/(\sigma_1 - s_1)$. Hence the following:

THEOREM II. *If two triangles inscribed in the same circle have a common line of images for the same point on their circumcircle, they have a common mean and a common inscribed parabola whose focus is the given point.*

Theorems I and II are equivalent to a theorem stated by Cwojdzinsky [6]: *When two triangles are inscribed in a circle and circumscribed to a parabola, the midpoint of the distance of their orthocenters is the orthopole of any side of one of the triangles for the other.*

The result stated in Theorem II may be verified otherwise.

It is well known that if, from a point t of the circumcircle, lines are drawn to the three sides of an inscribed triangle making equal angles θ with those sides, the feet of these lines lie on a line which may be considered a generalized pedal line of t under the chosen angle θ. We shall call it a skew pedal line. As the angle θ varies the pedal line envelopes a parabola inscribed in the fundamental triangle and having t as its focus. Letting $e^{2i\theta} = 1/k$ we obtain the equation of the skew pedal line of t under angle θ with respect to triangle $A_1A_2A_3$ to be

$$(k-1)tx + \frac{(k-1)\sigma_3\bar{x}}{k} + \frac{\sigma_3}{kt} - \sigma_2 - kt^2 + \sigma_1 t = 0. \tag{2.3}$$

As k varies we obtain the map equation of the envelope to be

$$x = t - \frac{(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)}{t^2(k-1)^2}, \tag{2.4}$$

which identifies the envelope to be a parabola. If $k=\alpha_i/t$, the equation (2.3) of the corresponding pedal line becomes

$$x + \alpha_j\alpha_k\bar{x} - \alpha_j - \alpha_k = 0, \qquad (i \neq j \neq k = 1, 2, 3),$$

showing that the parabola is inscribed in the triangle $A_1A_2A_3$. Its vertex tangent is the pedal line for $k=-1$, namely

$$2tx - 2\sigma_3\bar{x} + \sigma_3t^{-1} + \sigma_2 - t^2 - \sigma_1 t = 0. \tag{2.5}$$

If $k=\beta_i/t$, where $t=(\sigma_2-s_2)/(\sigma_1-s_1)$, and $\sigma_3=s_3$, equation (2.3) becomes $x+\beta_j\beta_k\bar{x}-\beta_j-\beta_k=0$ which is the equation of side B_jB_k of triangle $B_1B_2B_3$. The triangles A_i and B_i have a common mean by virtue of the assumption $\sigma_3=s_3$. Theorem II is thus verified by direct substitutions. From the results obtained above, we are led to the observation that if two triangles inscribed in the same circle have a common line of images for the same point of their circumcircle they are mutually orthopolar, which also implies that they have a common mean.

The line of images common to two triangles having the same mean is also the locus of points R mentioned in Musselman's generalization of Canon's theorem [7]. The generalized Canon theorem may be stated as follows: *If, for every point R on the line HP, where H is the orthocenter of $A_1A_2A_3$ and P is any point in the plane, we determine the images $C_1C_2C_3$ of R in the sides A_2A_3, A_3A_1, A_1A_2, the four circles $C_1C_2A_3$, $C_2C_3A_1$, $C_3C_1A_2$, and $A_1A_2A_3$ meet in a fixed point M.* Now if A_i and B_i are two triangles having the same mean, and if we choose P to be the orthocenter s_1 of B_i, the point M has the coördinate $(\sigma_2-s_2)/(\sigma_1-s_1)$. The symmetry of this result enables us to state the following modification of the generalization of Canon's theorem:

THEOREM III. *If H_a and H_b are the orthocenters of two triangles $A_1A_2A_3$ and $B_1B_2B_3$ having the same mean, and if for any point R on H_aH_b we determine the images A_i' and B_i' in the sides of triangles A_i and B_i respectively, the circles $A_1'A_2'A_3$, $A_2'A_3'A_1$, $A_3'A_1'A_2$, $B_1'B_2'B_3$, $B_2'B_3'B_1$, $B_3'B_1'B_2$ meet on the base circle at a point which is the focus of the parabola inscribed in the two triangles A_i and B_i.*

It can be readily shown, too, that A_i' and B_i' are images in the common line of images H_aH_b.

3. Co-mean triangles and a theorem of Blanc. If any transversal be drawn through O, the circumcenter of $A_1A_2A_3$, cutting the sides A_2A_3, A_3A_1, A_1A_2 in C_1, C_2, C_3, respectively, the three circles with A_iC_i as diameters meet in two points, one on the circumcircle and the other on the nine-point circle of $A_1A_2A_3$. Their common chord passes through H, the orthocenter of triangle A_i [8]. If the transversal is taken as the diameter through T on the circumcircle, Musselman [7] has shown that the three circles A_iC_i meet at

$$X_C = \frac{\sigma_3 + \sigma_1 T^2}{\sigma_2 + T^2} \qquad \text{and} \qquad X_N = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right)$$

on the circumcircle and nine-point circle respectively, and that the equation of the common chord of the three circles is

$$T^2(\sigma_2 + T^2)x - \sigma_3(\sigma_3 + \sigma_1 T^2)\bar{x} + \sigma_2\sigma_3 - \sigma_1 T^4 = 0. \tag{3.1}$$

Again we consider a second triangle $B_1B_2B_3$ having a common mean with triangle $A_1A_2A_3$, and we find that we can identify line (3.1) with the common line of images of point $(\sigma_2 - s_2)/(\sigma_1 - s_1)$, providing

$$\frac{T^2(\sigma_2 + T^2)}{\sigma_2 - s_2} = \frac{\sigma_3 + \sigma_1 T^2}{\sigma_1 - s_1} = \frac{\sigma_1 T^4 - \sigma_2\sigma_3}{\sigma_2 s_1 - \sigma_1 s_2}. \tag{3.2}$$

These equations are consistent when T is a root of

$$(\sigma_1 - s_1)T^4 - (\sigma_2 s_1 - \sigma_1 s_2)T^2 - \sigma_3(\sigma_2 - s_2) = 0. \tag{3.3}$$

Now the equation of the line of images of point $(\sigma_2 - s_2)/(\sigma_1 - s_1)$ as to triangles A and B is

$$x(\sigma_2 - s_2) - \sigma_3\bar{x}(\sigma_1 - s_1) - s_1\sigma_2 + \sigma_1 s_2 = 0. \tag{3.4}$$

This intersects the circumcircle $x\bar{x} = 1$ in two points x_1, x_2, roots of the quadratic

$$x^2(\sigma_2 - s_2) - (s_1\sigma_2 - \sigma_1 s_2)x - \sigma_3(\sigma_1 - s_1) = 0. \tag{3.5}$$

Let $x = -\sigma_3/T^2$; then equation (3.5) becomes identical with equation (3.3), enabling us to devise a means to construct the four points which represent the roots of equation (3.3). The equation $x = -\sigma_3/T^2$ identifies the points x, T, and $-T$ as vertices of a right triangle having the same mean as the fundamental triangle $A_1A_2A_3$. There are therefore two distinct diametral transversals OT which yield the same common chord of the system of circles mentioned in Blanc's theorem when applied to two triangles having a common mean. We have, then, the following construction for the points T: If L is a vertex of the mean triangle, and X_1 is an intersection of the common line of images with the circumcircle, locate a point K on the circumcircle so that arc $LX_1 = 2$ arc KL and point L lies between X_1 and K. Then one of the diametral transversals will be the diameter perpendicular to OK. The other diameter is found by using the second intersection X_2 of the common line of images with the base circle.

From equation (3.2) we have $T^2 = X_C\alpha$, where $\alpha = (\sigma_2 - s_2)/(\sigma_1 - s_1)$, the point whose line of images is (3.4). Then we may write, since there is a point X_C corresponding to a point T^2, $X_C'\alpha = T_1^2$ and $X_C''\alpha = T_2^2$. From these relations we find

$$X_C' + X_C'' = (T_1^2 + T_2^2)/\alpha = \frac{\sigma_2 s_1 - \sigma_1 s_2}{\sigma_2 - s_2} = x_1 + x_2,$$

$$X_C' X_C'' = T_1^2 T_2^2/\alpha^2 = -\frac{\sigma_3(\sigma_1 - s_1)}{\sigma_2 - s_2} = x_1 x_2,$$

where x_1 and x_2 are the roots of the equation (3.5). These results involve only a point on the circumcircle and its common line of images as to two triangles

having a common mean, and since, as we have seen, the point and line are focus and directrix, respectively, of a parabola common to the two triangles, we may state the following:

THEOREM IV. *All triangles in a poristic system of triangles inscribed in a circle and circumscribed to a parabola have a common mean. Each of the two diameters of the circle bisecting the arcs joining the focus to the points where the directrix cuts the circle, cuts the sides of any triangle of the poristic system in points such that the circles on the segments joining these points to the opposite vertices as diameters meet at a point common to the circle and the directrix.*

4. Isogonal conjugates. THEOREM. *The isogonal conjugates of the common orthopole of two mutually orthopolar triangles in those triangles are symmetric with respect to the circumcenter.*

Let the mutually orthopolar triangles be $A_1A_2A_3$ and $B_1B_2B_3$. Let α_i and β_i be their vertices, respectively. Then if $P(p)$ is their common orthopole, $2p=\sigma_1+s_1$ and $\sigma_3=s_3$. If P does not lie on the circumcircle, its isogonal conjugate x is given by $p+x+\sigma_3\bar{x}\bar{p}=\sigma_1$ [9] for the triangle $A_1A_2A_3$. Similarly, the isogonal conjugate y of p in triangle $B_1B_2B_3$ is given by $p+y+\sigma_3\bar{p}\bar{y}=s_1$. Adding we get, remembering that $2p=\sigma_1+s_1$, $x+y+\sigma_3\bar{p}(\bar{x}+\bar{y})=0$, which is not true unless $x+y=0$, *i.e.*, $x=-y$, and the points are symmetric with respect to the circumcenter.

5. Perspective triangles. Suppose triangles $A_1A_2A_3$ and $B_1B_2B_3$ are mutually orthopolar and perspective from a point $P(p)$. Then

$$\beta_i = \frac{\alpha_i - p}{\alpha_i\bar{p} - 1}, \ (i = 1, 2, 3), \ \text{ and } \ \beta_1\beta_2\beta_3 = \sigma_3 = \frac{(\alpha_1 - p)(\alpha_2 - p)(\alpha_3 - p)}{(\alpha_1\bar{p} - 1)(\alpha_2\bar{p} - 1)(\alpha_3\bar{p} - 1)};$$

hence, regarding triangle $A_1A_2A_3$ as fixed, and $B_1B_2B_3$ as variable,

$$p^3 - \sigma_1p^2 + \sigma_2p + \sigma_3^2\,\bar{p}^3 - \sigma_2\sigma_3\bar{p}^2 + \sigma_1\sigma_3\bar{p} = 2\sigma_3$$

is the equation of the locus of the centers of perspective. The locus is a cubic cutting the base circle $z\bar{z}=1$ at the vertices of the fundamental triangle $A_1A_2A_3$, and at the vertices of their common mean. The cubic cuts the sides A_jA_k where $p_i=\alpha_i(3\sigma_3-\alpha_i\sigma_2)/(\sigma_3-\alpha_i^3)$, *i.e.*, where the Lemoine axis crosses the sides of $A_1A_2A_3$. The clinant at any point is given by

$$\frac{dz}{d\bar{z}} = -\frac{3\sigma_3^2\,\bar{z}^2 - 2\sigma_2\sigma_3\bar{z} + \sigma_1\sigma_3}{3z^2 - 2\sigma_1z + \sigma_2},$$

which shows that the cubic cuts McCay's cubic [10] orthogonally [11]. The asymptotes are $z=-\omega_i\sigma_3^{2/3}\bar{z}+\frac{1}{3}(\sigma_1+\omega_i\sigma_2\sigma_3^{-1/3})$, where ω_i is any one of the cube roots of unity. The asymptotes intersect at the centroid of $A_1A_2A_3$, $z=\sigma_1/3$.

It may be remarked here that if the pair of perspective triangles $A_1A_2A_3$ and $B_1B_2B_3$ have opposite means in the sense defined by Delens [1], *i.e.*, if $\alpha_1\alpha_2\alpha_3=-\beta_1\beta_2\beta_3$, the locus of the centers of perspective is McCay's cubic [11].

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ON EXTREMA OF FUNCTIONS WHICH SATISFY CERTAIN SYMMETRY CONDITIONS

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1. Introduction. The following theorems on maxima and minima are well known.

I. *The function* $x_1 x_2 \cdots x_n$, *where the real variables* $x_1, x_2, \cdots, x_n$ *are subject to the condition* $x_1+x_2+\cdots+x_n=c>0$, *has a proper relative maximum at* $x_1=x_2=\cdots=x_n=c/n$.

II. *The function* $x_1+x_2+\cdots+x_n$, *where the real variables* $x_1, x_2, \cdots, x_n$ *are subject to the condition* $x_1x_2\cdots x_n=c>0$, *has a proper relative minimum at* $x_1=x_2=\cdots=x_n=\sqrt[n]{c}$.

R. H. Garver* has pointed out that most of the elementary problems on the applications of the theory of maxima and minima which are customarily encountered in textbooks on the calculus, can, by appropriate transformation of the variables, be put into forms to which Theorems I or II may be applied. One infers that Theorems I and II are rather fundamental results in the theory of maxima and minima.

Theorems I and II are, however, susceptible of a sweeping generalization. It is the purpose of this paper to call attention to this generalization.†

A mild study of Theorems I and II leads one rather naturally to suspect that the essence of those theorems may lie, not in the use of the particular functions $x_1+x_2+\cdots+x_n$ and $x_1x_2\cdots x_n$, but in the symmetry of the conditional relation and the function to be extremized, in the variables $x_1, x_2, \cdots, x_n$. This suspicion may be strengthened by the construction of examples which can not be made to fall under the jurisdiction of Theorems I or II by any transformation of the variables.

* This MONTHLY, vol. 42, 1935, pp. 435–437.

† I have been informed by Professor Tibor Radó that some form of this generalization is known to Fejér, who remarked about it several years ago in the course of a conversation with him. However, the result does not seem to be generally known, and I have not been able to find it in the literature.

The well known trigonometric inequality

$$\sin x/2 \sin y/2 \sin z/2 \leqq 1/8,$$

where x, y, and z are angles of a plane triangle, furnishes one such example. The equality sign holds if the triangle is equilateral. From this inequality we may state that the symmetric function $\sin x/2 \sin y/2 \sin z/2$, where x, y, and z are subject to the symmetric relation $x+y+z=\pi$, has a relative maximum at the set of values determined from the relation $x+y+z=\pi$ by setting $x=y=z$.

A second example of a similar nature is furnished by the function $\sin x \cos x$. That this function has a relative maximum at $x=\pi/4$ is equivalent to saying that the symmetric function $\cos x \cos y$, where x and y are subject to the symmetric condition $x+y=\pi/2$, has a maximum at $x=y$.

Neither of these examples can be put into a form to which Theorems I and II apply, yet the essence of those theorems still remains. Indeed, the theorem suggested by the foregoing discussion is true, under proper hypotheses as to the existence of derivatives, *etc.* It is possible, however, to prove a more general theorem, in which the number of conditions imposed on the variables is arbitrary (but of course less than the number of variables). Two essential modifications are to be made in such a theorem, one compulsory and one voluntary: (1) If k is the number of functionally independent relations among the n variables, then the set of values (corresponding to $x_1=x_2= \cdots =x_n=a$ in Theorems I and II) at which an extremum will occur will be determined by setting $n-k+1$ of the variables, say $x_1, \cdots, x_{n-k+1}$, equal in those relations and solving for the values of the $x_{n-k+2}, \cdots, x_n$, and the common value of the $x_1, \cdots, x_{n-k+1}$. (2) We may lighten the hypotheses of complete symmetry and require only that the functions involved be symmetric with respect to (at least) $x_1, \cdots, x_{n-k+1}$. It is this suggested theorem which we shall presently state and prove.

2. Definitions, notations, and lemmas. DEFINITION 1. *A region R of points $(x_1, x_2, \cdots, x_q, x_{q+1}, \cdots, x_n)$ in euclidean n-space will be said to be symmetric with respect to $x_1, \cdots, x_q, q\leqq n$, if, whenever a point $(a_1, a_2, \cdots, a_q, a_{q+1}, \cdots, a_n)$ is in R, so also is $(a_{i_1}, a_{i_2}, \cdots, a_{i_q}, a_{q+1}, \cdots, a_n)$, where $i_1, \cdots, i_q$ is any permutation of $1, \cdots, q$.*

DEFINITION 2. *Let $S(x_1, x_2, \cdots, x_q, x_{q+1}, \cdots, x_n)$ be a function defined in a region R which is symmetric with respect to $x_1, \cdots, x_q$. S is said to be symmetric with respect to $x_1, \cdots, x_q$ over R, if $S(x_1, x_2, \cdots, x_q, x_{q+1}, \cdots, x_n)$ is identically equal, in all of the variables, throughout R, to $S(x_{i_1}, x_{i_2}, \cdots, x_{i_q}, x_{q+1}, \cdots, x_n)$, for every permutation $i_1, \cdots, i_q$, of $1, \cdots, q$.*

We now define a notation which will be used in the sequel. If $f(x_1, x_2, \cdots, x_n)$ is a function of variables $x_1, \cdots, x_n$ which are not all independent, but are functions of m independent ones, say $x_1, \cdots, x_m$, then for the partial derivative of $f(x_1, \cdots, x_n)$ with respect to x_j, where all of the other variables of $x_1, \cdots, x_n$

are held fixed, we shall use the notation f_j. For the partial derivative of f with respect to x_j, $j \leqq m$, where only the remaining variables of the set $x_1, \cdots, x_m$ are held fixed, we shall use the notation $\partial f/\partial x_j$.

We now state without proof two lemmas which will be needed. These lemmas are almost immediate consequences of Definitions 1 and 2, and are intuitively obvious.

LEMMA 1. *If the function $S(x_1, x_2, \cdots, x_p, x_{p+1}, \cdots, x_n)$ is symmetric with respect to $x_1, \cdots, x_p$ over a region R, then for $i \leqq p$ and $j \leqq p$, $S_i = S_j$ at any point of R for which $x_i = x_j$, and for which S_i and S_j exist.*

LEMMA 2. *If the function $S(x_1, x_2, \cdots, x_p, x_{p+1}, \cdots, x_n)$ is symmetric with respect to $x_1, \cdots, x_p$ over a region R, then for indices i, j, l, m each less than or equal to p, it is true that $S_{ij} = S_{lm}$ at any point of R for which S_{ij} and S_{lm} exist, and for which $x_i = x_j = x_l = x_m$, if either, (1) $i = j$ and $l = m$, or (2) $i \neq j$ and $l \neq m$.*

3. Statement and proof of the theorem. We are now in a position to state and prove the theorem hinted at in §1.

THEOREM. *Let $S(x_1, x_2, \cdots, x_{n-k}, x_{n-k+1}, \cdots, x_n)$ be a function symmetric in the variables $x_1, x_2, \cdots, x_{n-k+1}$ over a region R. Let the variables $x_1, \cdots, x_n$ be subject to the k conditions*

$$V^{(i)}(x_1, x_2, \cdots, x_{n-k}, x_{n-k+1}, \cdots, x_n) = 0, \qquad (i = 1, 2, \cdots, k, k < n), \tag{1}$$

where each function $V^{(i)}$ is symmetric in $x_1, \cdots, x_{n-k+1}$ over R. Let equations (1) *admit a simultaneous solution in R $(x_1, x_2, \cdots, x_{n-k+1}, x_{n-k+2}, \cdots, x_n) = (a, a, \cdots, a, a_{n-k+2}, \cdots, a_n)$ denoted by P, and let the $V^{(i)}$ and S have continuous partial derivatives of the second order with respect to $x_1, \cdots, x_n$ in R. Further, let the Jacobian J of the $V^{(i)}$ with respect to $(x_{n-k+1}, \cdots, x_n)$ be different from zero at P. Then the function $S(x_1, \cdots, x_n)$, where the variables are subject to the conditions* (1), *has a relative extremum at P, provided the expression*

$$\Delta = \frac{1}{J}\begin{vmatrix} S_{11} - S_{12} & S_{n-k+1} & S_{n-k+2} & \cdots & S_n \\ V_{11}^{(1)} - V_{12}^{(1)} & V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ V_{11}^{(2)} - V_{12}^{(2)} & V_{n-k+1}^{(2)} & V_{n-k+2}^{(2)} & \cdots & V_n^{(2)} \\ \vdots & \vdots & \vdots & & \vdots \\ V_{11}^{(k)} - V_{12}^{(k)} & V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}$$

is different from zero at P. Moreover, P furnishes a maximum or a minimum according as $\Delta < 0$ or $\Delta > 0$, at P.

It should be noted that the theorem does not imply that P furnishes the only extremum, or that P furnishes an absolute extremum, to the function S, subject to the side conditions (1). Indeed, it is easy to construct examples to show that neither of these things is true in general. Thus the present theorem

is a special result in the general problem of extremizing the function S subject to conditions (1).

The proof we shall make is quite elementary, involving nothing more than the ordinary theory of maxima and minima as given in texts on advanced calculus.

Since $J\neq 0$ at P, and since the $V^{(i)}$ have continuous partial derivatives of the second order at P, the relations (1) define the $x_{n-k+1}, \cdots, x_n$ as single-valued functions of $x_1, \cdots, x_{n-k}$, which have continuous second partial derivatives in some region R' contained in R and containing P. (It is clear that R' may be taken to be symmetric with respect to $x_1, \cdots, x_{n-k+1}$ and such that $J\neq 0$ in R'.) Let $F(x_1, x_2, \cdots, x_{n-k})$ denote the function obtained by replacing each of $x_{n-k+1}, \cdots, x_n$ by its equivalent as a function of $x_1, \cdots, x_{n-k}$.

We begin by showing that the first necessary condition for an extremum, *i.e.*, the vanishing of all of the first partial derivatives of $F(x_1, \cdots, x_{n-k})$, is satisfied at $x_1=x_2=\cdots=x_{n-k}=a$.* By the usual method for differentiating composite functions, we obtain

$$\frac{\partial F}{\partial x_i} = S_i + \sum_{p=n-k+1}^{n} S_p \frac{\partial x_p}{\partial x_i}, \qquad (i = 1, 2, \cdots, n-k). \tag{2}$$

Now from (1)

$$V_i^{(m)} + \sum_{p=n-k+1}^{n} V_p^{(m)} \frac{\partial x_p}{\partial x_i} = 0, \qquad (m = 1, 2, \cdots, k), \tag{3}$$

whence, since

$$J = \begin{vmatrix} V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ V_{n-k+1}^{(2)} & V_{n-k+2}^{(2)} & \cdots & V_n^{(2)} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix} \neq 0$$

in R', we have, on solving the system of equations (3),

$$\frac{\partial x_{n-k+h}}{\partial x_i} = \frac{1}{J} \begin{vmatrix} V_{n-k+1}^{(1)} & \cdots & -V_i^{(1)} & \cdots & V_n^{(1)} \\ V_{n-k+1}^{(2)} & \cdots & -V_i^{(2)} & \cdots & V_n^{(2)} \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & \cdot & & \cdot \\ V_{n-k+1}^{(k)} & \cdots & -V_i^{(k)} & \cdots & V_n^{(k)} \end{vmatrix} = \frac{d_{n-k+h\ i}}{J}, \tag{4}$$

* The method of Lagrange multipliers has been intentionally avoided, because, although the method is elegant in showing that the first necessary conditions for an extremum are satisfied, it is much less elementary when applied to the question of determining whether or not the stationary point P furnishes a maximum, a minimum, or neither. (See Theory of Maxima and Minima, by H. Hancock, Ginn and Co., 1917.)

$(h=1, \cdots, k; i=1, 2, \cdots, n-k)$, where the $-V_i^{(m)}$, $(m=1, 2, \cdots, k)$, are in column h. Now since the $V^{(m)}$ are symmetric in $x_1, \cdots, x_{n-k+1}$, and since at P, $x_1=x_2= \cdots =x_{n-k+1}=a$, it follows from Lemma 1 that $V_i^{(m)}=V_{n-k+1}^{(m)}$ at P. Hence (4), evaluated at P, yields

$$\frac{\partial x_{n-k+h}}{\partial x_i} = 0, \qquad (i = 1, \cdots, n-k), \tag{5}$$

if $h>1$, and

$$\frac{\partial x_{n-k+1}}{\partial x_i} = -\frac{J}{J} = -1, \qquad (i = 1, \cdots, n-k), \tag{6}$$

if $h=1$. Therefore (2), evaluated at P, becomes

$$\left.\frac{\partial F}{\partial x_i}\right]_P = S_i]_P - S_{n-k+1}]_P = 0, \qquad (i = 1, 2, \cdots, n-k),$$

by Lemma 1. Thus the first necessary condition for the existence of an extremum at P is fulfilled.

To obtain a sufficient condition for an extremum at P we shall use the customary method of the calculus, for which we need the second partial derivatives of F. Before computing these, let us change the form of the expressions (2). From (2) and (4),

$$\frac{\partial F}{\partial x_i} = \frac{1}{J}\left(JS_i + \sum_{p=n-k+1}^{n} S_p d_{pi}\right), \qquad (i = 1, 2, \cdots, n-k).$$

The quantity in the parentheses is readily identified as the expansion by minors, according to the elements of the first row, of the $(k+1)$th order determinant

$$M^{(i)} = \begin{vmatrix} S_i & S_{n-k+1} & S_{n-k+2} & \cdots & S_n \\ V_i^{(1)} & V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ \vdots & \vdots & \vdots & & \vdots \\ V_i^{(k)} & V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}, \qquad (i = 1, 2, \cdots, n-k). \tag{7}$$

Hence

$$\frac{\partial F}{\partial x_i} = \frac{M^{(i)}}{J}, \qquad (i = 1, 2, \cdots, n-k). \tag{8}$$

From (8) we obtain

$$\frac{\partial^2 F}{\partial x_j \partial x_i} = M^{(i)} \frac{\partial}{\partial x_j}\left(\frac{1}{J}\right) + \frac{1}{J}\left(M_j^{(i)} + \sum_{p=n-k+1}^{n} M_p^{(i)} \frac{\partial x_p}{\partial x_j}\right),$$
$$(i, j = 1, 2, \cdots, n-k),$$

whence from (5), (6), (7), and Lemma 1, we have

$$\left.\frac{\partial^2 F}{\partial x_j \partial x_i}\right]_P = \frac{1}{J}\,(M_j^{(i)} - M_{n-k+1}^{(i)})]_P, \qquad (i, j = 1, 2, \cdots, n-k). \tag{9}$$

Now calculating $M_j{}^{(i)}$ by performing the differentiation with respect to columns, and evaluating at P, keeping in mind that from Lemma 1, $V_i^{(p)} = V_{n-k+1}^{(p)}$ and $S_i = S_{n-k+1}$, we obtain, omitting all determinants which are zero,

$$M_j^{(i)}]_P = \left[\begin{vmatrix} S_{ij} & S_{n-k+1} & S_{n-k+2} & \cdots & S_n \\ V_{ij}^{(1)} & V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ V_{ij}^{(k)} & V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix} + \begin{vmatrix} S_i & S_{n-k+1\,j} & S_{n-k+2} & \cdots & S_n \\ V_i^{(1)} & V_{n-k+1\,j}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ V_i^{(k)} & V_{n-k+1\,j}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}\right]_P, \tag{10}$$

$(i, j = 1, 2, \cdots, n-k)$. By Lemma 2, if $i \neq j$, $S_{ij} = S_{n-k+1\ i} = S_{12}$, $V_{ij}^{(p)} = V_{n-k+1\ i}^{(p)} = V_{12}^{(p)}$, and $S_{ii} = S_{11}$, $V_{ii}^{(p)} = V_{11}^{(p)}$ at P. Hence, according as $i \neq j$ or $i = j$, (10) yields $M_j{}^{(i)}]_P = 0$, or

$$M_i^{(i)}]_P = \left.\begin{vmatrix} S_{11} - S_{12} & S_{n-k+1} & S_{n-k+2} & \cdots & S_n \\ V_{11}^{(1)} - V_{12}^{(1)} & V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ V_{11}^{(k)} - V_{12}^{(k)} & V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}\right]_P.$$

In a similar manner the second term in the parentheses in (9) is found, from (10), to be

$$M_{n-k+1}^{(i)}]_P = -\left.\begin{vmatrix} S_{11} - S_{12} & S_{n-k+1} & S_{n-k+2} & \cdots & S_n \\ V_{11}^{(1)} - V_{12}^{(1)} & V_{n-k+1}^{(1)} & V_{n-k+2}^{(1)} & \cdots & V_n^{(1)} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ V_{11}^{(k)} - V_{12}^{(k)} & V_{n-k+1}^{(k)} & V_{n-k+2}^{(k)} & \cdots & V_n^{(k)} \end{vmatrix}\right]_P = -M_i^{(i)}]_P.$$

Hence, denoting $M_i{}^{(i)}/J]_P$ by $\Delta(P)$, (9) may be written

$$\left.\frac{\partial^2 F}{\partial x_j \partial x_i}\right]_P = \Delta(P), \quad i \neq j; \qquad \left.\frac{\partial^2 F}{\partial x_i^2}\right]_P = 2\Delta(P). \tag{11}$$

Now since F has continuous partial derivatives of the second order, by the well known criterion evolved from Taylor's theorem, a sufficient condition that F have a proper relative extremum at P is that the quadratic form

$$Q = \sum_{i,j=1}^{n-k} \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right]_P h_i h_j$$

be definite. But from (11),

$$Q = \Delta(P)\left[\sum_{i,j=1}^{n-k} h_i h_j + \sum_{i=1}^{n-k} h_i^2\right] = \Delta(P)\left[\left(\sum_{i=1}^{n-k} h_i\right)^2 + \sum_{i=1}^{n-k} h_i^2\right].$$

Obviously Q will be definite when, and only when, $\Delta(P) \neq 0$. Furthermore, F has a maximum or a minimum at P according as $\Delta(P)$ is negative or positive. This completes the proof of the theorem.

By the methods suggested in Garver's paper, a large class of problems on maxima and minima may be reduced to a symmetric form capable of being handled by this theorem with a considerable saving of labor.

We remark that in case $k=1$, the functions $V^{(i)}$ and S of the theorem are both symmetric in all of the variables $x_1, \cdots, x_n$. This is the case of the theorem which applies to the examples of §1. It is also the special case, of which Theorems I and II are special cases.

4. Further inferences from the theorem. Suppose that S and the $V^{(i)}$ satisfy the hypotheses of the theorem, so that S has the extremum value $S(a, a, \cdots, a, a, a_{n-k+2}, \cdots, a_n)$. Consider the extremum problem arising by replacing S by one of the $V^{(i)}$, say $V^{(p)}$, and putting the relation

$$S(x_1, x_2, \cdots, x_{n-k+1}, x_{n-k+2}, \cdots, x_n) - S(a, a, \cdots, a, a, a_{n-k+2}, \cdots, a_n) = 0$$

in place of $V^{(p)}=0$. In this second problem the determinant $M_i^{(i)}$ appearing in Δ will be equal to the corresponding determinant for the first extremum problem, with rows 1 and $p+1$ interchanged. Thus, if the Jacobian J in the second problem is different from zero at P, the hypotheses of the theorem are satisfied by this problem and $V^{(p)}$ has the extremum value zero at P. Further, if the Jacobians in the two problems have the same sign at P, the Δ's in the two problems will have opposite signs, so that if S, in the first problem, has a maximum (minimum) at P, then $V^{(p)}$, in the second problem, will have the minimum (maximum) value zero at P. If the two Jacobians are of opposite sign at P, then S and $V^{(p)}$ have the same type of extremum at P in the two cases.*

* This symmetric property of the rôles of the functions involved in the problem of restricted extrema is general, and is well known. (See, for instance, The Differential Calculus, by T. Chaundy, Clarendon Press, 1935, pp. 262–264.) The above independent demonstration of this property has been given because it entails so little effort in this special case.

The theorem of §3 is rather striking, but there is an inference to be drawn which is even more striking, *viz.*, the set of numbers P, at which an extremum occurs (provided $\Delta(P)\neq 0$), is determined by setting $x_1=x_2=\cdots=x_{n-k+1}$ in the k relations (1); that is, P does not depend on the function S. Thus, the set of numbers P will furnish an extremum to *every* function S which is symmetric in $x_1,\cdots,x_{n-k+1}$ and which satisfies the other meager hypotheses of the theorem (in particular the condition $\Delta(P)\neq 0$). It is apparent that the sign of $\Delta(P)$ will depend on the choice of S, so that some functions S will have a maximum and some a minimum at P, but every function S, for which $\Delta(P)\neq 0$, has one or the other. For instance, in the first example in §1, the relation $x+y+z=\pi$ determines the set of values $x=y=z=\pi/3$, which will furnish an extremum to any symmetric function of x, y, and z, subject to the above relation, provided the function is such that Δ is defined and different from zero for those values.

A THEOREM ON PARTITIONS OF THE SET OF POSITIVE INTEGERS

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The relation of congruence modulo an integer n generates a partition of the set of positive integers. This partition has the property that if A and B are any two of the n sub-sets thus defined, a_1 and a_2 any two integers in A, b_1 and b_2 any two in B, then a_1+b_1 and $a_1\cdot b_1$ are in the same sub-sets respectively as a_2+b_2 and $a_2\cdot b_2$. It follows that we can define $A+B$ as the sub-set in which lie all the sums of an integer in A and one in B. We define the product of two sub-sets in a similar manner. Then the algebra of sub-sets under addition and multiplication is a homomorphic image* of the system of positive integers under addition and multiplication. For this reason we call the partition homomorphic with respect to these two operations.

In this paper we shall investigate the question of whether there are any values of n for which the partition is also homomorphic with respect to exponentiation.† It will be shown that 1, 2, 6, 42, and 1806 are the only such values.

The problem can be put in the following form. We wish to find all positive integers n such that for all integers k_1 and k_2 satisfying $0<k_1\leqq n$ and $0<k_2\leqq n$, and for all non-negative integers m_1 and m_2,

$$(k_1+m_1n)^{k_2+m_2n}\equiv k_1^{k_2} \pmod{n}.$$

Using the binomial theorem we write this

$$k_1^{k_2+m_2n}+nA\equiv k_1^{k_2} \pmod{n},$$

where A is always an integer. This is clearly equivalent to

* *Cf.* van der Waerden, Moderne Algebra, vol. 1, second edition, page 41.

† This problem was suggested, and solved independently, by Professor G. Birkhoff.

$$k_1^{k_2+m_2n} \equiv k_1^{k_2} \pmod{n}, \tag{1}$$

or

$$k_1^{k_2}(k_1^{m_2n} - 1) \equiv 0 \pmod{n}. \tag{2}$$

With these statements of the problem given, we prove the following:

THEOREM 1. *If $p^2 | n$, p a prime, then n is not a solution.*

Proof. Suppose that $n = p^2n'$. In (2) let $k_1 = p$, $k_2 = 1$, $m_2 = 1$. Then clearly

$$p(p^n - 1) \not\equiv 0 \pmod{n},$$

since the left side is not even divisible by p^2.

From Theorem 1 we conclude that for possible moduli n, (2) is equivalent to

$$k_1(k_1^{m_2n} - 1) \equiv 0 \pmod{n}, \tag{3}$$

where k_1 and m_2 must satisfy the same conditions as before. Another trivial corollary is that no modulus of the form k^m, where $k>1$, $m>1$, is a solution.

THEOREM 2. *If $p|n$, p a prime, but $(p-1)\nmid n$, then n is not a solution.*

Proof. Assume $p \geqq 3$, since for $p=2$ the hypothesis is not satisfied. Let $n = n'(p-1)+r$, where $0<r<(p-1)$. If $p \nmid k_1$, then by Fermat's Theorem

$$k_1^{n'(p-1)} \equiv 1 \pmod{p}. \tag{4}$$

Next, there exists at least one k_1 with $0<k_1<p$, such that for $e<(p-1)$,

$$k_1^e \equiv f \not\equiv 1 \pmod{p}.^*$$

For such a k_1 we have

$$k_1^r \equiv g \not\equiv 1 \pmod{p}. \tag{5}$$

Multiplying (4) and (5) together we get

$$k_1^{n'(p-1)+r} \equiv g \not\equiv 1 \pmod{p}.$$

Since we know from our choice of k_1 that $p \nmid k_1$, (2) cannot hold, and n is not a solution.

THEOREM 3. *If $n = p_1p_2 \cdots p_r$, $p_i \neq p_j$ for $i \neq j$, and $(p_i - 1)|n$ for $i = 1, 2, \cdots, r$, then n is a solution of our problem.*

Proof. We note first that unless $n=1$, n is even. Theorem 2 shows at once that this is a necessary condition. Suppose first that $(k_1, n) = 1$. Let $n = (p_i - 1)n_i$, where by hypothesis n_i is an integer. Then (3) may be written

$$k_1[(k_1^{p_i-1})^{n_im_2} - 1] \equiv 0 \pmod{n}. \tag{6}$$

* *Cf.* L. E. Dickson, Introduction to the Theory of Numbers, pages 16–18.

By Fermat's Theorem the second factor is divisible by p_i for all permissible values of m_2. If k_1 has any p_i as a factor, p_i divides the first factor of (6). By the previous argument, the rest of the p_i divide the second factor.

Theorems 1, 2, and 3 together give a necessary and sufficient condition that an integer n be a solution. It remains only to find all the solutions explicitly, if this is possible. We do so in the following:

THEOREM 4. *The only values of n which satisfy* (2) *are* 1, 2, 6, 42, *and* 1806.

Proof. The prime factors of these numbers are 2, 3, 7, and 43. That the numbers above are all of them solutions can easily be seen from a consideration of their prime factorizations. Further, there are no other solutions. First note that any solution divisible by one of the primes above must be divisible by all the smaller primes of the set. Next we note that forming $2\cdot 3+1$, $2\cdot 7+1$, $2\cdot 3\cdot 7+1$, $2\cdot 3\cdot 43+1$, $2\cdot 43+1$, $2\cdot 7\cdot 43+1$, and $2\cdot 3\cdot 7\cdot 43+1$ we get no new primes. Finally we show by induction that no prime other than these four can divide a solution. Assume that of the first m primes $p_1, p_2, \cdots, p_m$ the only possible divisors of a solution are among our four. Let p_{m+k} be the next prime, not one of the four. Then $p_{m+k}-1$ is a product of primes, not all of which are possible divisors of a solution. Hence p_{m+k} cannot divide a solution.

AN ASYMPTOTIC FORMULA FOR THE AVERAGE SUM OF THE DIGITS OF INTEGERS

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In a recent book a statement is made for which the author says that he has no general proof.* The statement is essentially that the average sum of the digits of integers is least when they are written in the binary scale of notation. A simple asymptotic formula for the average sum of the digits of integers can be derived and by means of it the statement mentioned can be proved.

Let $S(r, N)$ be the sum of the digits of all non-negative integers less than N when these numbers are written in the scale of notation of radix r. Then $S(r, N)/N$ is the average sum of the digits of all integers less than N (including zero) when written in the r-scale.†

THEOREM 1.

$$\frac{S(r, N)}{N} \overset{N}{\sim} \frac{(r-1)\log N}{2\log r}.\ddagger$$

For, consider all integers from zero up to and including $N-1$ written in their

* Joseph Bowden, Special Topics in Theoretical Arithmetic, New York, 1936, p. 68.

† The inclusion of zero simplifies the notation and has no effect on the result.

‡ $f(x, y)\overset{y}{\sim}g(x, y)$ is used to mean $\lim_{y\to\infty} \dfrac{f(x, y)}{g(x, y)}=1$.

natural order in the r-scale. The digits in the ith place from the right repeat themselves in periods of r^i numbers, each period consisting of r^{i-1} of each of the digits $0, 1, 2, \cdots, r-1$. The last period will be complete if and only if N is divisible by r^i. Let s_i be the sum of the digits in the ith place from the right in all the numbers from 0 to $N-1$. Then

$$s_i \geqq \tfrac{1}{2}\left[\frac{N}{r^i}\right] r^i(r-1) > \tfrac{1}{2}(r-1)(N-r^i),^*$$

and

$$s_i < \tfrac{1}{2}\left[\frac{N}{r^i}+1\right] r^i(r-1) \leqq \tfrac{1}{2}(r-1)(N+r^i).$$

But

$$S(r, N) = \sum_{i=1}^{k} s_i, \quad \text{where} \quad r^{k-1} < N \leqq r^k.$$

Hence

$$S(r, N) > \tfrac{1}{2}(r-1)\sum_{i=1}^{k}(N-r^i) > \tfrac{1}{2}\{(r-1)k - r^2\}N,$$

$$S(r, N) < \tfrac{1}{2}(r-1)\sum_{i=1}^{k}(N+r^i) < \tfrac{1}{2}\{(r-1)k + r^2\}N.$$

Thus

$$\liminf_{N\to\infty} \frac{2\cdot S(r, N)\log r}{(r-1)N\log N} \geqq \liminf_{N\to\infty} \frac{\{(r-1)k - r^2\}\log r}{(r-1)\log N}.$$

Since

$$N \leqq r^k, \qquad k \geqq \frac{\log N}{\log r}, \quad \text{and} \quad r \geqq 2,$$

we have

$$\liminf_{N\to\infty} \frac{2\cdot S(r, N)\log r}{(r-1)N\log N} \geqq \liminf_{N\to\infty}\left\{1 - \frac{r^2\log r}{(r-1)\log N}\right\} = 1.$$

Also

$$\limsup_{N\to\infty} \frac{2\cdot S(r, N)\log r}{(r-1)N\log N} \leqq \limsup_{N\to\infty} \frac{\{(r-1)k + r^2\}\log r}{(r-1)\log N}.$$

Since

$$N > r^{k-1}, \qquad k < \frac{\log N}{\log r} + 1,$$

* $[x]$ is used to mean the greatest integer $\leqq x$.

we have

$$\limsup_{N\to\infty} \frac{2\cdot S(r, N) \log r}{(r-1)N \log N} \leqq \limsup_{N\to\infty} \left\{1 + \frac{(r^2+r-1)\log r}{(r-1)\log N}\right\} = 1.$$

Thus

$$\lim_{N\to\infty} \frac{2\cdot S(r, N) \log r}{(r-1)N \log N} = 1,$$

which proves the theorem.

In fact, for a fixed r and variable N, $S(r, N)/N$ attains a relative maximum for $N=r^n$, n an integer greater than $2/(r-1)$, and is there equal to the asymptotic representation of the theorem, *i.e.*, $S(r, r^n)/r^n = n(r-1)/2$ for integral values of n, but $S(r, N)/N$ is less than the asymptotic representation for both $N=r^n-1$ and $N=r^n+1$.

THEOREM 2. *For sufficiently large values of N the average sum of the digits of all non-negative integers less than N is least when the numbers are written in the binary scale of notation.*

For, from Theorem 1 and elementary properties of limits it is easily seen that

$$\lim_{N\to\infty} \frac{S(r, N)}{S(2, N)} = \frac{(r-1)\log 2}{\log r}.$$

Since

$$\frac{d}{dr}\left\{\frac{(r-1)\log 2}{\log r}\right\} = \frac{(\log r + r^{-1} - 1)\log 2}{\log^2 r} > 0 \quad \text{for} \quad r \geqq 2,$$

$(r-1)\log 2/\log r$ is a monotonic increasing function of r for $r \geqq 2$, and

$$\lim_{N\to\infty} \frac{S(r, N)}{S(2, N)} \geqq \frac{2\log 2}{\log 3} > \frac{5}{4},$$

for $r \geqq 3$, which proves the theorem. This theorem can also be obtained directly from the inequalities used in the proof of Theorem 1.

A proof similar to that of Theorem 2 gives the more general theorem:

THEOREM 3. *If* $2 \leqq r_1 \leqq r_2 - 1$, *then for sufficiently large values of* N, the inequality $S(r_2, N) > S(r_1, N)$ holds.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ELEMENTARY RIGOROUS TREATMENT OF THE EXPONENTIAL LIMIT

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In all elementary calculus texts seen by this writer there is a marked decrease in simplicity and rigor when irrational power, exponential, and logarithmic functions are introduced for differentiation. Since general power and exponential functions can be discussed by means of logarithmic functions, or ultimately by use of the relation

$$\lim_{h\to 0}(1+h)^{1/h} = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots, \tag{1}$$

the difficulties resolve themselves into proving (1). This note is intended to establish (1) in a simple and rigorous way, suitable at least for the better students of elementary calculus.

Let $G = |1/h|$. Then

$$(1+h)^{1/h} = \left(1 \pm \frac{1}{G}\right)^{\pm G}, \text{ where } G \to \infty \text{ as } h \to 0. \tag{2}$$

Note that

$$\left(1 - \frac{1}{G}\right)^{-G} = \left(\frac{G-1}{G}\right)^{-G} = \left(\frac{G}{G-1}\right)^{G} = \left(1 + \frac{1}{G-1}\right)^{G} > \left(1 + \frac{1}{G}\right)^{G}.$$

Therefore, if n is the last integer not exceeding G (that is, $n \leqq G < n+1$),

$$\left(1 + \frac{1}{n-1}\right)^{n+1} > \left(1 + \frac{1}{G-1}\right)^{G} = \left(1 - \frac{1}{G}\right)^{-G}$$
$$> \left(1 + \frac{1}{G}\right)^{G} > \left(1 + \frac{1}{n+1}\right)^{n},$$

and therefore, by (2),

$$\left(1 + \frac{1}{n-1}\right)^{n+1} > (1+h)^{1/h} > \left(1 + \frac{1}{n+1}\right)^{n}, \text{ where } n \to \infty \text{ as } h \to 0. \tag{3}$$

If it can be shown that both sides of (3) have the same limit

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots \qquad \text{(convergent by the ratio test)} \tag{4}$$

as $n \to \infty$, then (1) will follow immediately. Since

$$\left(1+\frac{1}{n-1}\right)^{n+1}=\left(1+\frac{1}{N}\right)^{2}\left(1+\frac{1}{N}\right)^{N}, \quad \text{with } N=n-1,$$

and

$$\left(1+\frac{1}{n+1}\right)^{n}=\left(1+\frac{1}{N}\right)^{-1}\left(1+\frac{1}{N}\right)^{N}, \text{ with } N=n+1,$$

it will be sufficient to prove that

$$\lim_{N\to\infty}\left(1+\frac{1}{N}\right)^{N}=e.$$

Using the binomial expansion, we have

$$\begin{aligned}\left(1+\frac{1}{N}\right)^{N} &= 1+\frac{N}{1!N}+\frac{N(N-1)}{2!N^2}+\cdots+\frac{N(N-1)(N-2)\cdots 1}{N!N^N}\\
&= 1+\frac{1}{1!}+\frac{\left(1-\frac{1}{N}\right)}{2!}+\cdots \qquad (5)\\
&\quad +\frac{\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{N-1}{N}\right)}{N!}.\end{aligned}$$

A typical term in (5) satisfies the inequality

$$\frac{1}{r!}>\frac{\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{r-1}{N}\right)}{r!}>\frac{1}{r!}-\frac{1}{2N(r-2)!} \quad \text{if } r>2,^* \qquad (6)$$

in proving which we use the inequalities

$$\left(1-\frac{p}{N}\right)\left(1-\frac{q}{N}\right)=1-\frac{p+q}{N}+\frac{pq}{N^2}>1-\frac{p+q}{N},$$

$$\begin{aligned}\left(1-\frac{1}{N}\right)\left(1-\frac{2}{N}\right)\cdots\left(1-\frac{r-1}{N}\right) &> 1-\frac{1+2+\cdots+(r-1)}{N}\\
&= 1-\frac{r(r-1)}{2N}.\end{aligned}$$

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* If $r=2$, then (6) is replaced by $\frac{1}{2!}>\frac{(1-1/N)}{2!}=\frac{1}{2!}-\frac{1}{2N}$.

From (5) and (6) it follows that

$$
(7)\qquad
\begin{aligned}
1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{N!} > \left(1+\frac{1}{N}\right)^N > 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{N!} \\
-\frac{1}{2N}\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(N-2)!}\right).
\end{aligned}
$$

Now as $N\to\infty$, each side of (7) approaches the limit e stated in (4), and therefore so does $(1+1/N)^N$, thus establishing (1).

The above proof has avoided the usual (not obviously valid) assumption that $\lim_{h\to 0}(1+h)^{1/h}$ can be found simply by letting h have reciprocal *positive integral values*, and that (5) converges to the exponential series since it does so termwise. It has also avoided explicit references to the theorem that a bounded increasing sequence always has a limit—a theorem which many students seem to find rather abstract.*

TANGENTS IN ELEMENTARY ANALYTIC GEOMETRY

L. S. Johnston, University of Detroit

There appears quite early in most texts in elementary analytic geometry a chapter bearing some such title as "Discussion of a Curve," or "Equations and Loci." In such a chapter are discussed symmetry, intercepts, asymptotes, and admissible or excluded values. But these items leave one with no method other than by direct plotting by which the nature of a curve at any particular point thereon may be determined, and direct plotting by points is a laborious and not always adequate method. The writer has found by frequent trial that certain very elementary and very powerful methods here are quite easily understood and applied by the average class—and, what is more, are quite provocative of interest and enthusiasm.

The concept of slope, defined by $(y-y_1)/(x-x_1)$, either is or can be introduced before the "discussion" mentioned, for it is one of the simplest and most quickly grasped concepts in the whole subject. Also, the concept of the slope of the tangent at a given point as the limit of the slope of the secant through the given point and a variable point on the curve as the variable point approaches the fixed point is assimilated quite as easily at this early stage. Then to find the slope of the tangent line at a given point, it is necessary merely to write the equation of the curve in such a way as to exhibit the ratio $(y-y_1)/(x-x_1)$ and then to calculate the limit of this ratio as x and y approach x_1 and y_1, respectively. For algebraic curves—practically the only kind discussed in such a chapter—this can be done by very simple algebraic operations.

A few examples will illustrate. We shall always write $s=(y-y_1)/(x-x_1)$ and let $m=\lim s$ as x and y approach their respective limits.

* *Editorial Note.* Is the proof rigorous? No one can tell until he knows what the reader is assumed to know. For my students in beginning calculus, who know little about limits, irrational numbers, and infinite series, the proof might appear "rigorous," but perhaps more in the sense of the climatologist than of the mathematician. E.J.M.

(i). Consider the curve $y=x^3-3x^2+2x-7$ at $(3, -1)$. We have

$$s=\frac{y+1}{x-3}, \quad y+1=x^3-3x^2+2x-6,$$

and therefore $s=x^2+2$; hence $m=11$, and the curve is tangent at $(3, -1)$ to a line whose slope is 11.

(ii). Consider the curve $y^2=x^2(a+x)/(a-x)$ at $(0, 0)$ and $(-a, 0)$.

For the point $(0, 0)$ we have $s=y/x$; we may write our equation

$$s^2=\frac{a+x}{a-x},$$

and therefore $m^2=1$. Hence the curve has a double point at $(0, 0)$ with slopes of positive and negative unity.

For the point $(-a, 0)$ we shall have $s=y/(x+a)$. Writing our equation $s^2=x^2/[(a+x)(a-x)]$, it at once appears that $s\to\infty$ as $x\to -a$, whence the curve has a vertical tangent at $(-a, 0)$.

(iii). Consider the curve $a^2y^2=x^3(2a-x)$ at $(0, 0)$. We have

$$s^2=\frac{x(2a-x)}{a^2}, \quad \text{and} \quad m^2=0,$$

whence the curve has a cusp at the origin.

For the same curve at $(2a, 0)$ we have

$$s^2=\frac{-x^3}{a^2(x-2a)},$$

and $s\to\infty$ as $x\to 2a$. Hence the curve has a vertical tangent at $(2a, 0)$.

For the same curve at (a, a) we must have

$$s=\frac{y-a}{x-a},$$

and we write the series of equations

$$y^2=\frac{x^3(2a-x)}{a^2},$$

$$y^2-a^2=\frac{x^3(2a-x)}{a^2}-a^2=\frac{-x^4+2ax^3-a^4}{a^2},$$

$$(y+a)(y-a)=\frac{(x-a)(-x^3+ax^2+a^2x+a^3)}{a^2},$$

$$s=\frac{y-a}{x-a}=\frac{-x^3+ax^2+a^2x+a^3}{a^2(y+a)},$$

and it follows at once that $m=1$.

(*iv*). Consider the curve $y^2=(x-2)^2(x^2-9)$ at $(2, 0)$. We have $s^2=x^2-9$, and $m^2=-5$. Hence the point $(2, 0)$ at once appears as an isolated point.

(*v*). Consider the curve $x^4+2x^3-x^2y-2xy^2+y^3=0$ at $(0, 0)$. Since $s=y/x$, we write our equation

$$x+2-s-2s^2+s^3=0,$$

whence

$$(s+1)(s-1)(s-2)=-x,$$

and, as $x\to 0$, we have $m=1, -1, 2$. Hence the origin is a triple point.

(*vi*). Consider the Folium of Descartes $x^3+y^3-3axy=0$ at $(0, 0)$. Since, $s=y/x$, the equation becomes

$$1+s^3-\frac{3as}{x}=0, \quad \text{or} \quad x=\frac{3as}{1+s^3};$$

we see that for $x\to 0$ we have both $s\to 0$ and $s\to\infty$. Hence the curve is tangent to both axes at the origin.

The general formula for slope is more easily handled by the definition

$$m=\lim_{\substack{x\to x_1\\ y\to y_1}}\frac{y-y_1}{x-x_1}$$

than by the usual h, k definition, and this definition avoids the confusion incident to the use of h and k in senses totally different from the conventional uses of the same symbols in translational transformation, where h and k are actual coördinates of points. For example, the function $y=3x^3-4x^2+5$ gives

$$y-y_1=3(x^3-x_1^3)-4(x^2-x_1^2),$$

whence

$$s=3(x^2+xx_1+x_1^2)-4(x+x_1)$$

and

$$m=9x_1^2-8x_1.$$

It will be noticed here that the rather troublesome binomial theorem is not used at all, and the much simpler form $(x^n-x_1^n)/(x-x_1)$ is used instead.

For implicit functions in which no term contains both variables, the method is quite as simple. For example, from

$$b^2x^2+a^2y^2=a^2b^2,$$

we have in succession

$$b^2(x^2 - x_1^2) + a^2(y^2 - y_1^2) = 0,$$

$$s = \frac{y - y_1}{x - x_1} = -\frac{b^2(x + x_1)}{a^2(y + y_1)},$$

$$m = -\frac{b^2 x_1}{a^2 y_1}.$$

Problems involving terms in which both variables appear are little if any more difficult. For example, from $x^2y^3 = a$ we have

$$x^2y^3 - x_1^2y_1^3 = 0 = x^2(y^3 - y_1^3) + y_1^3(x^2 - x_1^2).$$

Then

$$s = \frac{y - y_1}{x - x_1} = -\frac{y_1^3(x + x_1)}{x_1^2(y^2 + yy_1 + y_1^2)},$$

$$m = -2y_1/(3x_1).$$

While problems like finding the slope of the tangent to $x^2y^3 + x^3 - 3xy^2 = -81$ at $(3, -2)$ are more easily solved by first finding the general formula for the slope and substituting the coördinates of the given point, it is quite possible to solve it for the particular point in the manner already described. Writing under each term the value of that term for the given point we have

$$\begin{array}{c} x^2y^3 + x^3 - 3xy^2 = -81 \\ -72 + 27 - 36 = -81, \end{array}$$

whence in succession

$$x^2y^3 + 72 + x^3 - 27 - 3(xy^2 - 12) = 0,$$

$$x^2(y^3 + 8) - 8(x^2 - 9) + (x^3 - 27) - 3[x(y^2 - 4) + 4(x - 3)] = 0,$$

$$s = \frac{y + 2}{x - 3} = -\frac{-8(x + 3) + x^2 + 3x + 9 - 12}{x^2(y^2 - 2y + 4) - 3x(y - 2)},$$

and $m = 33/108$.

This problem was actually solved in this manner by a student of the writer's, but it must be admitted that this particular student was never a merely average student.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Superficie Razionali. By Fabio Conforto. Bologna, Nicola Zanichelli Editore, 1939–XVII. 15+554 pages. 80 Lire.

Elementary Statics. A text-book for engineers. By M. Appleby. Cambridge, The University Press, 1939. 8+164 pages. $2.25.

Mathematical Tables. Volume VII. British Association for the Advancement of Science. By W. F. Sheppard. Completed and edited by the Committee for the Calculation of Mathematical Tables. Published for the British Association at the University Press, Cambridge, 1939. 11+34 pages. $2.50.

A Supplement to Magic Squares of $(2n+1)^2$ *Cells.* By M. J. van Driel. London, Rider and Co., 1939. 31 pages.

Elementary Mathematics from an Advanced Standpoint. Geometry. By Felix Klein. Translated from the third German edition by E. R. Hedrick and C. A. Noble. New York, The Macmillan Company, 1939. 9+214 pages. $3.50.

Aspects of the Calculus of Variations. Notes by J. W. Green after lectures by Hans Levy. Berkeley, California, University of California Press, 1939. 96 pages. Mimeographed.

Portraits of Famous Philosophers who were also Mathematicians. With biographical notes by Cassius J. Keyser. New York, Scripta Mathematica, 1939. 12 folders. $3.00.

REVIEWS

Tests of Significance. What they mean and how to use them. By John H. Smith. Chicago, University of Chicago Press, 1939. 9+90 pages. $1.00.

This book of less than one hundred pages is a survey of the common tests of significance. Its purpose is to present in a simple manner an organized treatment of the theory and methods of the fundamental principles of sampling, and to show how they are applied to certain statistical problems. The author has written the main body of the book on the belief that it is far more important to clarify the type of reasoning in connection with the tests made, than it is to attempt an exposition of the rigorous mathematical techniques used in the theoretical development of the tests. Throughout the study emphasis is placed upon the uses of the various tests and their relationships. In each case the author has stated very clearly and definitely the assumptions on which the theory is developed, and takes every precaution not to lead the reader into erroneous uses of the tests.

The discussion in the text is supplemented by mathematical proofs and derivations which are presented in the Appendices. The technical methods em-

ployed are, as the author says, similar to those used by Pearson, "Student," and Fisher. The same nomenclature and symbols are employed as those used by these men.

The common tests of significance discussed are:

(1) U-test in which the areas under the normal curve are used to test the unusualness of the size of the means of samples of n variates, taken from a normally distributed universe.

(2) The chi-square test for more than one normally distributed statistic, where the standard errors are known.

(3) The F-test. Tests for sets of means using the statistical technique known as the analysis of variance in which F is called the variance ratio.

(4) The t-test, known as "Student's" t-test of unusualness for one mean, and also the t-test for regression coefficients.

(5) Fisher's Z-transformation of the coefficient of correlation: $Z = \frac{1}{2} \log (1+r) - \frac{1}{2} \log (1-r)$.

In each test presented and in the illustrations used to clarify the application of the test, the author gives a very clear statement of the problem and the statistical background for the application.

In all, it is the opinion of the reviewer that the book is well written, and would be found useful by students of statistics or those doing research work where methods of statistical analysis are employed.

C. C. WAGNER

La equacion de segundo grado a dos y tres variables. By Juan Bautista Kervor. Published by Libreria y Editorial "El Ateneo," Buenos Aires, 1938. 132 pages.

This is a study of the loci of the quadratic equation in two and three variables. By placing one or more coefficients in the general equation equal to zero, the author obtains classifications of 18 and 38 cases, respectively. The book has no table of contents, no index, and no problems for the student. A knowledge of the calculus is presupposed, but in illustrative problems the solution of cubic equations with integral coefficients and rational roots is given in detail.

L. A. DYE

Elementary Theory of Equations. By W. V. Lovitt. New York, Prentice-Hall, Inc., 1939. 11+237 pages. $2.50.

The selection of topics presented in this text follows closely the choice of Professor Dickson in his three books on the same subject. However, it is the aim of Professor Lovitt to present these topics in a manner acceptable to the undergraduate who has had no calculus and only one semester of analytic geometry. One would suppose, then, that the average student using this text would be in his sophomore year and that the calculus would parallel this course. In other words, a text of this kind enables the sophomore in mathematics to study in detail a subject ordinarily reserved for the junior or senior year.

The author has carefully given the *raison d'être* for each topic before the detailed development. In fact, throughout the book care has been taken to make the material readable and challenging to the younger students for whom it is intended. If at first glance there appears to be too much material included, it must be noticed that the author makes allowances for the personal preferences of individual instructors who may select some topics in place of others.

On the debit side:

Some terms are introduced without definition or explanation. For example, the term *polynomial*; the expressions *reducible* and *irreducible*; the symbols dy/dx and Σ.

Several illustrative figures occur without any direct reference to them in the text.

Page 20, line 9, should read $r \geqq 0$ instead of $n \geqq 0$.

Page 107, the statement of Theorem 1 omits the word *numerically* as given in the statement of the same theorem on page 40.

Page 179. Theorem XII should state $m > n$.

Certain errors in grammar were noted, but with no serious results.

There is an ample supply of problems with an emphasis on the numerical rather than the theoretical type.

HARRIET F. MONTAGUE

Descriptive Geometry. By A. V. Millar and K. G. Shiels. New York, D. C. Heath and Company, 1939. 10+187 pages (with Appendix). $2.25.

This text is the outgrowth of some twenty-five years of pioneering on the part of Dean Millar and several of his colleagues. This pioneering has been directed toward the presentation of descriptive geometry in ways conforming closely to those used in commercial drafting practice.

In order to accomplish this end, much of the drawing is done with the ground line omitted. This means that the object is shown with its relative distances from the horizontal and vertical planes. This use of relative distance does not alter the orthographic projection of the object on the particular plane in question. Aside from the fact that this omission is in accord with commercial practice, two other gains are sought; namely, concentration of the student's attention on the object itself instead of on the planes of projection, and analysis of the drawing rather than memorization of constructions. However, the ground line is shown when it is convenient to describe direction in space by its use.

Third quadrant constructions are used almost exclusively throughout the book. Again this is in accord with the work of commercial offices and many university drawing course practices.

In order to present a clearer idea of the form of the object portrayed, auxiliary views augment the top, front, and side views. These auxiliary views tend to increase the visualizing power of the student and lead him to accurate and concise constructions for the solution of the problems.

A further aid to the student is the fact that many of the problems, especially

those at the beginning of the text, are given in terms of a coördinate system. This practice of using coördinates should eliminate to a great extent the many sad experiences which the student usually has in discovering a satisfactory lay-out for the given exercises. By the time the student masters the basic ideas involved in descriptive geometry, he should be able to devise his own lay-out. The use of coördinates should therefore prove to be a valuable feature of the text.

Many readers are no doubt familiar with some of the earlier editions from which this present text has developed. I refer especially to the *Descriptive Geometry*, by A. V. Millar, E. S. Maclin, and L. J. Markwardt; Madison, Tracey and Kilgore, 1922. The present text is a revision and enlargement of the former. Several more figures have been added, problems have been changed, two additional chapters appear, and in general a fine book has been made much better. The figures merit special comment; they are beautifully and carefully prepared.

This text should appeal to colleges of engineering. It deserves special attention from those many colleges where engineering drawing and descriptive geometry are not taught because there is no engineering division. It merits the attention of mathematics teachers who offer descriptive geometry as an adjunct to the usual college major in mathematics.

G. E. Moore

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.

SOAP FILM EXPERIMENTS WITH MINIMAL SURFACES

Richard Courant, New York University*

In the development of mathematics there are several instances of valuable help rendered to mathematical theory by physical experiments, *e.g.*, imagined physical experiments with two-dimensional flows of electricity have greatly illuminated geometrical function theory, and the solution of boundary value problems for the Laplace equation and for other partial differential equations can be realized by electrical and also by statistical devices. But perhaps the most striking example of the theoretical mathematical value of simple physical experiments is the demonstration of minimal surfaces by means of soap film experiments as described in detail by the Belgian physicist Plateau,† and also used by mathematicians, particularly H. A. Schwarz.‡

The problem of finding the smallest surface bounded by a given closed contour was proposed by Euler early in the 18th century. It is easily reduced to a boundary value problem of a non-linear character for a system of partial differential equations. Solutions were given for special types of contours by Riemann, Schwarz, Weierstrass, and others in the 19th century, while general existence proofs were developed only recently by Radó, Douglas, McShane, Garnier, and Courant.

However, the Plateau experiments immediately yield solutions for very general contours. If one dips a wire forming any closed contour into a viscous liquid and then withdraws it, a film suspended in the wire will form, and this film will assume the shape of a minimal surface of least area spanned in the given contour. Thereby we assume that we may neglect the effects of gravity and other causes which interfere with the tendency of the film to attain the smallest possible area and thus the least possible value of the potential energy due to surface tension. A good recipe for such a viscous liquid, which can be easily produced, is the following:

Dissolve 10 grams of pure dry sodium oleate in 500 grams of distilled water, and mix 15 cubic units of the solution with 11 cubic units of glycerin.

Films obtained with this solution and with frames of half hard brass wire are

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* This paper is an extract from lectures and demonstrations given on different occasions, *e.g.*, at the meeting of the American Mathematical Society, March 26, 1937 in New York. Many of the models and demonstrations were devised in collaboration with Dr. I. Ritter. Attention should be called to an invited address by Professor W. T. Reid on a similar topic before the Mathematical Association in Madison, September, 1939.

† J. Plateau, Statique experimentale et theoretique des Liquides, Paris, 1873.

‡ Schwarz's Collected Papers, vol. I, *passim*.

relatively stable and last long enough for demonstrating and experimenting. However, the size of such frames should not exceed five or six inches in diameter, and of course the stability with smaller frames is much higher.

By using solutions of lacquer, instead of soap solution, one can obtain solid and to a certain degree permanent models, because lacquer, having formed the film, dries and solidifies.

With this method it is very easy to "solve" the problem simply by shaping the wire frame in the desired form. Beautiful models are obtained by polygonal wire frames formed by a sequence of edges of a regular polyhedron. In particular it is interesting to dip the whole frame of a cube into such a solution. The result is first a system of different surfaces meeting each other along lines of intersection. (If the cube is withdrawn very carefully, the result is a symmetric system of planes.) Then we have to pierce and to destroy enough of these different surfaces so that only one surface bounded by a closed polygon remains, which can be done in various ways leading to different surfaces.

But the scope and the informative value of soap film experiments with minimal surfaces is wider than these original demonstrations by Plateau. In recent years the problem of minimal surfaces has been studied when not only one but any number of contours is prescribed and when, in addition, the topological type might be more complicated, when for example, the surface is one-sided or is of a genus different from zero. Also the problem with free boundaries has been discussed. That is the problem where boundaries, or parts of boundaries, are not fixed curves, but are left free on given surfaces. Also these more general problems, which produce an amazing variety of geometrical phenomena, can be illuminated by soap film experiments. In this connection it was very useful to modify the simple experiments by making the system of wire frames or the prescribed boundary surfaces not rigid but flexible, and by studying the effect of deformations of the prescribed boundaries on the solution. Such deformations can be easily effected by attaching handles to the wire frames, or by other obvious mechanical devices.

There are essentially three general questions which should be studied in connection with the problem:

1. The existence of a solution of a given type.
2. The uniqueness of the solution.
3. The dependence of the solution on the prescribed data, particularly the question whether or not the solution depends continuously on the prescribed boundaries.

Of these questions only the first has so far been theoretically solved, while with respect to the second and third much remains to be done. However, physical experiments throw light on all three of them, and in addition suggest other mathematical questions and even their answers. This may be explained by a few typical examples.

1. If the contour is a circle we naturally obtain a plane circular disc. If we continuously deform the boundary circle we might expect that the minimal sur-

faces would always keep the topological character of a disc. However, this is not the case. If the boundary is deformed into a shape as indicated by Figure 1, the physical experiment yields a minimal surface which is no longer simply connected, but is a one-sided (non-orientable) Moebius strip. We may now start with this frame and with a soap film in the shape of a Moebius strip. By distorting the wire frame, pulling handles suitably soldered to it (Fig. 2), we reach a certain moment where suddenly the topological character of the film

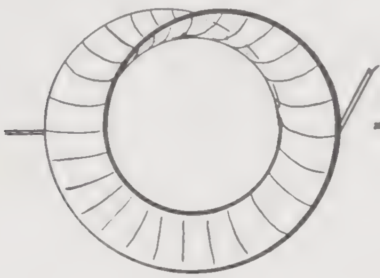

FIG. 1

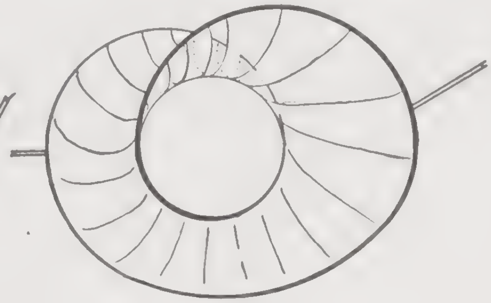

FIG. 2

changes so that the surface is again of the type of a simply connected disc (Fig. 3). Reversing this deformation process, one again obtains a Moebius strip. However, one observes with this alternating deformation process that, in the reverse process the mutation of the simply connected surface into the Moebius strip takes place at a later stage. This shows that there must be a range of shapes of the contour for which both the Moebius strip and the two-sided simply connected surface are stable (as indicated in Figures 2 and 3), one of which, of course, will furnish only a relative minimum. As a matter of fact, there always exists a two-sided simply connected surface which furnishes a relative minimum. But when the Moebius strip has a much smaller area than the two-sided surface, the latter is too unstable to be really formed.

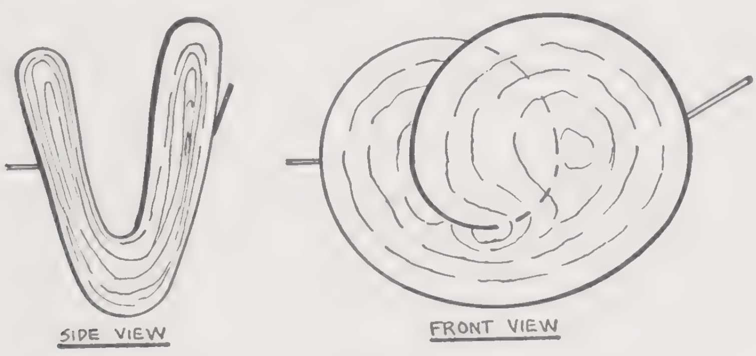

FIG. 3

2. The minimal surface of revolution spanned between two circles is very easily realized. In the actual experiment it happens that first, after the withdrawal of the wire frames, we find not one simple surface but a structure of three surfaces, one of which is a simple circular disc parallel to the prescribed

boundary circles (Fig. 4). In this case, only by destroying this intermediate surface is the classical catenoid produced. Then by pulling the two boundary circles apart one can easily demonstrate that there is a moment when the doubly connected minimal surface becomes unstable. At this moment the catenoid jumps discontinuously into the two separated boundary circles, the "Goldschmidt solution." This process is, of course, not reversible.

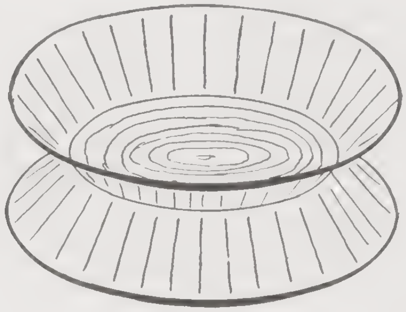

FIG. 4

3. A very instructive model is indicated by Figures 5 and 6. It is obtained from two parallel circles, such as would produce a catenoid, by cutting out two parallel segments and joining them by straight lines. Thus we have a simple closed contour. The interesting fact is that this frame produces essentially two different simply connected minimal surfaces. One is related by a slight deformation to the surface consisting of the two circular discs formed by the boundary

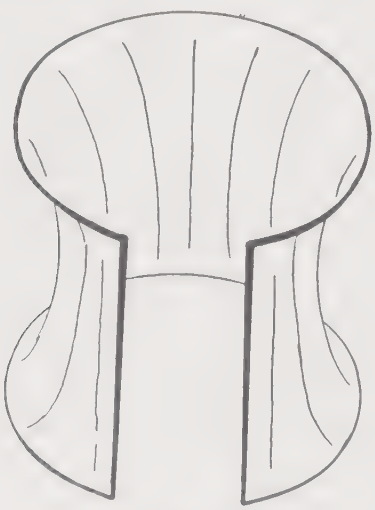

FIG. 5

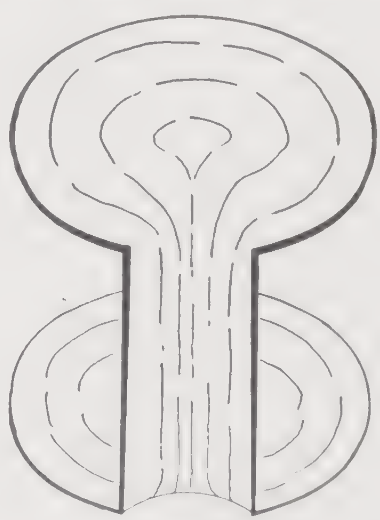

FIG. 6

circles plus the connecting small strip between the two connecting straight segments. The other is a surface obtained by deformation from the cylinder between the boundary circles, after a strip of the cylinder between the two straight lines is removed. In other words, the first corresponds to the Goldschmidt solution plus the connecting strip; the second to the catenoid minus the connecting strip. The dimensions of this frame can be chosen so that the two

different minimal surfaces described have approximately the same area; but even if the frame differs slightly from such a shape, both solutions are relatively stable positions of equilibrium. Therefore, we have here a case where the problem has two different simply connected solutions.

If we deform the frame, for example by pulling the two connecting lines apart or by pressing the opposite circular parts nearer together, we can make one of the solutions so much less stable than the other that at a certain moment of this deformation it becomes unstable and jumps into the other solution. By reverting the deformation of the frame, the transition from one solution to the other is reversed; one can repeat this alternating process very often without breaking the film. This is an instructive example of a problem for which the solution is not uniquely determined, and where the solution does not depend continuously on the data of the problem, namely, the frame.

4. Another even more significant example is provided by the frame of the Figures 7, 8, and 9. This frame again represents a simple curve. It permits the three different minimal surfaces as indicated in these diagrams. All of them are bounded by the same continuous curve; one of them (Fig. 9) has the genus 1,

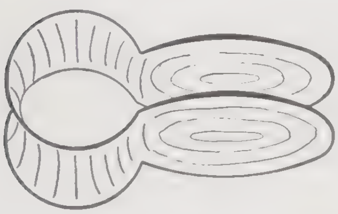

FIG. 7

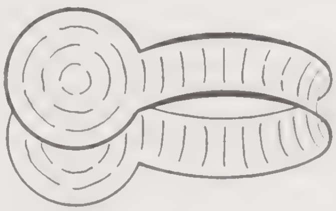

FIG. 8

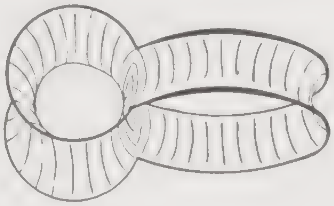

FIG. 9

while the other two are simply connected and in a way symmetrical to each other. The two latter ones have the same area if the contour is completely symmetrical. However, this need not be the case, and then only one of them gives the absolute minimum of the area while the other will give a relative minimum, provided that the minimum is sought among simply connected surfaces. The possibility of the solution of genus 1 depends on the fact that by admitting surfaces of genus 1 one may obtain a smaller minimum than by requiring that the surface shall be of genus zero or simply connected. By deforming our frame we must, if the deformation is radical enough, come to a point where the latter is no longer true. In this moment the surface of genus 1 becomes unstable and seemingly will transform itself by a discontinuous jump into a stable solution represented by one of the diagrams (Figs. 7 and 8) and simply connected. If we start with one of these simply connected solutions, as in Figure 7, we may deform it in such a way that the other simply connected solution in Figure 8 becomes much more stable. The consequence is that at a certain moment a discontinuous transition from type 7 into type 8 will take place. By reversing this deformation slowly we return to the initial position of the frame, but now with the solution of type 8 in it. Of course we can repeat the process in the other

direction, and in this way swing back and forth by discontinuous transitions between the types in these two diagrams. By careful handling one also is able to transform discontinuously either one of the simply connected solutions into that of genus 1. For this purpose, starting, *e.g.*, with the surface 7, we have to bring the disc-like parts very close to each other so that the surface of type 9 becomes markedly more stable. Sometimes in this process intermediate pieces of film appear first and have to be destroyed before the surface of genus 1 is obtained.

This example shows not only the possibility of different solutions of the same topological type, but also of another and different type in one and the same frame; moreover, it again illustrates the possibility of discontinuous transitions from one solution to another while the conditions of the problem are changed continuously.

It is very easy to construct more complicated models in a similar way and to study their behavior experimentally. This may be left to the reader.

5. A phenomenon of general interest occurs when the simply connected minimal surfaces in a single contour is necessarily self-intersecting. Such self-intersecting surfaces are always unstable because, as is easily seen, they can be replaced by other surfaces with smaller area and the same contour. Therefore in corresponding experiments the surfaces of a simple topological character will not form and instead we automatically obtain surfaces of a higher topological structure. This is the case, for example, if the given contour is knotted; a beautiful one-sided minimal surface is obtained by a frame in the form of a cloverleaf, as in Figure 10.

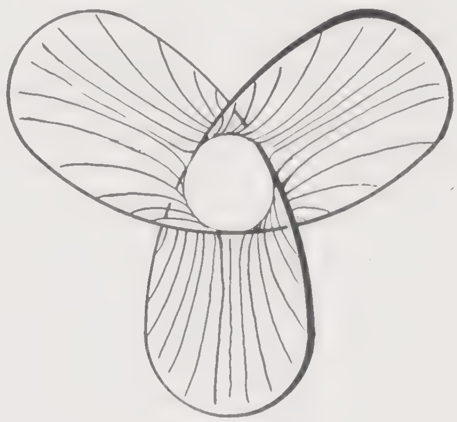

FIG. 10

Closely related to this phenomenon is the appearance of minimal surfaces bounded by two or more interlocking closed curves; for example, two circles. If we first consider the two simply connected minimal surfaces bounded by the two curves respectively, we can immediately see that they must intersect each other. Such an intersecting pair of surfaces does not give the smallest possible area. Therefore we obtain for interlocking curves doubly connected solutions bounded by the two curves. They can easily be realized by the experiment,

while it may be hard to visualize them *a priori*. Also in this experiment intermediate films will have to be pierced. Incidentally, it is recommended to move the two interlocking curves into different positions, and to study the changes and discontinuous transitions occurring. Figures 11 and 12 show two different surfaces for the same two interlocking circles in the same relative position; the different orientation of the circles with respect to the surfaces should be noticed.

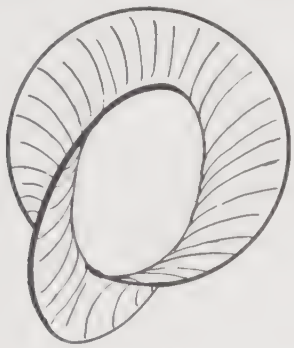

FIG. 11

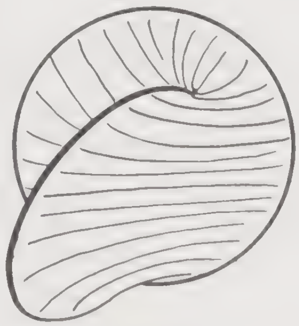

FIG. 12

6. Interesting minimal surfaces are obtained by experiments in which parts of the boundary are left free to move on a given surface S. As an example, it is recommended to use the frame indicated in Figures 13, 14, and 15. A curve joining two points on a surface S (in our diagram S is not indicated; it is supposed to be approximately a horizontal plane) is the fixed boundary, while the free boundary is on the surface. If the fixed curve stays far enough away from the surface, we obtain a simply connected minimal surface (Fig. 13). If, however, the flat part of this fixed curve comes down near enough to the sur-

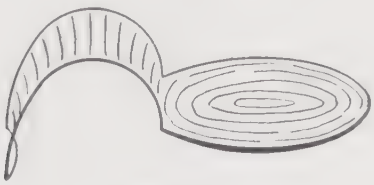

FIG. 13

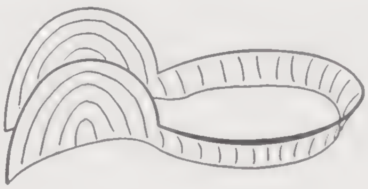

FIG. 14

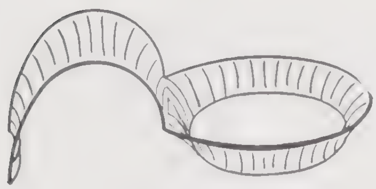

FIG. 15

(In the experiment the free boundary surface may be provided by a piece of celluloid. This surface S as well as the boundary wire should be subjected to deformations.)

face, the stable solution will be a doubly connected minimal surface which has one closed boundary line on the prescribed boundary surface besides the second boundary line on the surface, which completes the fixed boundary curve to a closed curve (Fig. 15). Both solutions can easily be shifted discontinuously into

each other. There is a second possibility of a simply connected surface (Fig. 14). Note the relationship of Figures 13, 14, and 15 to Figures 7, 8, and 9, respectively.

All these examples and many others, which the reader will easily construct himself after having some experience, illustrate a general principle which was proved in the recent theoretical investigations concerning minimal surfaces; namely, a solution of a given topological type always exists if for such a type we have greater stability (that means a smaller minimum of the area) than for lower types. For example, the Moebius strip in Example 1 exists as soon as the Moebius strip type gives a smaller area than the disc type. As regards uniqueness and continuity, no general theory has yet been developed. But it is obvious that discontinuous dependence and non-uniqueness are connected with each other.

Finally, it may be remarked that soap film experiments can serve to illustrate not only the theory of minimal surfaces but also that of other minimal problems. The spherical soap bubble, for example, is nothing but the physical realization of the isoperimetric character of the sphere. This isoperimetric problem can be generalized in the following way:

Among all surfaces through a given contour, which together with a prescribed surface in this contour include a given volume, that with least area is sought. The solution must be a surface of constant mean curvature (the minimal surfaces have mean curvature zero). Mathematically this problem is non-linear to a higher degree than that of the minimal surfaces. The theoretical solution has not yet been completed. But soap film experiments very easily "solve" such problems. The given volume simply has to be furnished by blowing air into soap bubbles which are forced into a prescribed boundary. For example, if our prescribed boundary is a square, and if the surface of the square is the prescribed boundary surface, then we may start with our cube. Dip it into the soap solution and blow up a soap bubble from inside until it fills the whole interior of the cube; then by continuing we obtain six surfaces of constant mean curvature as desired.

Interesting phenomena can be observed if we combine the original Plateau soap film experiments with the isoperimetric principle, *e.g.*, we may consider the problem: To find a minimal surface of least area bounded by k given fixed curves and one variable curve of prescribed length. In the corresponding experiments the undetermined curve may be represented by a silk thread; if we pierce the film inside of this curve, the remaining minimal surfaces must form the solution, while the silk thread takes on the optimal position. If the given boundary is plane then this position is, of course, a circle.

The analysis of the corresponding mathematical problems, as well as of problems referring to the phenomena of intermediate films, is still an open question.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 411. *Proposed by J. H. Butchart, Phillips University.*

Prove that, if the sides of a triangle form an arithmetic progression, the line joining the centroid to the incenter is parallel to one side.

E 412. *Proposed by R. A. Johnson, Brooklyn College.*

Solve the simultaneous equations

$$x_0x_1 = x_2,\ x_1x_2 = x_3,\ \cdots,\ x_{n-2}x_{n-1} = x_0,\ x_{n-1}x_0 = x_1.$$

E 413. *Proposed by H. T. R. Aude, Colgate University.*

The graph of a cubic function $y=x^3+ax^2+bx+c$ crosses the x-axis at three distinct points, two of which are A and B. The lines AP and BQ are drawn tangent to the humps of the curve, the points of contact being P and Q. Show that the ratio of the distance AB to the horizontal distance from P to Q is constant.

E 414. *Proposed by V. Thébault, Le Mans, France.*

Find a number of the form $abbbb$ whose square, diminished by unity, has ten digits, all different.

E 415. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

In a triangle of sides a, b, c, prove that the distance from the centroid G to the incenter I is given by the formula

$$3(a+b+c)GI = \left\{\Sigma a^2(b-c)^2 - \Sigma(b^2+c^2-a^2)(c-a)(a-b)\right\}^{1/2}.$$

SOLUTIONS

E 372 [1939, 168]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

The variable point Q moves on a circle through the fixed point A, and B is another fixed point in the same plane. The points R and S are the feet of the perpendiculars from A and Q on BQ and AB, respectively. The line through B, parallel to RS, meets AQ at P. Find the locus of P.

Solution by D. F. Johnson, Boston University.

The locus of P is a straight line, namely the radical axis of the given circle and the null-circle at B.

Let RS meet AQ at T. Since ARQ and ASQ are right angles, the quadrangle $ARQS$ is inscriptible; hence the angles RQA and RSA are equal, and the triangles RTQ and ATS are similar. Since PB is parallel to RS, these triangles are similar to BPQ and APB (respectively). Hence $BP/PQ=AP/PB$, *i.e.*,

$$PB^2 = PQ \cdot PA.$$

Thus the power of P is the same for the given circle as for the null-circle at B, and its locus is the radical axis.

E 373 [1939, 168]. *Proposed by David Segal, Kosow Huculski, Poland.*

Show that, for every odd positive integer n, there exists a denumerable set of number pairs, (a, b), such that a^2+b^2 is the nth power of an integer.

Solution by William Forman, Brooklyn College.

Suppose $n=2k+1$, and let x, y be any two integers. Then such a number pair is given by $a=z^kx$, $b=z^ky$, where $z=x^2+y^2$. For,

$$a^2 + b^2 = z^{2k}(x^2 + y^2) = z^{2k+1} = z^n.$$

Note. A similar proof applies when n is even. If $n=2k$, we write $a=z^{k-1}x$, $b=z^{k-1}y$, where (x, y, z) is any solution of the Pythagorean equation $x^2+y^2=z^2$.

Also solved by Richard Bellman (with various generalizations), B. A. Hausmann, and C. W. Trigg.

E 374 [1939, 168]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

What relationship exists between the sides of a triangle ABC if the bisector of angle A, the median from vertex B, and the altitude from vertex C are concurrent? Can the three sides be commensurable if the triangle is not equilateral?

Solution by C. W. Trigg, Los Angeles City College.

Let CF be the altitude from C. By the solution to E 263 [1937, 600], we have $b \cos A = AF = bc/(b+c)$. Hence

$$a^2 = b^2 + c^2 - 2bc^2/(b + c) = b^2 - c^2 + 2c^3/(b + c).$$

If the triangle has commensurable sides, and if the proper unit of measurement is chosen, a, b, c will be integers with no common factor. Moreover, b and c must be relatively prime, since any common factor of b and c would divide a. Hence $c^3/(b+c)$ cannot be an integer, and the only way to make a an integer is to put $b=c=1$, in which case the triangle is equilateral. Therefore in all other cases the three sides are incommensurable.

By putting $-c$ for c, we can make the same proof apply to the case when the *external* bisector of angle A is used.

Partially solved by W. B. Clarke, Wm. Forman, and the proposer.

E 375 [1939, 236]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

A rectangular block or beam has for its three dimensions different odd prime numbers of inches. The numbers expressing its volume and total surface area in cubic and square inches are respectively a three- and a four-place number. Find the dimensions and show that the solution is unique.

Solution by J. E. Trevor, Cornell University.

We arrange the three primes in the order $x<y<z$. The surface area is $2(yz+zx+xy)=2xyz(x^{-1}+y^{-1}+z^{-1})$, and xyz is the volume. Since the area is to exceed the volume, we have

$$x^{-1}+y^{-1}+z^{-1}>\tfrac{1}{2}.$$

To obtain the largest possible values of the sum of reciprocals, we must give x, y, z the smallest possible values, such as 3, 5, 7. Since $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}<\frac{1}{2}$, we must have $x=3$; and since $\frac{1}{3}+\frac{1}{13}+\frac{1}{17}<\frac{1}{2}$, we must have $y=5$ or 7 or 11.

Solving the inequalities

$$xyz<1000\leqq 2(yz+zx+xy)$$

for z, we obtain

$$(500-xy)/(x+y)\leqq z<1000/xy. \tag{1}$$

Putting $x=3$, we reduce this to

$$9y^2-500y+3000>0.$$

Since the larger root of the corresponding equation is obviously irrelevant, y must be less than the smaller root, namely

$$y<10(25-\sqrt{355})/9=6.8\cdots.$$

Putting $x=3$ and $y=5$ in (1), we obtain $60\frac{5}{8}\leqq z<66\frac{2}{3}$. Hence the unique solution $x=3$, $y=5$, $z=61$.

Also solved by W. E. Buker, Wm. Douglas, Wm. Forman, Henry Lacrampe, Walter Penny, G. W. Petrie, W. R. Talbot, and C. W. Trigg.

E 376 [1939, 236]. *Proposed by V. W. Graham, Harcourt Street High School, Dublin, Ireland.*

Show that if $1/[(1-x)(1-x^2)(1-x^4)(1-x^8)]$ is expanded in a series of positive powers of x, then the coefficients will run as follows:

x^{8n} and x^{8n+1} will have $(n+1)(2n+1)(2n+3)/3$,

x^{8n+2} and x^{8n+3} will have $(n+1)(n+2)(4n+3)/3$,

x^{8n+4} and x^{8n+5} will have $(n+1)(n+2)(4n+6)/3$, and

x^{8n+6} and x^{8n+7} will have $(n+1)(n+2)(4n+9)/3$.

Solution by the Proposer.

We have

$$1/[(1-x)(1-x^2)(1-x^4)(1-x^8)] = (1+x)(1+x^2)^2(1+x^4)^3(1-x^8)^{-4}$$
$$= (1+x)(1+2x^2+4x^4+6x^6+6x^8+6x^{10}$$
$$+ 4x^{12}+2x^{14}+x^{16})\cdot\sum_0^\infty\binom{n+3}{3}x^{8n}.$$

Hence the required coefficients are:

$$\binom{n+3}{3}+6\binom{n+2}{3}+\binom{n+1}{3},\qquad 2\binom{n+3}{3}+6\binom{n+2}{3},$$
$$4\binom{n+3}{3}+4\binom{n+2}{3},\qquad 6\binom{n+3}{3}+2\binom{n+2}{3}.$$

Also solved by M. J. Gottlieb, and Henry Lacrampe.

E 377 [1939, 236]. *Proposed by V. Thébault, Le Mans, France.*

Find two perfect cubes which, considered jointly, contain the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 once each. Is the solution unique?

Solution by C. W. Trigg, Los Angeles City College.

We observe that $N^3\equiv 0$, 1, or 8 (mod 9) according as $N\equiv 0$, 1, or 2 (mod 3). Since the conditions of the problem require $N^3+M^3=(N+M)(N^2-NM+M^2)\equiv 0$ (mod 9), both N and M are of the form $3k$, or else one is of the form $3k+1$ and the other of the form $3k+2$. Moreover, neither N nor M may exceed 999; neither may end with 0, 14, 42, 64, nor 92; nor may they both have the same terminal digit. These properties restrict the field of search. However, it seems most expeditious to extract from Barlow's Tables the 41 cubes whose digits are distinct, namely the cubes of 0, 1, 2, 3, 4, 5, 6, 8, 9, 12, 13, 16, 17, 18, 19, 21, 22, 24, 27, 29, 32, 35, 38, 41, 59, 66, 69, 73, 75, 76, 84, 88, 93, 97, 135, 145, 203, 289, 297, 302, and 319. By examination of these we find the unique solution to be

$$21^3 = 9261,\qquad 93^3 = 804357.$$

The following by-products are also obtained:

I. No N exists for which N^3 contains the ten digits, or the nine digits excluding zero, once each.

II. There are three pairs of cubes which contain the nine digits excluding zero, once each:

$$2^3 = 8,\qquad 289^3 = 24137569;$$
$$2^3 = 8,\qquad 319^3 = 32461759;$$
$$5^3 = 125,\qquad 76^3 = 438976.$$

III. There is no set of three, but there is a set of four cubes containing the ten digits once each, namely

$$1^3 = 1, \qquad 2^3 = 8, \qquad 4^3 = 64, \qquad 59^3 = 205379.$$

IV. There is no N such that N and N^3 together contain the ten digits, or the nine digits excluding zero, once each.
V. There are but three pairs of cubes, each composed of the same digits: 289^3 and 319^3; $35^3=42875$ and $38^3=54872$; $5^3=125$ and $8^3=512$.
VI. There is but one cube which is a permutation of consecutive digits: $203^3=8365427$.
VII. There are four N's such that the digits of N and N^3 are distinct: $2^3=8$, $3^3=27$, $8^3=512$, $27^3=19683$.
VIII. There are two ninth powers with distinct digits: 2^9 and 3^9.

Also solved by W. E. Buker, and the proposer.

E 378 [1939, 297]. *Proposed by J. L. Brenner, University of Minnesota.*

Find the number of integral values of B which make B^2+m a perfect square, for any given, fixed, integer m.

Solution by C. W. Trigg, Los Angeles City College.

If $B^2+m=A^2$, then $m=rs$ where $r=A+B$ and $s=A-B$. Since $B=(r-s)/2$, r and s must be both odd or both even. Since the values of B include negative integers and zero, no restrictions need be placed on the relative values of r and s except that they be complementary factors of m. If $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, where the p's are distinct primes and the α's are positive integers, then every divisor of m is included once and only once among the terms of the expansion of

$$(1+p_1+p_1^2+\cdots+p_1^{\alpha_1})(1+p_2+p_2^2+\cdots+p_2^{\alpha_2})\cdots (1+p_n+p_n^2+\cdots+p_n^{\alpha_n}).$$

It follows that, if m is odd, the number of divisors of m, and hence the required number of integral values of B, will be

$$(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_n+1).$$

If m is even and $p_1=2$, then r and s must both be even; so the number of integral values of B for which B^2+m is a perfect square is then the number of divisors of $m/4$, namely

$$(\alpha_1-1)(\alpha_2+1)\cdots(\alpha_n+1).$$

Also solved by W. R. Talbot and V. Thébault.

E 379 [1939, 297]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Find a trapezoid whose sides, altitude, diagonals, and area are rational.

Solution by the Proposer.

Let AC and BD be the parallel sides of a trapezoid $ABDC$ whose sides, alti-

tude, and diagonals are integers. We write $c_1=AB$, $c_2=BC$, $c_3=CD$, $c_4=DA$, and denote the projections of these lines on BD by b_1, b_2, b_3, b_4, with appropriate signs, so that

$$b_1+b_2+b_3+b_4=0. \tag{1}$$

Then $AC=b_1+b_2$, $BD=b_2+b_3$, and the altitude a is given by

$$a^2=c_1^2-b_1^2=c_2^2-b_2^2=c_3^2-b_3^2=c_4^2-b_4^2.$$

The c's and a are integers by definition, and the b's also are integers, since otherwise they would be simple quadratic surds, and then $b_1+b_2(=-b_3-b_4)$ could not be an integer. Thus the c's are the hypotenuses of four Pythagorean triangles with a common side a, the remaining sides (made negative when necessary) satisfying (1) (with $b_1+b_2\neq 0$, $b_2+b_3\neq 0$). If a is odd, all the b's are even; hence the area $a(AC+BD)/2$ is always an integer.

Since (1) is satisfied by $b_1=9$, $b_2=96$, $b_3=-30$, $b_4=-75$, the four triangles

$$(40,\ 9,\ 41),\qquad (40,\ 96,\ 104),\qquad (40,\ 30,\ 50),\qquad (40,\ 75,\ 85)$$

give rise to a trapezoid having parallel sides $b_1+b_2=105$, $b_2+b_3=66$, other sides $c_1=41$, $c_3=50$, altitude $a=40$, diagonals $c_2=104$, $c_4=85$, and area 3420. Other trapezoids can be derived from the same triangles by taking them in a different order.

It is permissible for one of the b's to vanish. If $b_2=0$ or $b_4=0$, one diagonal is perpendicular to the bases. If $b_1=0$ or $b_3=0$, one side is perpendicular to the bases. A general formula for rectangular trapezoids has been given by Germain Clarux (*Sphinx*, 1932, p. 31). This amounts to putting

$$\begin{aligned} a &= 4pq(p^4-q^4), \qquad b_1=0,\\ b_2 &= (p^3-p^2q+3pq^2+q^3)(p^3+3p^2q-pq^2+q^3),\\ b_3 &= (p^3+p^2q+3pq^2-q^3)(p^3-3p^2q-pq^2-q^3),\\ b_4 &= -2(p^2-q^2)(p^4-q^4). \end{aligned}$$

Editorial Note. Although we cannot have $b_1=b_3$ without the trapezoid reducing to a parallelogram, there is no objection to taking $b_1+b_3=0$ (and therefore $b_2+b_4=0$). In fact, a symmetrical (or isosceles) trapezoid, having parallel sides $b_2\pm b_1$, other sides both c_1, altitude a, diagonals both c_2, and area ab_2, occurs whenever there are two Pythagorean triangles $(a,\ b_1,\ c_1)$ and $(a,\ b_2,\ c_2)$ with $b_1<b_2$. The integral trapezoid of smallest altitude, arising from triangles (8, 6, 10) and (8, 15, 17), has parallel sides 9 and 21, other sides 10, altitude 8, diagonals 17, and area 120. The integral trapezoid of smallest area, arising from triangles (12, 5, 13) and (12, 9, 15), has parallel sides 4 and 14, other sides 13, altitude 12, diagonals 15, and area 108.

Even if we stipulate that $b_1+b_3\neq 0$, so as to rule out symmetry, it is still not necessary for more than three of the four Pythagorean triangles to be different;

any two consecutive b's may be equal. For instance, when $a=15$, we have three triangles

$$(15, 8, 17), \qquad (15, 20, 25), \qquad (15, 36, 39),$$

which give rise to a trapezoid since $8+8+20-36=0$. Putting $b_1=b_2=8$, $b_3=20$, $b_4=-36$, we find parallel sides 16 and 28, other sides 17 and 25, altitude 15, diagonals 17 and 39, and area 330.

Readers may be interested to compare the various asymmetrical trapezoids of altitude 24 which arise from the triangles

$$(24, 7, 25), \qquad (24, 18, 30), \qquad (24, 32, 40), \qquad (24, 45, 51), \qquad (24, 70, 74),$$

observing that $7+7+18-32=7+18+45-70=7-32-45+70=0$.

A general formula, along the lines suggested by Clarux, will still be welcomed by this department.

In a solution received just before going to press, Free Jamison points out that the problem is equivalent to that of finding four rational numbers whose sum is equal to the sum of their reciprocals. In fact, for any Pythagorean triangle (a, b, c) we have $2b/a=k-k^{-1}$, where k is rational; hence (1) reduces to

$$k_1 + k_2 + k_3 + k_4 = k_1^{-1} + k_2^{-1} + k_3^{-1} + k_4^{-1},$$

where the k's are unrestricted (apart from their rationality, and certain inequalities). For instance, the above trapezoid of altitude 15 is given by

$$k_1 = k_2 = \tfrac{5}{3}, \qquad k_3 = 3, \qquad k_4 = \tfrac{1}{5}.$$

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at the least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3947. *Proposed by N. A. Court, University of Oklahoma.*

If M, M' are two isogonal conjugate points for the tetrahedron $DABC$, S the projection of M upon the plane ABC, and S' the point common to the planes perpendicular to the lines $M'A$, $M'B$, $M'C$ at the points A, B, C, show that the line SS' and its three analogs PP', QQ', RR' have a point in common.

3948. *Proposed by Michael Goldberg, Washington, D. C.*

Suppose that n slotted discs are freely mounted on the same axis. If the portion of the circumference subtended by the slot of the ith disc is p_i, show that

the probability that light, parallel to the axis, can pass through the slots is

$$p_1p_2 \cdots p_n\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n}\right),$$

provided that $p_i + p_j \leqq 1$ for every i and j.

3949. *Proposed by P. Turán, Budapest, Hungary.*

Given $0 \leqq \phi_1 < \phi_2 < \cdots < \phi_n < 2\pi$, show that there exists an angle β with the properties: $\beta \geqq \pi/2^{(n/2)+1}$, and there exist no integers k and ν such that $\phi_\nu + \beta < \phi_k < \phi_\nu + 2\beta$ or $\phi_\nu - 2\beta < \phi_k < \phi_\nu - \beta$.

3950. *Proposed by P. Turán, Budapest, Hungary.*

Given

$$(1 + z + z^2 + \cdots + z^k)^n = c_0^{(k,n)} + c_1^{(k,n)} z + \cdots + c_{kn}^{(k,n)} z^{kn},$$

show that for kn odd

$$c_0^{(k,n)} \leqq c_1^{(k,n)} \leqq \cdots \leqq c_{(kn-1)/2}^{(k,n)} = c_{(kn+1)/2}^{(k,n)} \geqq c_{(kn+3)/2}^{(k,n)} \geqq \cdots \geqq c_{kn}^{(k,n)},$$

and for kn even

$$c_0^{(k,n)} \leqq c_1^{(k,n)} \leqq \cdots \leqq c_{kn/2}^{(k,n)} \geqq c_{(kn/2)+1}^{(k,n)} \geqq \cdots \geqq c_{kn}^{(k,n)}.$$

3951. *Proposed by V. V. Johnston, Ellwood City, Pa.*

Does the equation $m^3 + 3m^2 + 2m = 2n^3 + 3n^2 + n$ admit positive integral solutions in m and n other than $m = n = 1$?

SOLUTIONS

3856 [1938, 53]. *Proposed by J. H. M. Wedderburn, Princeton University.*

If the fundamental units h_{ij} $(i, j, = 1, 2, \cdots, n)$ of an algebra have the law of combination $h_{ij}h_{pq} = k_{jp}h_{iq}$, where k_{jp} is a scalar, the algebra is associative; and a necessary and sufficient condition that it shall be equivalent to the algebra of matrices of order n is that the matrix $K = \|k_{jp}\|$ is non-singular, or that it contains an identity element.

Solution by John Williamson, The Johns Hopkins University.

Since $h_{ij}h_{pq} = k_{jp}h_{iq}$, $(h_{ij}h_{pq})h_{rs} = k_{jp}h_{iq}h_{rs} = k_{jp}k_{qr}h_{is}$ and $h_{ij}(h_{pq}h_{rs}) = h_{ij}k_{qr}h_{ps} = k_{qr}k_{jp}h_{is}$. Hence the algebra is associative.

Let the algebra contain a unit element

$$e = e_{\alpha\beta}h_{\alpha\beta}, \tag{1}$$

where a repeated Greek suffix denotes summation from 1 to n. Since $eh_{pq} = h_{pq}$, $(p, q = 1, 2, \cdots, n)$,

$$e_{\alpha\beta}h_{\alpha\beta}h_{pq} = e_{\alpha\beta}k_{\beta p}h_{\alpha q}. \tag{2}$$

As the elements h_{pq} form a basis of the algebra, (2) is equivalent to

$$e_{p\beta}k_{\beta q} = \delta_{pq}, \qquad (p, q = 1, 2, \cdots, n), \tag{3}$$

where δ_{pq} is the Kronecker δ. But (3) implies that the matrix (e_{ij}) is the reciprocal of K so that K is non-singular. Conversely, if K is non-singular, and $(e_{ij}) = K^{-1}$, the element e defined by (1) is a left unit. Further, since

$$k_{p\beta}e_{\beta q} = \delta_{pq}, \qquad (p, q = 1, 2, \cdots, n),$$

$$h_{pq}e = h_{pq},$$

so that e is also a right unit. Hence the algebra contains the unit element e. The n^2 elements $\epsilon_{ij} = e_{\alpha i}h_{\alpha j}$ form a basis for the algebra since the determinant of the coefficient matrix has the value $[K^{-1}]^n$ and therefore is not zero. Since

$$\begin{aligned} \epsilon_{ij}\epsilon_{rs} &= e_{\alpha i}h_{\alpha j}e_{\beta r}h_{\beta s} = e_{\alpha i}e_{\beta r}k_{j\beta}h_{\alpha s} \\ &= \delta_{jr}e_{\alpha i}h_{\alpha s} \\ &= \delta_{jr}\epsilon_{is}, \end{aligned}$$

the algebra is equivalent to the algebra of all matrices of order n.

The necessity of the condition follows immediately from the fact that a total matrix algebra contains a unit element.

Solved also by S. A. Jennings, and the proposer.

Editorial Note. It appears from the above solution that the first statement in the problem should contain some kind of restriction upon the k_{ij} such as belonging to a *commutative* ring R. The second and last statement appears to mean what is called a total matrix algebra where R must be a field. The above solution is with this meaning.

The proposer's solution contains the remark that, if none of the k_{ij} is zero, the result gives another solution of 3700 [1936, 378]. See the article by the proposer entitled *A special linear associative algebra*, in the Proceedings of the Edinburgh Mathematical Society, ser. 2, vol. 5, Part IV, p. 169.

3857 [1938, 53]. *Proposed by V. Thébault, Le Mans, France.*

In a triangle the minor auxiliary circle of the Brocard ellipse is tangent to the nine-point circle.

Note. The Brocard ellipse for a triangle is tangent to its sides and has for its foci the Brocard points of the triangle.

I. *Solution by R. Goormaghtigh, Bruges, Belgium.*

The property considered here is a special case of the following well known generalization of Feuerbach's theorem: When, in a triangle, two counter-points are on the same circumdiameter, their common pedal circle touches the nine-point circle at the orthopole of that diameter.

The Brocard points, real foci of the Brocard ellipse, being symmetric with

respect to the Brocard diameter, the imaginary foci of that ellipse lie on that diameter and therefore their common pedal circle, *i.e.*, the minor auxiliary circle of the considered ellipse, touches the nine-point circle at the orthopole of the Brocard diameter.

On the general theorem here mentioned, see Weill, *Nouvelles Annales de Mathématiques*, 1880, p. 250; Mac Cay, *Irish Academy*, 1889; Fontené, *Nouvelles Annales de Mathématiques*, 1906, p. 505; the locus of the considered pairs of counter-points is Mac Cay's cubic. About these theorems, see also our paper on the orthopole, *Tôhoku Mathematical Journal*, 1926, pp. 77–125.

II. *Solution by Otto J. Ramler, Catholic University of America.*

We shall base our proof on a comparison of the sum or difference of the radii of the two circles with the distance between their centers.

Let O be the circumcenter, K the symmedian point, S and S' the Brocard points of the triangle ABC, and ω its Brocard angle. Then, as is well known, S and S' lie on the Brocard circle whose diameter is OK, and situated symmetrically with respect to OK. Let SS' intersect OK in M. Then M is the center of the Brocard ellipse and of its minor auxiliary circle. If p and q are the radii of the major and minor auxiliary circles respectively, we have

$$p^2 - q^2 = \overline{MS}^2.$$

Now p is the circumradius of the pedal triangle PQR of S, and since this triangle is similar to ABC we have

$$p/R = PR/AB = \sin \omega$$

(Johnson, *Modern Geometry*, §441).

Moreover, the area of PQR is proportional to the power of S with regard to the circumcircle of ABC (Johnson, §198). Hence, letting F be the area of PQR, and Δ the area of ABC (Johnson, §447b),

$$R^2 - \overline{OS}^2 = 4FR^2/\Delta = 4R^2 \sin^2 \omega.$$

Hence $\overline{OS}^2 = R^2(1-4 \sin^2 \omega)$ and (Johnson, §447c),

$$MS = SS'/2 = R \sin \omega \sqrt{1 - 4 \sin^2 \omega}.$$

Whence the radius of the minor auxiliary circle of the Brocard ellipse is given by $q^2 = p^2 - \overline{MS}^2 = R^2 \sin^2 \omega - R^2 \sin^2 \omega(1-4 \sin^2 \omega) = 4R^2 \sin^4 \omega$. And hence $q = 2R \sin^2 \omega$.

Now by Aiyar's theorem (Gallatly, *Modern Geometry of the Triangle*, §108), the product of the distances of two isogonal points from the circumcenter is equal to the product of the circumdiameter and the distance of the center of the nine-point circle from the center of the inscribed conic having the isogonal points as foci. Hence we may write $\overline{OS}^2 = 2R \cdot NZ$, where N and Z are the centers of the nine-point circle and conic, respectively. Whence we readily obtain

$\overline{NZ}=\overline{OS}^2/2R=R^2(1-4\sin^2\omega)/2R=R/2-2R\sin^2\omega=R/2-q$. Hence the minor auxiliary circle is tangent internally to the nine-point circle.

Editorial Note. The proposer stated that the theorem of the problem is a particular case of the following generalization of Weill's theorem, and that he expects to publish a simple synthetic proof of the generalization in *Mathesis*:

If a conic is inscribed in a triangle so that one of its principal axes passes through the circumcenter of the triangle, the auxiliary circle for that axis is tangent to the nine-point circle.

We shall give a direct determination of the value of q in Ramler's solution, since it gives the absolute normal coördinates of the Brocard points, and then OS is easily found. If S and S' are any pair of isogonal conjugates with respect to triangle ABC; if r is the radius of the common pedal circle of S and S', and M is its center; and if (x, y, z), (x', y', z') are the absolute normal coördinates of S, S'; then from the solution of 3658 [1935, 258] we have $xx'=yy'=zz'=r^2-c^2$, where $c=SM$. In Ramler's solution the right side of the above equation is denoted by q^2, and r is the same as p. In order to determine q when S, S' are Brocard points, we consider a figure similar to the one in Johnson, p. 266. Let $B\overline{C}$ be parallel to CA cutting the tangent at A to the circumcircle (O) of ABC in $\overline{C}$; then it is easily shown that $C\overline{C}$ cuts the circle determined by A, $\overline{C}$, B in the Brocard point S. Let this circle be denoted by (O_c) with the center O_c and radius r_c. The triangles ABC, $B\overline{C}A$ are similar, and hence $r_c/R=AB/CA=c/b$. Also $\angle BO_cS=2\angle B\overline{C}S=2\angle BAS=2\omega$. Then $BS=2r_c\sin\omega=2R\sin\omega(c/b)$, and $x=2Rc\sin^2\omega/b$; and similarly for y, z, and x', y', z'. Thus the absolute normal coördinates for S, S', and the value of q are

$$2R\sin^2\omega(c/b),\qquad 2R\sin^2\omega(a/c),\qquad 2R\sin^2\omega(b/a);$$
$$2R\sin^2\omega(b/c),\qquad 2R\sin^2\omega(c/a),\qquad 2R\sin^2\omega(a/b);$$
$$q = 2R\sin^2\omega.$$

Let BS cut (O) again in β; then $BS\cdot S\beta=R^2-(OS)^2$. Since βCS and ABC are similar, $S\beta/b=SC/a=2R\sin\omega/c$. Hence $BS\cdot S\beta=4R^2\sin^2\omega$, and $(OS)^2=R^2-4R^2\sin^2\omega$. Since we get the same result for $(OS')^2$, we have $OS=OS'$. Then the Aiyar theorem finishes the proof without the use of p.

3862 [1938, 122]. *Proposed by V. Thébault, Le Mans, France.*

Through the vertices of a triangle ABC perpendiculars to its plane are drawn

$$AA_1=-AA_2=BC,\qquad BB_1=-BB_2=CA,\qquad CC_1=-CC_2=AB.$$

Show that the points A_1, B_1, C_1, A_2, B_2, C_2 and the vertices of the triangle anticomplementary to ABC lie on the same sphere.

Solution by L. M. Kelly, Northeastern Univ., Boston, Mass.

Let A', B', C' be the anticomplementary triangle of the given triangle ABC, and let H be the orthocenter of ABC. Then a sphere with center H and radius

HC' will pass through the nine points mentioned in the problem. For $AC'=CB=AA_1$. Hence the two right triangles HAC' and HAA_1 are congruent, and thus $HA_1=HC'$. Similarly, the rest of the nine points are at the same distance from H.

3863 [1938, 189]. *Proposed by S. B. Townes, University of Oklahoma.*

The vertices of a simplex in n dimensions are $O, P_1, P_2, \cdots, P_n$. Let $OP_i=(a_{ii})^{1/2}$ and the cosine of the angle between OP_i and OP_j be $a_{ij}/(a_{ii}a_{jj})^{1/2}$, $(i, j=1, 2, \cdots, n)$. Show that r_n, the radius of the circumscribed hypersphere, is given by

$$r_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ii}a_{jj}A_{ij}/4\,|a_{ij}|,$$

where A_{ij} is the cofactor of a_{ij} in the symmetric determinant $|a_{ij}|$. This result in different notation, for $n=2$, appears in (10) page 160 of Dickson's *Studies in the Theory of Numbers.*

Solution by H. S. M. Coxeter, University of Toronto.

Let $l_{ij}(=l_{ji})$ denote the edge P_iP_j of the simplex $P_0P_1\cdots P_n$, so that, identifying P_0 with O,

$$l_{ij}^2 - l_{i0}^2 - l_{0j}^2 = -2l_{i0}l_{0j}\cos(P_iOP_j) = -2a_{ij}.$$

By a known formula* for the circumradius in terms of the edges,

$$-2r_n^2 = \begin{vmatrix} 0 & l_{01}^2 & l_{02}^2 & \cdots & l_{0n}^2 \\ l_{10}^2 & 0 & l_{12}^2 & \cdots & l_{1n}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_{n0}^2 & l_{n1}^2 & l_{n2}^2 & \cdots & 0 \end{vmatrix} \div \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & l_{01}^2 & l_{02}^2 & \cdots & l_{0n}^2 \\ 1 & l_{10}^2 & 0 & l_{12}^2 & \cdots & l_{1n}^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & l_{n0}^2 & l_{n1}^2 & l_{n2}^2 & \cdots & 0 \end{vmatrix}.$$

When the first row and column are subtracted from every subsequent row and column, respectively, the former determinant becomes

$$\begin{vmatrix} 0 & a_{11} & a_{22} & \cdots & a_{nn} \\ a_{11} & -2a_{11} & -2a_{12} & \cdots & -2a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{nn} & -2a_{1n} & -2a_{n2} & \cdots & -2a_{nn} \end{vmatrix} = -(-2)^{n-1}\Sigma\Sigma a_{ii}a_{jj}A_{ij}.$$

Similarly, when the second row and column are subtracted from every subsequent row and column, respectively, the latter determinant becomes

* Mathematical Gazette, December, 1930.

$$\begin{vmatrix} 0 & 1 & 0 & 0 & & 0 \\ 1 & 0 & a_{11} & a_{22} & \cdots & a_{nn} \\ 0 & a_{11} & -2a_{11} & -2a_{12} & \cdots & -2a_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & a_{nn} & -2a_{n1} & -2a_{n2} & \cdots & -2a_{nn} \end{vmatrix} = -(-2)^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Hence

$$-2r_n^2 = (\Sigma\Sigma a_{ii}a_{jj}A_{ij})/(-2|a_{ij}|),$$

as required.

Editorial Note. The proposer stated that his proof was obtained by induction using a rectangular coördinate system.

The desired formula is an almost immediate consequence of the results given in the *Note* to the Solution of 3752 [1937, 404]. Let $\mathbf{a}_i$ be the vector of the vertex P_i, $(i=1, 2, \cdots, n)$, with the origin at the remaining vertex O of the simplex S; in the reference the letter $\mathbf{b}_i$ is used in place of $\mathbf{a}_i$. Let $\mathbf{c}$ be the vector of the circumcenter of S. We then have directly from the reference $2\mathbf{c} = \Sigma\mathbf{a}_i^2\mathbf{a}_i'$; and hence $4r_n^2 = \Sigma\Sigma\mathbf{a}_i^2\mathbf{a}_j^2\mathbf{a}_i'\cdot\mathbf{a}_j'$. Then by (1) $|\mathbf{a}_i\cdot\mathbf{a}_j| = V^2$, and the cofactor A_{ij} of $\mathbf{a}_i\cdot\mathbf{a}_j$ in this determinant is given by $A_{ij} = V^2\mathbf{a}_i'\cdot\mathbf{a}_j'$, see the first line after (3). Finally

$$4V^2r_n^2 = \Sigma\Sigma\mathbf{a}_i^2\mathbf{a}_j^2A_{ij},$$

which is the desired result.

In order to show the relation of this result to the solution above, we write it in determinant form

$$4|\mathbf{a}_i\cdot\mathbf{a}_j|r_n^2 = -\begin{vmatrix} 0 & \mathbf{a}_1^2 & \mathbf{a}_2^2 & \cdots & \mathbf{a}_n^2 \\ \mathbf{a}_1^2 & \mathbf{a}_1\cdot\mathbf{a}_1 & \mathbf{a}_1\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{a}_n \\ \mathbf{a}_2^2 & \mathbf{a}_2\cdot\mathbf{a}_1 & \mathbf{a}_2\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_2\cdot\mathbf{a}_n \\ \cdot & & & & \\ \cdot & & & & \\ \mathbf{a}_n^2 & \mathbf{a}_n\cdot\mathbf{a}_1 & \mathbf{a}_n\cdot\mathbf{a}_2 & \cdots & \mathbf{a}_n\cdot\mathbf{a}_n \end{vmatrix}.$$

Using now on each side the simple determinant reductions of the above solution in reverse order, we easily obtain the formula with which that solution begins.

3864 [1938, 189]. *Proposed by H. D. Grossman, New York City.*

Prove that the condition that $ax^2+bx\pm c$ are both factorable, where a, b, c are integers whose G. C. D. is unity, is $ac=rs(r^2-s^2)$, $\pm b=r^2+s^2$, where r and s are relatively prime integers.

Solution by W. V. Parker, Louisiana State University.

The condition is obviously sufficient since

$$b_2 \pm 4ac = (r^2+s^2)^2 \pm 4rs(r^2-s^2) = (r^2-s^2 \pm 2rs)^2.$$

The condition is not necessary, for both expressions are factorable if $b=k(r^2+s^2)$ and $ac=k^2rs(r^2-s^2)$. For example

$$2x^2 + 15x + 27 = (2x + 9)(x + 3),$$
$$2x^2 + 15x - 27 = (2x - 3)(x + 9),$$

but 15 is not expressible as the sum of two squares. When $b=r^2+s^2$ it is not necessary that r and s be relatively prime. For example

$$15x^2 + 52x + 32 = (3x + 8)(5x + 4),$$
$$15x^2 + 52x - 32 = (15x - 8)(x + 4),$$

and $52=4^2+6^2$. If a, b, c are prime each to each, the conditions as stated are necessary for then we must have $k=\pm 1$.

Solved also by E. P. Starke.

Editorial Note. After the first sentence in the above proof, we may add for greater precision that the given two quadratic expressions are the products of linear factors with integral coefficients. This was shown by Starke, and he also proved that r and s need not be relatively prime.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Dr. J. E. Eaton has been promoted to an assistant professorship at Hofstra College.

The retirement of Professor Arnold Emch of the University of Illinois is announced.

Dr. W. K. Feller of the University of Stockholm has been appointed to a lectureship at Brown University.

Dr. D. G. Fulton of Armour Institute of Technology has been appointed to an assistant professorship at Ohio Northern University.

Professor W. C. Graustein of Harvard University has been appointed assistant dean of the faculty of arts and sciences.

Assistant Professor J. A. van Groos of Oregon State College has been promoted to an associate professorship.

Professor E. H. Hadlock, formerly of Chulalonghorn University, Siam, has been appointed to an assistant professorship at Boston University.

Dr. Witold Hurewicz has been appointed to an assistant professorship at the University of North Carolina.

After a two-year leave of absence for work with the Home Owners' Loan Corporation, Assistant Professor W. J. Kirkham has returned to Oregon State College.

Dr. M. S. Macphail of Acadia University has been promoted to an assistant professorship.

J. C. Oxtoby has been appointed to an assistant professorship at Bryn Mawr College.

Dr. H. V. Park of North Carolina State College has been promoted to an assistant professorship.

Professor R. D. Perry of State Teachers College, Bowling Green, Kentucky, has been given a leave of absence for six months to serve as chief statistician at Baylor University.

Professor H. R. Phalen, formerly of Bard College, is at Brown University for the current academic year.

Professor G. Y. Rainich and Assistant Professor L. J. Rouse of the University of Michigan are on leave of absence for the second semester of 1939–40.

Assistant Professor Francis Regan of St. Louis University has been promoted to an associate professorship.

Dr. M. S. Robertson of Rutgers University has been promoted to an assistant professorship.

The following appointments to instructorships are announced:

Baldwin-Wallace College: E. R. Stabler
Berkeley Institute, Brooklyn, New York: Miss Melita A. Holly
Blue Ridge College: M. E. Terry
Brooklyn College: Dr. Hyman Serbin
Brown University: O. H. Schmidt
University of California: Dr. A. P. Morse
Catholic University of America: Dr. Joseph Daly
Chicago Teachers College: Ralph Mansfield
College of the City of New York: Dr. Max Shiffman
Compton Junior College, Compton, California: R. W. Rector
Duke University: A. V. Martin
Elmhurst College: Miss Mary M. Handel
Georgia School of Technology: C. L. Carroll
Herzl Junior College, Chicago: Dr. B. H. Gere
Hunter College: F. C. Hall
Los Angeles City College: Dr. C. P. Brady
University of Minnesota: Dr. Isaac Opatowski
University of Nebraska: Dr. D. H. Rock
University of Notre Dame: Dr. A. O. Linstrom

Oregon State College: Dr. C. B. Smith, Dr. Andrew Sobczyk
Pennsylvania State College at Altoona: E. H. Umberger
Pennsylvania State College at Pottsville: R. B. Kleinschmidt
Pennsylvania State College at Uniontown: Dr. C. E. Sealander
University of Rochester: H. P. Atkins
Swarthmore College: Dr. Herbert Buseman
University of Tennessee: D. M. Seward
Agricultural and Mechanical College of Texas: Dr. G. L. Gross
Tulane University: Dr. M. L. Kales
University of Utah: W. H. Myers
Wright Junior College, Chicago: Dr. W. H. Erskine
Yale University: Dr. S. A. Jennings, J. C. Montgomery, R. R. Stoll, W. G. Swope.

The death of Dr. Charles Hopkins of Tulane University on September 15, 1939 is reported.

The death of Dr. W. C. Mitchell of George Washington University is reported.

Professor A. W. Smith of Colgate University, a charter member of the Mathematical Association, died February 11, 1940. He had taught at Colgate since 1902 and was head of the department since 1920.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va., April 20.
Illinois, Bloomington, May 10–11.
Indiana, Richmond.
Iowa, Mt. Vernon, April 19–20.
Kansas, Wichita, March 30.
Kentucky, Lexington, April 27.
Louisiana-Mississippi, Oxford, Miss., March 8–9.
Maryland-District of Columbia-Virginia, Richmond, Va., May 11.
Michigan, Ann Arbor, April 26–27.
Minnesota
Missouri, Warrensburg, April 19–20.
Nebraska, Omaha, May 9–11.
Northern California, Berkeley, January 27.
Ohio, Columbus, April 4.
Oklahoma
Philadelphia, November 23 or 30.
Rocky Mountain, Fort Collins, Colo., April 19.
Southeastern, Athens, Ga., March 29–30.
Southern California, Compton, March 2.
Southwestern, Tucson, Ariz., April 22–23.
Texas, Dallas, March 29–30.
Wisconsin, Milwaukee.

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics.

THE ANNUAL MEETING OF THE NORTHERN CALIFORNIA SECTION

The second annual meeting of the Northern California Section of the Mathematical Association of America was held at the University of California, Berkeley, California, on Saturday, January 27, 1940. Mr. A. L. McCarty, chairman of the Section, presided at both morning and afternoon sessions, with Professor Sophia H. Levy, vice-chairman, taking the chair for a short time during the afternoon session. During the noon recess luncheon was served for members and visitors at the Men's Faculty Club.

The attendance at the two sessions was approximately one hundred ten, including the following twenty members of the Association: H. M. Bacon, T. J. Bass, Jr., B. A. Bernstein, W. D. Cairns, G. C. Evans, Emma V. Hesse, A. A. Koch, Sophia H. Levy, A. L. McCarty, F. R. Morris, T. M. Putnam, Adeline M. Scandrett, Ethel Spearman, Pauline Sperry, Falka G. Sturges, Ruth G. Sumner, Gabor Szegö, Harriet A. Welch, A. R. Williams, B. C. Wong, and two applicants for membership: Grace A. Fuller, Marjorie Ley.

The following officers were elected for the coming year: Chairman, Sophia H. Levy, University of California; Vice-Chairman, F. R. Morris, Fresno State College; Secretary-Treasurer, H. M. Bacon, Stanford University. Mrs. Ruth G. Sumner, Oakland High School, was re-elected to represent the Section as associate editor of the *California Journal of Secondary Education*. During the afternoon session Professor C. B. Morrey of the University of California described the new publication, *Mathematical Reviews*. Considerable discussion followed the presentation of Miss Welch's paper on "Shall we defer the teaching of algebra to the tenth grade?" and the following resolution was adopted without dissent:

> "The Northern California Section of the Mathematical Association of America and those in attendance at the meeting of the Section wish to go on record as favoring that a program of mathematics be provided in the secondary schools, beginning normally with algebra in the ninth year, to be available for those who wish to elect it or who otherwise need it in preparation for college work. It is felt that a capable student should be able to secure solid geometry and trigonometry in the secondary school."

The main part of the program was given over to an address by W. F. Durand, Professor of Mechanical Engineering, Emeritus, Stanford University. Five other papers were presented. The list of titles and speakers follows:

1. "A simple mathematical theory of economic relief" by Professor G. C. Evans, University of California.

2. "Mathematics and the constructive arts" by Professor W. F. Durand, Stanford University, introduced by the Secretary.

3. "Geometric representation of certain magnetic fields" by Professor F. R. Morris, Fresno State College.

4. "Shall we defer the teaching of algebra to the tenth grade?" by Harriet A. Welch, Lowell High School, San Francisco.

5. "Some difficulties with mathematics in a core curriculum" by Dr. Vern James, Menlo Junior College, introduced by the Secretary.

6. "A general solution of $x_1^2+x_2^2+\cdots+x_n^2=m^2$" by A. L. McCarty, San Francisco Junior College.

Abstracts of the papers follow, numbered in accordance with their listing above:

1. Professor Evans considered the place of the term representing the amount for direct relief in a simplified economic system which is in equilibrium with respect to the conditions of strict competition, in particular with regard to the equation of exchange, where terms for hoarding or investment and incomplete consumption also occur. Labor, capital, and consumption commodity are the significant variables. The analysis is carried over into a system which is not in equilibrium but where a fixed interval is assumed as an average length of the process of production. In this case the equation of exchange, instead of being an identity which concerns merely price ratios, becomes one which relates present to past prices, and thus provides an essential element in the determination of the system.

2. Professor Durand's address has been submitted for publication in the MONTHLY.

3. Professor Morris discussed the following problem: Upon a plane section of the earth's magnetic field, an artificial field containing two poles is superimposed with the poles in a north and south line. By choosing either pole positive or negative, four distinct cases occur. A study was made of the lines of force and equipotential lines with particular attention to the analysis of multiple points where the compass behaves peculiarly. The potential, the third dimension, is in each case an algebraic function, involving two radicals, of the Cartesian coördinates of the plane. Rationalization of any one of the four equations yields a twelfth degree equation including all cases. A plane section of this surface through the poles and the potential axis is composed of four cubic curves, rational cubic equations, though they do not correspond to the four cases.

4. It was pointed out by Miss Welch that the movement to defer elementary algebra to the tenth year is unwise if applied to the better pupils. It deprives them of an opportunity to study a new subject within their capabilities, and handicaps them in their later courses.

5. Dr. James stressed the fact that if two or more subjects are to merge successfully into a larger unit, they must be mutually helpful, and the teachers in each field must be aware of the needs for such a merger. Mathematics is essential in almost every phase of modern life. Moreover, it is enriched by its many applications in modern society. Mathematics teachers lack knowledge and and training in other fields, and teachers in other fields are woefully ignorant of mathematics. Our conventional courses have not met the needs of the new social order. The vast majority of American students have little knowledge of mathematical principles. Three-fourths of the secondary teachers in Pennsylvania who are teaching other subjects have had six or less units of college mathemat-

ics. We must reorganize our courses with new objectives, and teach the fundamental concepts without requiring proficiency in the manipulation of mathematical symbols.

6. Mr. McCarty showed that if

$$x_1 + x_2 i_1 + x_3 i_2 + \cdots + x_n i_{n-1} = (b_1 + b_2 i_1 + b_3 i_2 + \cdots + b_n i_{n-1})^2;$$

$$i_a i_m = -\, i_m i_a \quad \text{and} \quad i_1^2 = i_2^2 = \cdots = i_{n-1}^2 = -1,$$

then

$$m = \lambda(b_1^2 + b_2^2 + b_3^2 + \cdots + b_n^2), \quad x_1 = \lambda(b_1^2 - b_2^2 - b_3^2 - \cdots - b_n^2),$$

$$x_2 = 2\lambda b_1 b_2,\ x_3 = 2\lambda b_1 b_3,\ \cdots,\ x_n = 2\lambda b_1 b_n.$$

H. M. BACON, *Secretary*

PROJECTIVE CONSTRUCTIONS FOR CERTAIN ALGEBRAIC CURVES

I. C. FISCHER, Milwaukee School of Engineering

The projective constructions to be described were suggested by a certain metric curve that can be constructed by use of a well known geometric figure. It may be of interest to define this curve.

Given a circle of radius a, with the line l_1 a tangent, and the line l_2 passing through the center A of the circle and cutting it in the point of tangency with l_1. We draw a ray through A cutting the circle at M from which perpendiculars are dropped to points L_1 and L_2 on l_1 and l_2, respectively. The line L_1L_2 cuts the ray through A at a point P on the required metric curve. If the origin is A and l_1 and l_2 are the lines $x=a$, $y=0$, respectively, then the polar and cartesian equations of the curve are:

$$r = a/(2 - \sec\theta),$$

$$4x^2(x^2 + y^2) - (x^2 + y^2 + ax)^2 = 0.$$

The above quartic is a special case of a more general construction. The projective generalization (see Figure 1) may be obtained as follows: Replace the circle by any conic K, and the two base lines by any two lines l_1 and l_2. Let the triangle of reference be ABC. Draw a ray through A cutting K at M. From M rays are drawn through B and C, cutting l_1 and l_2, respectively, at L_1 and L_2. The line through L_1 and L_2 cuts the ray through A at point P on the required curve. It will be shown that the curve is, in general, a proper sextic which may be reduced to branches of lower degree by proper orientation of the base elements. To this end, we use the homogeneous coördinates x, y, z; the reference

triangle defined by the vertices $A(0, 0, 1)$, $B(1, 0, 0)$, $C(0, 1, 0)$; and the following expressions for the generating and base elements:

(1) $x-hy=0$, the flat pencil (A), on A;
(2) $y-mz=0$, the flat pencil (B), on B;
(3) $x-nz=0$, the flat pencil (C), on C;
(4) $a_1x+b_1y+c_1z=0$, the line l_1;
(5) $a_2x+b_2y+c_2z=0$, the line l_2;
(6) $Ax^2+By^2+Cz^2+2Dxy+2Exz+2Fyz=0$, the conic K.

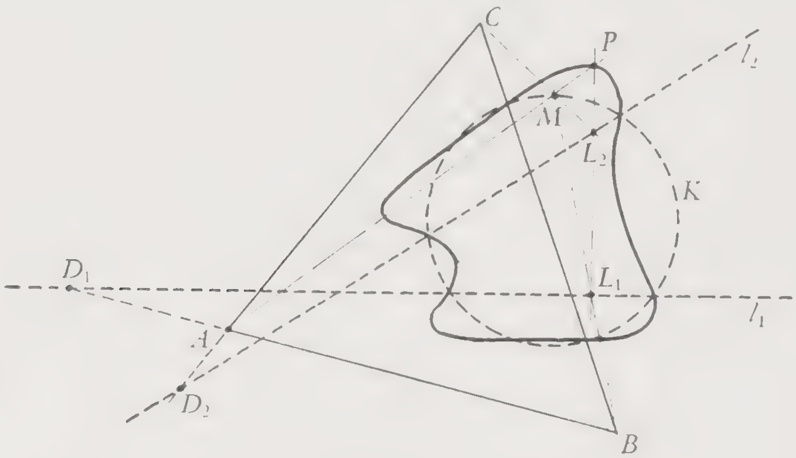

FIG. 1

Now these formulations enable us to express the equations for the conic and the points L in terms of the parameters, and we have

(7) $K(n, m, 1)=0$, by combining (2), (3), and (6);
(8) $L_1(-b_1m-c_1, a_1m, a_1)$, from (2) and (4);
(9) $L_2(b_2n, -a_2n-c_2, b_2)$, from (3) and (5).

The equation for line L_1L_2 can now be obtained, and elimination of the parameters with the help of (7), (1), (2), (3) will yield the sextic

(10) $Q^2s^2+Q(Cr^2-2Prs+2aCst)+2aP(Cr-2Ps)t+a^2C^2t^2=0$,

where

$$P=Ex+Fy,\ Q=Ax^2+2Dxy+By^2,$$
$$r=a_1a_2x^2+b_1b_2y^2+2a_1b_2xy+a_2c_1xz+b_1c_2yz,$$
$$s=a_1c_2x+b_2c_1y+c_1c_2z,\ t=xyz,$$
$$a=a_1b_2-a_2b_1.$$

The curve (10) is plotted in Figure 1. The reader can easily verify the fact that A is, in general, an isolated quadruple point, and D_1 and D_2 are isolated double points. It may be of interest to note several special cases.

(a). If A lies on K, $C=0$ and (10) degenerates to a quintic branch and the line $s=0$; and $s=0$ is the line D_1D_2,

$$s(Q^2s - 2PQr - 4aP^2t) = 0.$$

It will be noted that when A is on K and M is at A, L_1 is D_1 and L_2 is D_2 so that L_1L_2 is the line D_1D_2; but the line AM is indeterminate so that P is anywhere on D_1D_2.

(b). If l_1 is the line $x=0$, (10) has the factor x^2 and hence the degenerate sextic has a quartic branch and the coincident lines $x^2=0$. If l_2 is taken as $y=0$, the factor y^2 is obtained, and again a quartic branch results, together with the lines $y^2=0$. The metric curve discussed at the beginning of this paper is a special case of this reduction, provided that $y=0$ is taken as the line l_2.

(c). If, in case (a), l_2 is taken as the line $x=0$, (10) takes the form

$$sPx[Q(a_1x + c_1z) - 2b_1Pyz] = 0.$$

Thus we have a cubic branch together with $s=0$, $P=0$, $x=0$. $P=0$ is the tangent to K at A for this case. Figure 2 shows the cubic branch.

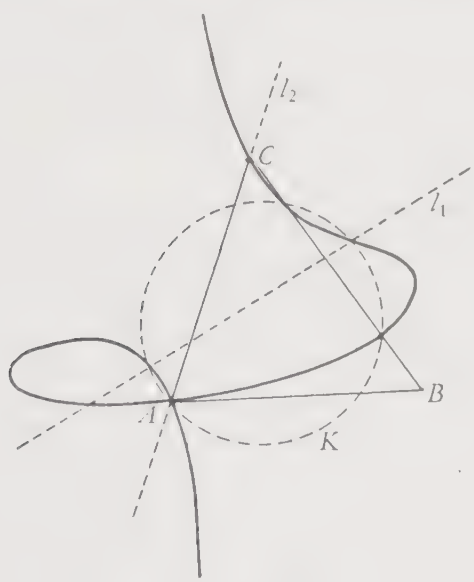

FIG. 2

(d). Now if, in (10), we let l_1 be $x=0$ and l_2 be $y=0$, two pairs of coincident lines and a conic results, since

$$x^2y^2(4Q + 4Pz + Cz^2) = 0.$$

The processes described will no doubt suggest a further generalization for the construction of curves of degree other than the sixth, by replacing the conic and base lines with curves of various degree.

NOTE ON COMPOUND INTEREST

RICHARD KERSHNER, University of Wisconsin

The fundamental principle on which the solution of problems in the mathematics of investments is based is the so-called principle of equivalence, according to which two sums of money at different dates are considered as equivalent, under a given interest rate, if the earlier sum would amount to the later sum, at the given interest rate, during the intervening time. Thus an interest rate may be said to establish an equivalence relation on the set of points (s, t) of a euclidean plane, where s represents a sum of money and t a date on which a transaction involving this sum of money takes place.

A binary relation

$$(s_1, t_1) \sim (s_2, t_2) \tag{1}$$

is usually called an equivalence relation if the following three assumptions are valid:

A_1: *The reflexive law:*

$$(s_1, t_1) \sim (s_1, t_1).$$

A_2: *The symmetric law:*

$$(s_1, t_1) \sim (s_2, t_2) \textit{ implies } (s_2, t_2) \sim (s_1, t_1).$$

A_3: *The transitive law:*

$$(s_1, t_1) \sim (s_2, t_2);\ (s_2, t_2) \sim (s_3, t_3) \textit{ implies } (s_1, t_1) \sim (s_3, t_3).$$

Actually, of the types of interest in actual use, only compound interest converted continuously establishes a relation (1) which satisfies this transitive law A_3. The fact that the relation based on simple interest does not satisfy A_3 is usually expressed, in the elementary text-books, by saying that the solution of "equation of value" problems depends on the focal date. For the usual compound interest law, the transitive law will hold if and only if the two time intervals involved, $|t_2 - t_1|$ and $|t_3 - t_2|$, each consist of an integral number of conversion periods. In fact, compound interest is more useful than simple interest in business practice precisely because the transitive law A_3 more nearly holds for compound interest.

Thus it is seen that the equivalence relation established by the law of compound interest converted continuously is by far the most satisfactory of those in common use and, in fact, is the only one which deserves to be called an equivalence relation as the term is usually used. The fact that continuous conversion is not more widely used can be explained only on the basis of the fact that its calculation involves something more than grammar school arithmetic. It is with this equivalence relation that this note will be concerned. It is most symmetrically defined by the following:

$$(E): \qquad (s_1, t_1) \sim (s_2, t_2) \textit{ if and only if } s_1e^{-jt_1} = s_2e^{-jt_2}.$$

Obviously this relation (E) has much to recommend it for use in financial connections beyond the fact that it satisfies A_1, A_2, A_3 and is thus a true equivalence relation. Some of the more fundamental properties of (E), from an abstract point of view, may be expressed in the present notation as follows:

A_4: $(s_1, 0) \sim (s_2, 0)$ *implies* $s_1 = s_2$.
A_5: *For any* (s_1, t_1) *there exists an* s_2 *such that* $(s_1, t_1) \sim (s_2, 0)$.
A_6: *There exists an* $\epsilon > 0$ *such that* $(s_1, t_1) \sim (1, 0)$; $|t_1| < \epsilon$; *implies* $\frac{1}{2} < s_1 < 2$.
A_7: $(s_1, t_1) \sim (s_2, t_2)$ *implies* $(as_1, t_1) \sim (as_2, t_2)$ *for any* a.
A_8: $(s_1, t_1) \sim (s_2, t_2)$ *implies* $(s_1, t_1 + h) \sim (s_2, t_2 + h)$ *for any* h.

Of these axioms, A_4, A_5, and A_6 are self-explanatory uniqueness, existence, and "reasonableness" axioms; A_7 states that the equivalence relation is not affected by a change in the unit of currency; and A_8 expresses the fact that the choice of the date $t = 0$ is arbitrary.

It has already been pointed out that the compound interest law (E) is the only relation (1) in common use which obeys all the axioms $A_1, \cdots, A_8$. The object of this note is to show that actually (E) is the only such relation, *i.e.*, that from the abstract laws $A_1, \cdots, A_8$ alone it is possible to derive the precise formula (E) of compound interest. This fact seems to provide a more rational justification of the use of compound interest than those usually given. In fact it is very easy to convince oneself of the desirability of each of the axioms $A_1, \cdots, A_8$. Actually, the relation (E) would only be called compound interest in case $j > 0$, since it is desirable for money to increase, rather than decrease, with increasing time; but it is clear that the relation defined by (E) satisfies all axioms $A_1, \cdots, A_8$ for any j. Thus it is not quite correct to say that $A_1, \cdots, A_8$ completely characterize compound interest. The precise statement is as follows:

THEOREM. *The only binary relation* (1) *satisfying* $A_1, \cdots, A_8$ *is the relation defined by* (E) *for some* j.

Proof. For any (s, t) there is, by A_5, an s' such that $(s, t) \sim (s', 0)$. Now if $(s, t) \sim (s'', 0)$ then, by A_2 and A_3, $(s', 0) \sim (s'', 0)$ so that $s' = s''$, by A_4. Thus, there is a unique number $s' = \phi(s, t)$ defined by the relation

$$(2) \qquad (s, t) \sim (\phi(s, t), 0).$$

In particular,

$$(3) \qquad (1, t) \sim (\psi(t), 0),$$

where $\psi(t) = \phi(1, t)$.

Thus, by A_7, with $a = s$, $(s, t) \sim (s\psi(t), 0)$ so that, by (2) and A_2, A_3, and A_4,

$$(4) \qquad \phi(s, t) = s\psi(t).$$

Now, by (3),

(5) $$(1, t + h) \sim (\psi(t + h), 0).$$

On the other hand, by (3) and A_8,

(6) $$(1, t + h) \sim (\psi(t), h).$$

But, by (3) again, $(1, h) \sim (\psi(h), 0)$, so that, by A_7 with $a = \psi(t)$,

(7) $$(\psi(t), h) \sim (\psi(t)\psi(h), 0).$$

Now, according to A_3, (6) and (7) imply

(8) $$(1, t + h) \sim (\psi(t)\psi(h), 0).$$

Comparison of (8) and (5) gives, in view of A_2, A_3, A_4,

(9) $$\psi(t + h) = \psi(t)\psi(h).$$

Now, from (3) and A_7 with $a = 1/\psi(t)$,

$$(1/\psi(t), t) \sim (1, 0),$$

so that, by A_6,

(10) $$\tfrac{1}{2} < \psi(t) < 2 \quad \text{if} \quad |t| < \epsilon.$$

But it is well known that the only solution of the functional equation (9) which satisfies the boundedness condition (10) is the exponential

$$\psi(t) = e^{-jt}$$

for some j. Thus, by (2) and (4),

$$(s, t) \sim (se^{-jt}, 0).$$

The relation (E) follows immediately.

SPACE ANALOGS OF FUNCTION-THEORETIC RESULTS*

E. F. BECKENBACH, The Rice Institute

1. Introduction. That fundamental branch of mathematics known as function-theory, or the theory of analytic functions of a complex variable, is interrelated with many other branches of mathematics. The Fundamental Theorem of Algebra is most easily proved as a theorem in function-theory. Potential theory and function-theory coincide through a long chapter, for both deal with harmonic functions. The topology of Riemann surfaces is a beautiful discipline. On the formal side, analogies between function-theory and the theory of subharmonic functions have been pursued recently by Privalof, Saks, Radó and others; and numerous systems of differential equations have been studied, as being generalizations of the Cauchy-Riemann differential equations.

I am confining my remarks to analogies concerning the theory of surfaces. I trust this will lend some orientation and coherence to what I have to say; but more important is the fact that the most stimulating and fruitful purely analytical problems and results often arise naturally in the course of the investigation of problems having physical settings.

In looking for space analogs of function-theoretic results, one might make any of several different starts. For example, he might seek analogs of

1.1. a theorem having an analytic setting,

1.2. a theorem having a geometric setting, or

1.3. a function-theoretic method.

We shall look successively at these three approaches.

2. Theorems having analytic settings. The Cauchy and Morera theorems. A fundamental theorem of function-theory states that the function

$$w = f(z) = x_1(u, v) + ix_2(u, v), \qquad z = u + iv,$$

defined in a simply connected domain D, is analytic in D if and only if the Cauchy-Riemann differential equations

$$x_{1,u} = x_{2,v}, \qquad x_{1,v} = -x_{2,u} \tag{1}$$

are satisfied there. Any space analog of these equations should, I think, involve three (or n) [1] functions

$$x_1(u, v), \qquad x_2(u, v), \qquad x_3(u, v),$$

and should be symmetric. Equations (1) imply

$$x_{1,u}^2 + x_{2,u}^2 = x_{1,v}^2 + x_{2,v}^2, \qquad x_{1,u}x_{1,v} + x_{2,u}x_{2,v} = 0,$$

which are symmetric, and which we can write as

* An address delivered before the Mathematical Association of America at Columbus, Ohio, December 30, 1939, and at the Texas Section meeting, Dallas, March 29, 1940.

$$\sum_{j=1}^{2}(x_{j,u}^2 + 2ix_{j,u}x_{j,v} - x_{j,v}^2) \equiv \sum_{j=1}^{2}\left[\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)x_j\right]^2 \equiv \sum_{j=1}^{2}(\lambda x_j)^2 = 0.$$

Now these equations generalize to

$$\sum_{j=1}^{3} x_{j,u}^2 = \sum_{j=1}^{3} x_{j,v}^2, \qquad \sum_{j=1}^{3} x_{j,u}x_{j,v} = 0, \quad \text{or} \quad \sum_{j=1}^{3}(\lambda x_j)^2 = 0, \tag{2}$$

which are the familiar $E=G$, $F=0$ characterizing conformal, or isothermic, maps. This is nothing new, but it indicates how one might in general go about setting up space analogs of function-theoretic formulas. A principal difference between (1) and (2) is that (1) implies that the $x_j(u, v)$ are harmonic, $(j=1, 2)$, while (2) does not imply that the $x_j(u, v)$ are harmonic, $(j=1, 2, 3)$. Indeed, by a theorem of Weierstrass, *if* (2) *holds, then the* $x_j(u, v)$ *are harmonic,* $(j=1, 2, 3)$, *if and only if the* $x_j(u, v)$ *are coördinate functions of a minimal surface.*

On the other hand, we do obtain something new [**2**] if we treat the Cauchy integral equation

$$\int_\gamma f(z)dz = 0, \tag{3}$$

which by the Cauchy and Morera theorems also characterizes analytic functions, as we did the Cauchy-Riemann differential equations. We have

$$\int_\gamma f(z)dz \equiv \int_\gamma (x_1 + ix_2)(du + idv) \equiv \int_\gamma (x_1du - x_2dv) + i\int_\gamma (x_2du + x_1dv),$$

so that (3) is equivalent to

$$\int_\gamma x_1du = \int_\gamma x_2dv, \qquad \int_\gamma x_1dv = -\int_\gamma x_2du, \tag{4}$$

which are integral equations quite analogous to the Cauchy-Riemann differential equations. Like the Cauchy-Riemann equations, equations (4) imply a symmetric set of equations, and the symmetric set generalizes to a set of integral equations analogous to the differential equations $E=G$, $F=0$:

$$\sum_{j=1}^{3}\left(\int_\gamma x_jdu\right)^2 = \sum_{j=1}^{3}\left(\int_\gamma x_jdv\right)^2, \qquad \sum_{j=1}^{3}\left(\int_\gamma x_jdu\right)\left(\int_\gamma x_jdv\right) = 0,$$

or, combined,

$$\sum_{j=1}^{3}\left[\int_\gamma x_j(du + idv)\right]^2 \equiv \sum_{j=1}^{3}\left(\int_\gamma x_jdz\right)^2 = 0. \tag{5}$$

Now the study of (5) yields several results. If we restrict the closed rectifiable Jordan curves γ to being circles C of radius r in D, and denote by $\sigma(r^\alpha)$ any quan-

tity (not always the same quantity) such that

$$\lim_{r\to 0} \frac{\sigma(r^\alpha)}{r^\alpha} = 0,$$

we have the following results. Suppose the $x_j(u, v)$ have continuous third derivatives in the simply connected domain D; we shall say they are coördinate functions of a surface S; *then*

$$\sum_{j=1}^{3}\left(\int_C x_j dz\right)^2 = \sigma(r^\alpha)$$

holds at each point D

2.1. *for $\alpha=4$ if and only if the map is conformal, or isothermic;*

2.2. *for $\alpha=6$ but not for $\alpha=8$ if and only if the map is an isothermic map on a spherical surface, such that circles are not mapped on circles;*

2.3. *for $\alpha=8$ if and only if the map is an isothermic map either on a sphere such that circles are mapped on circles, or on a minimal surface.*

2.4. *The sum of squares of integrals has the value 0 for all circles C in D, i.e.,*

$$\sum_{j=1}^{3}\left(\int_C x_j dz\right)^2 = 0,$$

if and only if condition 2.3 is satisfied, i.e.,

$$\sum_{j=1}^{3}\left(\int_C x_j dz\right)^2 = \sigma(r^8).$$

2.5. *Finally,* (5) *holds for all closed rectifiable Jordan curves γ in D if and only if the map is an isothermic map on a plane surface.* Thus in generalizing from plane isothermic maps (*i.e.*, analytic functions $f(z)$), we are at last led back in a broader field to a characterization of these same maps.

There remains the problem of lessening the differentiability assumptions in the above discussion, as has been done most brilliantly in the case of the Cauchy-Riemann differential equations by Looman, Menchoff, Menger and others. A first result in this direction is the following. The averaging functions

$$x_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2}\iint_{\xi^2+\eta^2\leqq\rho^2} x_j(u+\xi, v+\eta)d\xi d\eta \tag{6}$$

are immediately suggested, since they have one more derivative than the given functions $x_j(u, v)$; denote by S_ρ the surface of which the $x_{j,\rho}(u, v)$ are coördinate functions. *Then the above condition* 2.3 *holds if and only if for all ρ for which the functions* (6) *are defined, the map on S_ρ is isothermic.* This brings up the question of what sort of surface S_ρ is, in the two cases occurring in 2.3. For the case of the minimal surface in isothermic representation, the coördinate functions of S are harmonic, so that the averaging functions are equal to the functions themselves.

$$x_{j,\rho}(u, v) \equiv x_j(u, v);$$

hence S_ρ coincides with the minimal surface S. But for the case of the spherical surface in isothermic representation, with circles mapped on circles, it appears that we shall have the result that S_ρ is not a spherical surface, but lies on a surface of revolution interior to the sphere S.

Taking a new departure, we treat the coefficients e, f, g; $\mathcal{E}, \mathcal{F}, \mathcal{G}$ of the second and third fundamental quadratic differential forms as we have treated the coefficients E, F, G of the first fundamental form. Since the coefficients of the second and third forms involve derivatives of the components of the unit normal vector ζ of S,

$$e = -\sum_{j=1}^{3} x_{j,u}\zeta_{j,u}, \qquad \mathcal{E} = \sum_{j=1}^{3} \zeta_{j,u}^{2}, \; etc.,$$

and since ζ is constant for a plane surface, these forms are without interest in function-theory; but they are of great importance in the general theory of surfaces. Integral equations analogous to

$$e = g, f = 0 \quad \text{and} \quad \mathcal{E} = \mathcal{G}, \mathcal{F} = 0$$

are

$$\sum_{j=1}^{3}\left(\int_\gamma x_j dz\right)\left(\int_\gamma \zeta_j dz\right) = 0 \quad \text{and} \quad \sum_{j=1}^{3}\left(\int_\gamma \zeta_j dz\right)^2 = 0,$$

respectively. These last equations can be discussed as was (5), but we shall not now take the time to list the results.

3. Theorems having geometric settings. The principle of reflection; Bloch's theorem; schlicht functions. We shall consider in this section three examples of generalizations to space of theorems having geometric formulations. The first is a very interesting result of J. Douglas [**3**], and is to the effect that if a piece of minimal surface is bounded in part by a segment of straight line, then the surface can be continued analytically by reflection in the line, so that the line is an axis of symmetry of the surface. This result might be compared with the less general theorem of Schwarz, that if a minimal surface contains a straight line in its interior, then the line is an axis of symmetry of the surface. As Douglas points out, his result is a generalization of a special case of the general reflection principle, and accordingly it suggests several other interesting problems.

As a second example of a generalization of a function-theoretic result having a geometric formulation, let us consider the theorem of Bloch:

There exists a positive absolute constant B with the following property. Let $w=f(z)$ be analytic for $|z| \leq 1$, with $|f'(0)| \geq 1$; then in the w-plane there is an open circle of radius at least B, which is the single-sheeted map of a portion of the circle $|z| < 1$.

Bloch's theorem may be generalized as follows [4]:

There exists a positive absolute constant M with the following property. Let the circle $u^2+v^2\leqq 1$ be mapped isothermically on a minimal surface, with $E_0\geqq 1$, where E_0 denotes the area deformation ratio at the origin; then on the minimal surface there is an open geodesic circle of radius at least M, containing no singular points, which is the one-to-one map of a portion of the circle $u^2+v^2<1$.

Now the above result suggests several other questions. If we let $\mathcal{B}$ be the least upper bound of values B that will serve in Bloch's theorem, and let $\mathcal{M}$ be the least upper bound of values M, then $\mathcal{M}\leqq\mathcal{B}$, since a plane is a particular minimal surface. But is $\mathcal{M}<\mathcal{B}$ or $\mathcal{M}=\mathcal{B}$? And what classes of surfaces admit Bloch's constants, and how does the Bloch's constant vary with the characterizing properties of such classes of surfaces?

Our third example comes from the following consideration. If the functions $x_j(u,v)$, $(j=1, 2, 3)$, are coördinate functions of a minimal surface in isothermic representation for (u, v) in a simply connected domain D, so that the $x_j(u, v)$ are harmonic in D, then these functions are the real parts of analytic functions of the complex variable $z=u+iv$,

$$x_j(u, v) = \mathcal{R} f_j(z), \qquad (j = 1, 2, 3).$$

If the three functions $f_j(z)$ are *schlicht* (*i.e.*, yield single-sheeted maps of D), we might say that the minimal surface is *of schlicht derivation* [5]; this is a large but by no means exhaustive class of minimal surfaces. Many of the geometric properties of *schlicht* plane maps are possessed by maps on minimal surfaces of *schlicht* derivation. Thus we have the following generalization of Bieberbach's theorem:

If the circle $u^2+v^2<1$ is mapped isothermically on a minimal surface of schlicht derivation, with $E_0\geqq 1$, where E_0 denotes the area deformation ratio at the origin, then the minimum distance on the surface from the image of $(0, 0)$ to the boundary is $\geqq 1/4$. No closer inequality holds for all surfaces of this type, and the sign of equality can hold only if the minimal surface is a plane.

4. The method of coefficients. Cluster values; coincident surfaces. We shall consider extensions of three general function-theoretic methods. The first is the sensitive *method of coefficients*. Let functions $x_j(u, v)$ be harmonic in a simply connected domain D,

$$x_j(u, v) = \frac{a_{j,0}}{2} + \sum_{n=1}^{\infty} r^n(a_{j,n} \cos n\theta + b_{j,n} \sin n\theta), \tag{7}$$
$$u = u_0 + r \cos \theta, \qquad v = v_0 + r \sin \theta.$$

Then the harmonic function $x_2(u, v)$ is conjugate to $x_1(u, v)$, as is well known, if and only if

$$a_{1,n} = b_{2,n}, \qquad a_{2,n} = - b_{1,n}, \qquad (n = 1, 2, \cdots). \tag{8}$$

Analogous though less simple relations among the coefficients can be shown to be necessary and sufficient conditions that three such harmonic functions be *a triple of conjugate harmonic functions*; that is, satisfy $E=G$, $F=0$; that is, be coördinate functions of a minimal surface in isothermic representation. These conditions are [6]

$$
(9) \qquad \begin{aligned}
&\sum_{l=1}^{k-1} l(k-l)\sum_{j=1}^{3}(a_{j,l}a_{j,k-l}-b_{j,l}b_{j,k-l})=0, \qquad (k=2,3,4,\cdots),\\
&\sum_{l=1}^{k-1} l(k-l)\sum_{j=1}^{3}(a_{j,l}b_{j,k-l}+b_{j,l}a_{j,k-l})=0, \qquad (k=2,3,4,\cdots).
\end{aligned}
$$

It might be interesting to note that conditions

$$
\begin{aligned}
&\sum_{j=1}^{3}(a_{j,l}a_{j,k-l}-b_{j,l}b_{j,k-l})=0,\\
&\sum_{j=1}^{3}(a_{j,l}b_{j,k-l}+b_{j,l}a_{j,k-l})=0, \qquad l<k, \qquad (k=2,3,4,\cdots),
\end{aligned}
$$

which are more restrictive than (9) but less than (8), turn out to be necessary and sufficient conditions that three harmonic functions (7) be coördinate functions of a plane surface in isothermic representation.

From a study of conditions (9) we can obtain various theorems concerning the area and length of boundary of a piece of minimal surface [7], and theorems analogous to the isolation of zeros of an analytic function, and so on [6]. We shall content ourselves by stating three results obtained by this method.

4.1. *If the functions* (7) *are coördinate functions of a minimal surface S in isothermic representation for (u, v) in D, and if $x_j(u, v)=c_j$, $(j=1, 2, 3)$, where each c_j is a constant, on a set R of points with a cluster point P interior to D, then $x_j(u, v)\equiv c_j$ in D; that is, S is a point.*

4.2. *Let the functions* (7) *be coördinate functions of a minimal surface S in isothermic representation for (u, v) in D, and let the direction cosines of the normal to the surface at the image of (u, v) be denoted by $X_j(u, v)$, $(j=1, 2, 3)$. If $X_j(u, v)=c_j$, where each c_j is a constant, on a set R of points with a cluster point P interior to D, then $X_j(u, v)\equiv c_j$ in D; that is, S is a plane surface.*

4.3. *Let S and S' be minimal surfaces given in isothermic representation respectively by*

$$x_j=x_j(u,v), \qquad y_j=y_j(u,v), \qquad (j=1,2,3), \qquad (u,v) \text{ in } D,$$

and let the directions cosines of the normals to the surfaces at the images of (u, v) be denoted respectively by $X_j(u, v)$, $Y_j(u, v)$; if $x_j(u, v)=y_j(u, v)$, $(j=1, 2, 3)$, on a set R of points having a cluster point P interior to D, and if $X_j(u, v)=Y_j(u, v)$, $(j=1, 2, 3)$, on a set R' of points having the same cluster point P, then $x_j(u, v)\equiv y_j(u, v)$ in D; that is, S and S' are coincident surfaces.

5. Normal families. Lindelöf's theorem. The second method we shall consider is an extension to minimal surfaces of the function-theoretic method based on the notion of *normal families.* We shall say that an infinite set, or family, of triples of conjugate harmonic functions, having a common domain of definition D, constitutes a normal family of triples of conjugate harmonic functions in D provided every infinite sequence of triples of the family contains a subsequence of triples which converges uniformly to a triple of conjugate harmonic functions, or for which $x_1^2+x_2^2+x_3^2$ converges uniformly to infinity, in every closed region in D.

By using the notion of normal families, we are able to prove, for example, the following generalization of a theorem of Lindelöf [**8**]:

Let $x_j(u, v)$, $(j=1, 2, 3)$, *be a bounded triple of conjugate harmonic functions in the sector*

$$0 < \operatorname{arc}\tan (v/u) < \alpha, \qquad 0 < u^2 + v^2 < r_0^2.$$

Suppose the $x_j(u, v)$, $(j=1, 2, 3)$, *remain continuous on the ray* $0<u<r_0$, $v=0$, *and* $x_j(u, 0)\to x_{j,0}$, $(j=1, 2, 3)$, *as* $u\to+0$. *Then in every sector*

$$0 < \operatorname{arc}\tan (v/u) < \alpha - \sigma, \qquad 0 < u^2 + v^2 < r_0^2, \quad \textit{where} \quad \sigma > 0,$$

we have $x_j(u, v)\to x_{j,0}$, $(j=1, 2, 3)$, *as* $(u, v)\to(0, 0)$ *in any manner.*

6. Sub-harmonic functions and the theory of surfaces. The last method we shall consider is the *principle of the maximum.* If $f(z)$ is analytic in a region R, then $|f(z)|$ takes on its maximum on the boundary of R. The same is true of the distance in space, $[\sum_{j=1}^{3}(x_j-\alpha_j)^2]^{1/2}$, if the $x_j(u, v)$ are harmonic, as they are for example if they are isothermic coördinate functions of a minimal surface. But the effectiveness of the principle of the maximum in the case of analytic functions of a complex variable depends essentially on the fact that sums and products of analytic functions are analytic functions. This situation does not seem to admit of any direct generalization, for example, to minimal surfaces. Nevertheless, many important applications of the principle of the maximum can be generalized to minimal surfaces and to other classes of surfaces: this depends on the fact that *sums and products of non-negative functions having sub-harmonic logarithms are again functions having sub-harmonic logarithms.*

The notion of sub-harmonic function is a generalization, to functions of several variables, of the notion of convex function. A sub-harmonic function of two variables is a function $x(u, v)$, continuous (it may be defined more liberally) in a domain D, and such that the following condition is satisfied: for each domain D' comprised together with its boundary B' in D, and for each function $h(u, v)$ harmonic in D', continuous in $D'+B'$, and satisfying $h(u, v)\geqq x(u, v)$ on B', we have also $h(u, v)\geqq x(u, v)$ in D'. These functions are characterized by the mean-value inequality

$$x(u_0, v_0) \leqq \frac{1}{2\pi}\int_0^{2\pi} x(u_0 + \rho\cos\theta, v_0 + \rho\sin\theta)d\theta, \tag{10}$$

where ρ and (u_0, v_0) are the radius and the coördinates of the center of an arbitary circle lying together with its interior in D. Further, if $x(u, v)$ has continuous second derivatives, it is sub-harmonic if and only if

$$\Delta x \equiv x_{uu} + x_{vv} \geqq 0. \tag{11}$$

We note that (10) and (11), with the inequality signs replaced by the sign of equality, characterize harmonic functions. For any region in the domain of definition, a sub-harmonic function assumes its maximum on the boundary. Also, if $x(u, v)$ is non-negative and $[x(u, v)]^\alpha$ is sub-harmonic, where α is a positive constant, then $[x(u, v)]^\beta$ is sub-harmonic for all $\beta > \alpha$; and $\log x(u, v)$ is sub-harmonic if and only if $[x(u, v)]^\alpha$ is sub-harmonic for all positive α.

We now give characterizations of certain classes of surfaces in terms of sub-harmonic functions. Let $x_j(u, v)$, $(j = 1, 2, 3)$, continuous in a domain D, be the coördinate functions of a surface S. Then:

6.1. *The coördinate functions are harmonic if and only if* $[\sum_{j=1}^3 (x_j - \alpha_j)^2]^{1/2}$ *is sub-harmonic for all real constants* α_j [9]. We say that S is a harmonic surface in normal representation. That the "viewpoint" parameters α_j are indispensable follows from the fact that for the surface

$$x_1 = u, \qquad x_2 = v, \qquad x_3 = (1 - u^2 - v^2)^{1/2},$$

$[\sum_{j=1}^3 x_j^2]^{1/2} \equiv 1$ is sub-harmonic, while $x_3(u, v)$ is not harmonic.

6.2. *The coördinate functions are a triple of conjugate harmonic functions, that is, S is a minimal surface in isothermic representation, if and only if* $\log [\sum_{j=1}^3 (x_j - \alpha_j)^2]^{1/2}$ *is sub-harmonic for all real constants* α_j [9].

6.3. *If the $x_j(u, v)$ have continuous third derivatives and satisfy $E = G$, $F = 0$, then the Gaussian curvature K of S is $\leqq 0$ if and only if* $\log E$ *is sub-harmonic* [10].

In section 8, we shall indicate further characterizations of the surfaces in 6.2 and 6.3 in terms of sub-harmonic functions.

We note in passing that the class of surfaces considered in 6.2 obviously is a sub-class of the class of surfaces considered in 6.1. The class 6.2 is a sub-class also of the class 6.3.

7. The principle of the maximum: harmonic surfaces. Length of boundary. As an application of the characterization in 6.1, we give a theorem of T. Radó [11] generalizing a function-theoretic result of Csillag and Bieberbach:

Let $x_j(u, v)$, $(j = 1, 2, 3)$, be continuous for $u^2 + v^2 \leqq 1$ and harmonic for $u^2 + v^2 < 1$. Let $l(\rho)$ be the length of the map of $u^2 + v^2 = \rho^2$, for $0 \leqq \rho \leqq 1$. Then $l(\rho)$ is a monotonically increasing function of ρ, for $0 \leqq \rho \leqq 1$. The difficult and tricky point here lies in establishing the result for $\rho \leqq 1$ rather than only for $\rho < 1$.

8. The principle of the maximum: surfaces of negative curvature. Length and area; the isoperimetric inequality; the Lemma of Schwarz; Liouville's theorem. The theorem of section 7 holds, for $0 \leqq \rho < 1$, for isothermic maps on surfaces of negative curvature:

Let $S: x_j = x_j(u, v)$, $(j = 1, 2, 3)$, *have the properties that in* $u^2 + v^2 \leqq 1$ *the* $x_j(u, v)$ *are continuous, and in* $u^2 + v^2 < 1$ *they have continuous third derivatives, with* $E = G$, $F = 0$, *and the Gaussian curvature* K *of* S *is* $\leqq 0$. *Then* $l(\rho)$ *is a monotonically increasing function of* ρ, $0 \leqq \rho < 1$. This leaves the problem of investigating $l(1)$.

Under the above hypotheses concerning the surface S *with* $K \leqq 0$, *we have also*

$$l(\rho) \geqq 2\pi\rho E_0^{1/2}, \tag{12}$$

where E_0 *denotes the value of* E *at the center of the circle. The sign of equality holds in* (12) *if and only if the map is an isometric map on a developable piece of surface.*

Suppose conversely that (12) holds for every circle in the domain of definition of the functions $x_j(u, v)$. *It follows that* $E^{1/2}$ *is sub-harmonic.* Now what class of surfaces does the condition that $E^{1/2}$ be sub-harmonic characterize? It turns out not to characterize any class of surfaces; for it holds for some isothermic maps of plane domains on a piece of spherical surface and not for others. However, $E^{1/2}$ is sub-harmonic for *all* isothermic representations of plane domains on a piece of surface S if and only if $K \leqq 0$ on S; this is a second characterization (see 6.3) of surfaces of negative curvature in terms of sub-harmonic functions. *It follows that* (12) *holds for all isothermic maps of plane domains* D *on the surface* S *and for all circles* C *in* D, *if and only if* $K \leqq 0$ *on* S [**12, 13**].

If $a(\rho)$ denotes the area of the map on S of a circle of radius ρ, then a discussion analogous to that of (12) holds for the inequality

$$a(\rho) \geqq \pi\rho^2 E_0.$$

Also, just as we have here given a characterization, different from that given in 6.3, of surfaces of negative curvature in terms of sub-harmonic functions, we can similarly, by considering a class of conformally equivalent maps on S, give a second characterization of minimal surfaces in terms of sub-harmonic functions.

The *isoperimetric inequality* is related to surfaces of negative curvature by the present method. We note that for a plane circle,

$$a = \pi r^2 = \frac{1}{4\pi}(2\pi r)^2 = \frac{1}{4\pi} l^2,$$

and for other plane Jordan regions,

$$a < \frac{1}{4\pi} l^2.$$

Carleman has shown [**14**] that the isoperimetric inequality,

$$a \leqq \frac{1}{4\pi} l^2 \tag{13}$$

holds for all Jordan regions on minimal surfaces. Blaschke in his text on differential geometry pointed out that Carleman's result is almost immediate, provided the minimal surface has a minimum area; for he pointed out that one can compare the area of the minimal surface with that of an inscribed developable, for which the inequality surely holds. But the inequality actually holds more generally than either Blaschke or Carleman showed. It holds for all simply connected regions bounded by analytic curves on surfaces of negative curvature, and actually characterizes these surfaces:

Let S be an analytic surface. Then a necessary and sufficient condition that (13) *hold for all simply connected regions bounded by analytic curves on S is that $K \leqq 0$ on S. Further, if $K \leqq 0$ but $K \not\equiv 0$ on S, then the strict sign of inequality holds in* (13), *while if $K \equiv 0$ on S, then the sign of equality holds in* (13) *only for geodesic circles on S* [**10**].

Analytically, the above result concerning the isoperimetric inequality can be stated as follows: *Let $g(u, v)$ be continuous and $\geqq 0$ in a domain D, and let $A(g^2; u_0, v_0; \rho)$ and $L(g; u_0, v_0; \rho)$ be area and circumference means*:

$$A(g^2; u_0, v_0; \rho) = \frac{1}{\pi\rho^2} \iint_{\xi^2+\eta^2<\rho^2} g^2(u_0 + \xi, v_0 + \eta) d\xi d\eta,$$

$$L(g; u_0, v_0; \rho) = \frac{1}{2\pi} \int_0^{2\pi} g(u_0 + \rho \cos \theta, v_0 + \rho \sin \theta) d\theta.$$

Then

$$[A(g^2; u_0, v_0; \rho)]^{1/2} \leqq L(g; u_0, v_0; \rho)$$

for every point (u_0, v_0) in D and for every ρ, such that the disc $[(u-u_0)^2+(v-v_0)^2] \leqq \rho^2$ is comprised in D, if and only if $\log g(u, v)$ *is sub-harmonic in D.*

Various other theorems in function-theory admit generalizations by means of sub-harmonic functions to surfaces of negative curvature. One further example is a generalization [**15**] of a theorem of Fejér and Riesz:

If D is mapped isothermically on a surface S with $K \leqq 0$, and if C is a circle in D, then the length of the image on S of any diameter of C is less than or equal to half the length of the image on S of the circumference of C.

With each such geometric theorem there is of course an associated analytical theorem expressing a property of sub-harmonic functions.

It is not surprising that conversely the consideration of these theorems on surfaces should occasionally suggest new theorems in function-theory, and that sub-harmonic functions should furnish the key to the proofs of the new function-

theoretic results. One such result is an analog [16] of the Lemma of Schwarz. Let $w=f(z)$ be analytic for $|z|<1$, let $d(r, \theta; f)$ be the length of the segment on the w-plane between the image of the point $z=0$ and the image of the point $z=re^{i\theta}$,

$$d(r, \theta; f) = |f(re^{i\theta}) - f(0)| = \left| \int_0^r f'(\rho e^{i\theta}) d\rho \right|,$$

and let $l(r, \theta; f)$ be the length of the image on the w-plane of the segment between the points $z=0$ and $z=re^{i\theta}$,

$$l(r, \theta; f) = \int_0^r |f'(\rho e^{i\theta})| d\rho.$$

The Lemma of Schwarz may be stated as follows:

If $d(r, \theta; f) \leqq 1$ *for all* (r, θ) *with* $r<1$, *then*

$$d(r, \theta; f) \leqq r \tag{14}$$

and

$$|f'(0)| \leqq 1. \tag{15}$$

The sign of equality holds in (14) (*for* $r \neq 0$) *and in* (15), *if and only if* $|f'(z)| \equiv 1$.

The analog is obtained by replacing $d(r, \theta; f)$ by $l(r, \theta; f)$ throughout:

If $l(r, \theta; f) \leqq 1$ *for all* (r, θ) *with* $r<1$, *then*

$$l(r, \theta; f) \leqq r \tag{16}$$

and

$$|f'(0)| \leqq 1. \tag{17}$$

The sign of equality holds in (16) (*for* $r \neq 0$) *and in* (17), *if and only if* $|f'(z)| \equiv 1$.

There is a corresponding theorem concerning sub-harmonic functions, a generalization to surfaces of negative curvature, and a consequent generalization to space of the theorem of Liouville on bounded entire functions:

Let the functions $x_j = x_j(u, v)$ *map the entire finite plane isothermically on a surface S for which the Gaussian curvature K is less than or equal to zero wherever K is defined on S. If the length* $l(r, \theta)$ *of the image on S of the segment between the points* $(0, 0)$ *and* $(r \cos \theta, r \sin \theta)$ *is bounded,*

$$l(r, \theta) \equiv \int_0^r [E(\rho \cos \theta, \rho \sin \theta)]^{1/2} d\rho \leqq M,$$

for all (r, θ), *then S reduces to a point.*

9. The principle of the maximum: minimal surfaces. The Lemma of Schwarz; the problem of Plateau; the Osgood-Carathéodory theorem. Since $K \leqq 0$ on minimal surfaces, the theorems we have proved for surfaces of negative curvature hold in particular for minimal surfaces. But also the characterization 6.2 of minimal surfaces in isothermic representation yields a powerful tool for obtaining additional results concerning minimal surfaces. In particular, the Lemma of Schwarz generalizes directly to minimal surfaces. While the analog of the Lemma of Schwarz which we have been considering (see section 8) is concerned with distances on the surface, the Lemma of Schwarz itself is concerned with distances in the containing space. Our generalization to minimal surfaces is the following:

Let $S: x_j = x_j(u, v)$, $(j = 1, 2, 3)$, *be a minimal surface in isothermic representation, with* $(0, 0)$ *carried into* $(0, 0, 0)$. *If S is contained in the unit sphere,*

$$\sum_{j=1}^{3} [x_j(u, v)]^2 \leqq 1,$$

then

$$\sum_{j=1}^{3} [x_j(u, v)]^2 \leqq u^2 + v^2,$$

and

$$E_0 \leqq 1.$$

The equality signs hold if and only if S is a simply covered circular disc with unit radius.

The problem of Plateau may be stated as follows: Given a Jordan curve Γ in the $x_1x_2x_3$-space, determine functions $x_j(u, v)$, $(j = 1, 2, 3)$, which are a triple of conjugate harmonic functions (*i.e.*, the coördinate functions of a minimal surface in isothermic representation) for $u^2 + v^2 < 1$, continuous for $u^2 + v^2 \leqq 1$, and which carry $u^2 + v^2 = 1$ in a topological way into the Jordan curve Γ.

Solutions of the above problem may be stated as follows:

9.1. *If* Γ *bounds some surface, of the type of the circular disc, with finite area, then the problem of Plateau is solvable for* Γ.

9.2. *The problem of Plateau is solvable for an arbitrary* Γ.

Theorem 9.1 has been proved by several different methods, and Theorem 9.2 then proved on the assumption that Theorem 9.1 is valid. But the proof of Theorem 9.2 is not necessarily dependent on the method of proof of Theorem 9.1. Using the principle of the maximum, one can readily prove the above men-

tioned generalization of the theorem of Lindelöf (see section 5) and also the following generalization of a well known function-theoretic result:

Let $x_j(u, v)$, $(j=1, 2, 3)$, be a triple of conjugate harmonic functions for $u^2+v^2<1$. Suppose the $x_j(u, v)$ remain continuous on an arc σ of $u^2+v^2=1$, and

$$x_j(u, v) = \text{const.} = x_{j,0}, \qquad (j = 1, 2, 3),$$

on σ. Then $x_j(u, v) \equiv x_{j,0}$, $(j=1, 2, 3)$.

These two theorems make the passage from Theorem 9.1 to Theorem 9.2 particularly easy and concise [**8**].

Finally, the principle of the maximum allows us to establish [**9**] a generalization to minimal surfaces of the famous Osgood-Carathéodory theorem. The following result holds for *all* minimal surfaces, not only for those whose existence is established in Theorem 9.2.

A minimal surface S, bounded by a Jordan curve Γ, admits a representation

$$x_j = x_j(u, v), \qquad (j = 1, 2, 3), \qquad u^2 + v^2 \leqq 1, \tag{18}$$

with the following properties:

(i) *the $x_j(u, v)$, $(j=1, 2, 3)$, form a triple of conjugate harmonic functions in $u^2+v^2<1$;*

(ii) *the $x_j(u, v)$ are continuous in $u^2+v^2 \leqq 1$, and the equations* (18) *carry $u^2+v^2=1$ in a topological way into the Jordan curve Γ.*

One might very well ask the general question as to what further classes of surfaces admit Osgood-Carathéodory theorems.

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CURVES ON A SURFACE

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1. Introduction. Differential geometry has long been recognized as one of the most interesting branches of pure mathematics, and has lately gained new significance through its applications in mathematical physics. As C. E. Weatherburn has shown in his two-volume work, *Differential Geometry*, the use of vector methods permits great elegance in the proof of many classical results. It is the object of the present paper to give elementary proofs of a number of standard results, which are not proved by Weatherburn, concerning the relation of a curve to a surface which contains it. The reader, who need not have studied differential geometry hitherto, should be acquainted with line integrals and vector analysis including the differentiation of vectors.

2. Vectors. The *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a}\cdot\mathbf{b}$ and is equal to $ab\cos\alpha$, where a and b are the magnitudes of $\mathbf{a}$ and $\mathbf{b}$, respectively, and α is the angle between them.

The *vector product* $\mathbf{a}\times\mathbf{b}$ is a vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$, of magnitude $ab\sin\alpha$. The sense of the vector product is such that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{a}\times\mathbf{b}$ constitute a positively oriented triad. We recall that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$.

The *triple scalar product* $[\mathbf{a},\mathbf{b},\mathbf{c}]$ is equal to $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$, and it may be shown that

$$[\mathbf{a},\mathbf{b},\mathbf{c}]=[\mathbf{b},\mathbf{c},\mathbf{a}]=[\mathbf{c},\mathbf{a},\mathbf{b}]=-[\mathbf{a},\mathbf{c},\mathbf{b}]=-[\mathbf{c},\mathbf{b},\mathbf{a}]=-[\mathbf{b},\mathbf{a},\mathbf{c}]. \tag{2.1}$$

The *triple vector product* $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ may be expanded as follows:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{2.2}$$

From (2.1) we find that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=[\mathbf{a},\mathbf{b},(\mathbf{c}\times\mathbf{d})]=\mathbf{a}\cdot\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}.$$

Therefore by (2.2) we have,

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})=(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}). \tag{2.3}$$

Finally, we recall a principle that will be very useful. The derivative of a vector of constant magnitude is perpendicular to the vector.

3. Line integrals. Assuming that the reader remembers the meaning of a line integral, we merely recall Green's Theorem in two dimensions which may be expressed as follows:

$$\iint \left(\frac{\partial Q}{\partial u^1} - \frac{\partial P}{\partial u^2}\right) du^1 du^2 = \int P\,du^1 + Q\,du^2, \tag{3.1}$$

where the line integral is taken along the curve which bounds the area over which the surface integral is taken.

4. Curves. The vector **r** drawn out from the origin of coördinates determines a definite point P, say. If **r** depends on the single parameter u, then as u varies P traces out a single infinity of points. If, as we shall assume, there is a one-to-one correspondence between the values of the parameter and the points of the curve, we may take the arc-length of the curve as the parameter. A dot over a vector will always denote differentiation with respect to the arc-length of the curve to which the vector is attached.

Since the limit of the ratio of the length of an arc to the length of its chord is unity, $\mathbf{t}=\dot{\mathbf{r}}$ is a unit vector tangent to the curve. Since **t** has a constant magnitude, $\dot{\mathbf{t}}$ is perpendicular to it and we set

$$\dot{\mathbf{t}} = \kappa \mathbf{p}, \tag{4.1}$$

where **p** is a unit vector called the *principal normal* to the curve. The *first curvature*, κ, provides a measure of the arc-rate of turning of the tangent.

5. Surfaces. When **r** depends on two independent parameters u^1 and u^2 such that there is a one-to-one correspondence between the points P and the pairs of values of the parameters, P has two degrees of freedom and traces out a surface. A relation such as

$$f(u^1, u^2) = 0$$

between the parameters restricts P to a one-dimensional sequence of points, that is, to a curve on the surface.

The equation, $u^2=$constant, determines a curve on the surface and when the constant is varied a family of curves is generated which we shall call the u^1-parameter curves, since along any one of them u^1 alone varies. Similarly we shall have a family of u^2-parameter curves. It follows from our insistence on the one-to-one correspondence between pairs of parameter values and the points of the surface that each member of one family cuts each member of the other once and only once. Any two families of curves satisfying the above conditions will provide us with a coördinate system which could be used to specify our surface.

If A is any function, scalar or vector, defined for the points of our surface, we shall denote

$$\partial A/\partial u^1 \quad \text{and} \quad \partial A/\partial u^2$$

by A_1 and A_2, respectively. Thus $\mathbf{r}_1$ is a vector in the direction in which u^1 alone varies, or $\mathbf{r}_1$ is tangent to the u^1-parameter curve and therefore tangent to the surface. We write

$$\mathbf{r}_1 \cdot \mathbf{r}_1 = a_{11}, \qquad \mathbf{r}_1 \cdot \mathbf{r}_2 = a_{12} = a_{21}, \qquad \mathbf{r}_2 \cdot \mathbf{r}_2 = a_{22}. \tag{5.1}$$

Thus

$$\mathbf{r}_1 = \sqrt{a_{11}}\,\mathbf{v}, \qquad \mathbf{r}_2 = \sqrt{a_{22}}\,\mathbf{k}, \tag{5.2}$$

where $\mathbf{v}$ and $\mathbf{k}$ are unit vectors.

The $a_{\alpha\beta}$ are important because the distance between two neighboring points of the surface may be expressed in terms of them. The magnitude of the displacement $d\mathbf{r}$ from the point (u^1, u^2) to the point (u^1+du^1, u^2+du^2) is ds where

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{r}_1 du^1 + \mathbf{r}_2 du^2) \cdot (\mathbf{r}_1 du^1 + \mathbf{r}_2 du^2),$$

or

$$(ds)^2 = a_{11}(du^1)^2 + 2a_{12}du^1du^2 + a_{22}(du^2)^2. \tag{5.3}$$

Formula (5.3) is known as the *first ground form* of the surface.

The vector $\mathbf{r}_1 \times \mathbf{r}_2$ is orthogonal to two surface vectors and is therefore parallel to the unit surface normal $\mathbf{n}$. From (2.3) we have

$$(\mathbf{r}_1 \times \mathbf{r}_2) \cdot (\mathbf{r}_1 \times \mathbf{r}_2) = \mathbf{r}_1^2\, \mathbf{r}_2^2 - (\mathbf{r}_1 \cdot \mathbf{r}_2)^2 = a_{11}a_{22} - a_{12}^2.$$

Thus

$$\mathbf{n} = (\mathbf{r}_1 \times \mathbf{r}_2)/\sqrt{a}, \tag{5.4}$$

where $a = a_{11}a_{22} - a_{12}^2$. By taking the positive square root in (5.4) we determine the positive direction of the normal, making it depend on the manner in which the coördinate curves are numbered.

6. Curves on a surface. We find that two quantities known as the *normal curvature* and the *geodesic curvature* are of peculiar importance in the discussion of the relation of a curve to a surface in which it lies.

The normal curvature of a curve whose tangent is $\mathbf{t}$ is denoted by κ_n and is defined by the equation

$$\kappa_n = \mathbf{n} \cdot \dot{\mathbf{t}} = \kappa \mathbf{n} \cdot \mathbf{p} = \kappa \cos \theta, \tag{6.1}$$

where θ is the angle between the normal to the surface and the principal normal to the curve. For any curve on the surface, the curvature depends on the values of $(\dot{u}^1, \dot{u}^2)$ and $(\ddot{u}^1, \ddot{u}^2)$ at the point considered but, as we shall prove below, the normal curvature depends only on the value of $(\dot{u}^1, \dot{u}^2)$. Thus all curves with the same tangent at a point of the surface have the same normal curvature; so the normal curvature is more a characteristic of the surface than of the curve. By (4.1) we have

$$\kappa\mathbf{p} = \ddot{\mathbf{r}} = \frac{d}{ds}(\mathbf{r}_1\dot{u}^1 + \mathbf{r}_2\dot{u}^2)$$
$$= (\mathbf{r}_{11}\dot{u}^1 + \mathbf{r}_{12}\dot{u}^2)\dot{u}^1 + (\mathbf{r}_{21}\dot{u}^1 + \mathbf{r}_{22}\dot{u}^2)\dot{u}^2 + \mathbf{r}_1\ddot{u}^1 + \mathbf{r}_2\ddot{u}^2$$
$$= \mathbf{r}_1\ddot{u}^1 + \mathbf{r}_2\ddot{u}^2 + \mathbf{r}_{11}(\dot{u}^1)^2 + 2\mathbf{r}_{12}\dot{u}^1\dot{u}^2 + \mathbf{r}_{22}(\dot{u}^2)^2.$$

Therefore, by (6.1), we have

$$\kappa_n = b_{11}(\dot{u}^1)^2 + 2b_{12}\dot{u}^1\dot{u}^2 + b_{22}(\dot{u}^2)^2, \tag{6.2}$$

where $b_{\alpha\beta} = \mathbf{n}\cdot\mathbf{r}_{\alpha\beta}$. Since $\mathbf{n}\cdot\mathbf{r}_\alpha = 0$, we have

$$b_{\alpha\beta} = \mathbf{n}\cdot\mathbf{r}_{\alpha\beta} = -\mathbf{n}_\alpha\cdot\mathbf{r}_\beta = -\mathbf{n}_\beta\cdot\mathbf{r}_\alpha. \tag{6.3}$$

The formula

$$\kappa_n(ds)^2 = b_{11}(du^1)^2 + 2b_{12}du^1du^2 + b_{22}(du^2)^2 \tag{6.4}$$

is known as the *second ground form* of the surface.

We define the geodesic curvature, κ_g, by the equation

$$\kappa_g = \mathbf{m}\cdot\dot{\mathbf{t}}, \tag{6.5}$$

where $\mathbf{m}$ is the unit vector $\mathbf{n}\times\mathbf{t}$.

Since $\mathbf{t}$ is a unit vector, the magnitude of $\dot{\mathbf{t}}$ gives the rate at which the tangent is changing direction. Thus the normal curvature is a measure of the rate at which the curve departs from the tangent plane, and the geodesic curvature indicates the rate of turning in the tangent plane.

7. Christoffel symbols. Any vector may be expressed as a linear combination of three independent, (non-coplanar) vectors. Thus we can express $\mathbf{r}_{\alpha\beta}$ in terms of $\mathbf{n}$, $\mathbf{r}_1$, $\mathbf{r}_2$. We write

$$\mathbf{r}_{\alpha\beta} = b_{\alpha\beta}\mathbf{n} + \left\{ {1 \atop \alpha\beta} \right\}\mathbf{r}_1 + \left\{ {2 \atop \alpha\beta} \right\}\mathbf{r}_2, \tag{7.1}$$

where $\left\{ {\gamma \atop \alpha\beta} \right\}$ are defined as the coefficients in this equation. Taking the scalar product of both sides of (7.1) by $\mathbf{n}$ we see that the $b_{\alpha\beta}$ are the same as those defined in (6.3). Taking the scalar product of (7.1) by $\mathbf{n}\times\mathbf{r}_2$, we have

$$[\mathbf{n}, \mathbf{r}_2, \mathbf{r}_{\alpha\beta}] = [\mathbf{n}, \mathbf{r}_2, \mathbf{r}_1]\left\{ {1 \atop \alpha\beta} \right\}.$$

By a similar argument we obtain $\left\{ {2 \atop \alpha\beta} \right\}$. Thus noting (5.4) we have

$$\left\{ {1 \atop \alpha\beta} \right\} = -[\mathbf{n}, \mathbf{r}_2, \mathbf{r}_{\alpha\beta}]/\sqrt{a},$$
$$\left\{ {2 \atop \alpha\beta} \right\} = [\mathbf{n}, \mathbf{r}_1, \mathbf{r}_{\alpha\beta}]/\sqrt{a}. \tag{7.2}$$

The $\left\{ {\gamma \atop \alpha\beta} \right\}$ are the Christoffel symbols of the second kind, and are very im-

portant in modern differential geometry since they occur in the fundamental formulas of the absolute differential calculus. Formulas (7.2) give convenient expressions for them in the vector notation.

The Christoffel symbols of the first kind are defined as follows:

$$[\alpha\beta, \gamma] = \mathbf{r}_{\alpha\beta}\cdot\mathbf{r}_{\gamma}. \tag{7.3}$$

Definition (7.3) may be immediately generalized to apply to a space of n dimensions, but formulas (7.2) are valid only for a two-dimensional surface immersed in a three-space.

8. Darboux formula for geodesic curvature. At each point of any curve C of a surface there is a triad of mutually orthogonal unit vectors $\mathbf{t}$, $\mathbf{m}$, $\mathbf{n}$, where $\mathbf{m}=\mathbf{n}\times\mathbf{t}$. Also, if we set $\mathbf{w}=\mathbf{n}\times\mathbf{v}$, where $\mathbf{v}$ is the unit vector tangent to the u^1 coördinate curve, we have at each point of C the triad of mutually orthogonal unit vectors $\mathbf{v}$, $\mathbf{w}$, $\mathbf{n}$.

Let θ denote the angle between C and the u^1 coördinate curve. Then

$$\mathbf{v}\cdot\mathbf{t} = \cos\theta, \qquad \mathbf{v}\cdot\mathbf{m} = \cos(\theta+\pi/2) = -\sin\theta,$$
$$\mathbf{w}\cdot\mathbf{t} = \sin\theta, \qquad \mathbf{w}\cdot\mathbf{m} = \cos\theta.$$

Thus

$$\mathbf{t} = \cos\theta\,\mathbf{v} + \sin\theta\,\mathbf{w}, \tag{8.1}$$

$$\mathbf{w} = \sin\theta\,\mathbf{t} + \cos\theta\,\mathbf{m}. \tag{8.2}$$

From (8.1) we get $\dot{\mathbf{t}}$, and hence

$$\kappa_g = \mathbf{m}\cdot\dot{\mathbf{t}} = (\sin^2\theta + \cos^2\theta)\dot{\theta} + \cos\theta\,\dot{\mathbf{v}}\cdot\mathbf{m} + \sin\theta\,\dot{\mathbf{w}}\cdot\mathbf{m}. \tag{8.3}$$

Since $\mathbf{m}$, $\dot{\mathbf{n}}$, $\mathbf{v}$ are coplanar, $[\mathbf{m}, \dot{\mathbf{n}}, \mathbf{v}]=0$. Therefore

$$\mathbf{m}\cdot\dot{\mathbf{w}} = [\mathbf{m}, \dot{\mathbf{n}}, \mathbf{v}] + [\mathbf{m}, \mathbf{n}, \dot{\mathbf{v}}] = [\mathbf{m}, \mathbf{n}, \dot{\mathbf{v}}] = \mathbf{t}\cdot\dot{\mathbf{v}}.$$

Thus

$$\kappa_g = \dot{\theta} + (\cos\theta\,\mathbf{m} + \sin\theta\,\mathbf{t})\cdot\dot{\mathbf{v}},$$

or using (8.2) we have

$$\kappa_g = \dot{\theta} + \mathbf{w}\cdot\dot{\mathbf{v}}. \tag{8.4}$$

This formula has a simple geometric meaning and could perhaps have been written down without the above calculations. For $\dot{\theta}$ is the rate of change of direction of C with respect to the u^1 coördinate curve and $\mathbf{w}\cdot\dot{\mathbf{v}}$ is the rate of change of direction in the tangent plane of the u^1 curve, and thus their sum is the geodesic curvature of C.

Referring to the definition of $\mathbf{w}$ and to (5.2) we transform (8.4) obtaining

$$\kappa_g = \dot{\theta} + [\mathbf{n}, \mathbf{r}_1, \mathbf{r}_{11}]\dot{u}^1/a_{11} + [\mathbf{n}, \mathbf{r}_1, \mathbf{r}_{12}]\dot{u}^2/a_{11}.$$

Thus using the formulas for the Christoffel symbols (7.2) we have

$$\kappa_g = \dot{\theta} + \frac{\sqrt{a}}{a_{11}} \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \dot{u}^1 + \frac{\sqrt{a}}{a_{11}} \left\{ \begin{matrix} 2 \\ 12 \end{matrix} \right\} \dot{u}^2. \tag{8.5}$$

In the same manner we could show that

$$-\kappa_g = \dot{\phi} + \frac{\sqrt{a}}{a_{22}} \left\{ \begin{matrix} 1 \\ 21 \end{matrix} \right\} \dot{u}^1 + \frac{\sqrt{a}}{a_{22}} \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\} \dot{u}^2, \tag{8.6}$$

where ϕ is the angle which C makes with the u^2 coördinate curve.

If C is the u^1 curve, then $\dot{u}^1 = 1/\sqrt{a_{11}}$, $\dot{u}^2 = 0$, and $\dot{\theta} = 0$; so (8.5) gives

$$\overset{(1)}{\kappa_g} = \frac{\sqrt{a}}{(a_{11})^{3/2}} \left\{ \begin{matrix} 2 \\ 11 \end{matrix} \right\} \tag{8.7}$$

for the geodesic curvature of the u^1 coördinate curve. Similarly

$$\overset{(2)}{\kappa_g} = -\frac{\sqrt{a}}{(a_{22})^{3/2}} \left\{ \begin{matrix} 1 \\ 22 \end{matrix} \right\}. \tag{8.8}$$

When the coördinate curves are orthogonal, $\cos\theta = \sqrt{a_{11}}\dot{u}^1$ and $\sin\theta = \sqrt{a_{22}}\dot{u}^2$ and we obtain *Liouville's Formula* from (8.5), (8.7), and (8.8):

$$\kappa_g = \dot{\theta} + \overset{(1)}{\kappa_g} \cos\theta + \overset{(2)}{\kappa_g} \sin\theta. \tag{8.9}$$

9. Principal directions and lines of curvature. At any point P of a surface there is a normal $\mathbf{n}$. If we move P an infinitesimal amount $d\mathbf{r}$, $\mathbf{n}$ will change by $d\mathbf{n}$, say. Is it possible to choose the direction of $d\mathbf{r}$ so that $d\mathbf{n}$ will be parallel to it? In other words, can we satisfy

$$d\mathbf{r} + \rho d\mathbf{n} = 0, \tag{9.1}$$

where ρ is a scalar? Equation (9.1) will be satisfied if its projections on three independent directions are satisfied. The projection of the left-hand side of the equation on $\mathbf{n}$ is identically zero. The projections on $\mathbf{r}_1$ and $\mathbf{r}_2$ give the conditions

$$\begin{aligned} \mathbf{r}_1 \cdot d\mathbf{r} + \rho \mathbf{r}_1 \cdot d\mathbf{n} &= 0, \\ \mathbf{r}_2 \cdot d\mathbf{r} + \rho \mathbf{r}_2 \cdot d\mathbf{n} &= 0, \end{aligned} \tag{9.2}$$

which, by the use of (5.1) and (6.3), may be written

$$\begin{aligned} (a_{11} - \rho b_{11}) du^1 + (a_{12} - \rho b_{12}) du^2 &= 0, \\ (a_{21} - \rho b_{21}) du^1 + (a_{22} - \rho b_{22}) du^2 &= 0. \end{aligned} \tag{9.3}$$

These equations can be satisfied only if

$$\begin{vmatrix} a_{11} - \rho b_{11}, & a_{12} - \rho b_{12} \\ a_{21} - \rho b_{21}, & a_{22} - \rho b_{22} \end{vmatrix} = 0. \tag{9.4}$$

This quadratic equation in ρ will, in general, have two distinct roots $\rho^{(1)}$ and $\rho^{(2)}$.

Corresponding to these roots there will be two directions at each point called the *principal directions*. We shall leave to the reader the pleasant task of showing that the principal directions give the maximum and minimum values of the normal curvature for the point considered, and that these values are the reciprocals of $\rho^{(1)}$ and $\rho^{(2)}$.

A curve which is tangent at each of its points to a principal direction is called a *line of curvature*. There will be two families of lines of curvature, and if we take them as the coördinate curves we have, in view of (9.1),

$$\mathbf{r}_1 = -\rho^{(1)}\mathbf{n}_1, \qquad \mathbf{r}_2 = -\rho^{(2)}\mathbf{n}_2, \tag{9.5}$$

which are known as *Rodrigues' Formulas*.

From $\mathbf{n}\cdot\mathbf{r}_1 = \mathbf{n}\cdot\mathbf{r}_2 = 0$, we deduce

$$\mathbf{n}_2\cdot\mathbf{r}_1 + \mathbf{n}\cdot\mathbf{r}_{12} = 0 = \mathbf{n}_1\cdot\mathbf{r}_2 + \mathbf{n}\cdot\mathbf{r}_{21},$$

and hence

$$\mathbf{n}_2\cdot\mathbf{r}_1 - \mathbf{n}_1\cdot\mathbf{r}_2 = 0.$$

But from (9.5) we see that the left-hand side of this equation is equal to $(\rho^{(2)} - \rho^{(1)})\, \mathbf{n}_1\cdot\mathbf{n}_2$. Thus $\mathbf{r}_1\cdot\mathbf{r}_2 = 0$, and therefore the principal directions at any point are orthogonal.

10. Gaussian curvature. The quantities $\rho^{(1)}$ and $\rho^{(2)}$ are characteristic of the surface. Thus the reciprocal of their product, which is called the *Gaussian curvature* and is denoted by K, is independent of the coördinate system. From (9.4) and (6.3) we find that

$$\begin{aligned} K &= (b_{11}b_{22} - b_{12}^2)/a \\ &= (\mathbf{r}_1\cdot\mathbf{n}_1\mathbf{r}_2\cdot\mathbf{n}_2 - \mathbf{r}_1\cdot\mathbf{n}_2\mathbf{r}_2\cdot\mathbf{n}_1)/a. \end{aligned} \tag{10.1}$$

Thus by (2.3) and (5.4) we have

$$K = [\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2]/\sqrt{a}. \tag{10.2}$$

Let us consider $[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2]$, where the subscripts refer to any coördinate system. We shall use the notation of §8. We have

$$\begin{aligned} [\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2] &= (\mathbf{v} \times \mathbf{w})\cdot(\mathbf{n}_1 \times \mathbf{n}_2) \\ &= \mathbf{v}\cdot\mathbf{n}_1\mathbf{w}\cdot\mathbf{n}_2 - \mathbf{v}\cdot\mathbf{n}_2\mathbf{w}\cdot\mathbf{n}_1 \\ &= \mathbf{v}_1\cdot\mathbf{n}\mathbf{w}_2\cdot\mathbf{n} - \mathbf{v}_2\cdot\mathbf{n}\mathbf{w}_1\cdot\mathbf{n}, \end{aligned}$$

since $\mathbf{v}\cdot\mathbf{n} = \mathbf{w}\cdot\mathbf{n} = 0$. Also, since $\mathbf{v}\cdot\mathbf{v}_1 = \mathbf{v}\cdot\mathbf{v}_2 = 0$, we have

$$[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2] = \mathbf{v}_1\cdot(\mathbf{w}_2\cdot\mathbf{n}\mathbf{n} + \mathbf{w}_2\cdot\mathbf{v}\mathbf{v}) - \mathbf{v}_2\cdot(\mathbf{w}_1\cdot\mathbf{n}\mathbf{n} + \mathbf{w}_1\cdot\mathbf{v}\mathbf{v}).$$

Thus

$$[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2] = \mathbf{v}_1\cdot\mathbf{w}_2 - \mathbf{v}_2\cdot\mathbf{w}_1. \tag{10.3}$$

We readily see from this that

$$[\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2] = (\mathbf{w}\cdot\mathbf{v}_1)_2 - (\mathbf{w}\cdot\mathbf{v}_2)_1. \tag{10.4}$$

From (10.4) we can immediately deduce a well known expression for the Gaussian curvature, the usual proofs of which require elaborate calculations. From (10.2) and (5.2) we have

$$\begin{aligned} K &= [(\mathbf{w}\cdot\mathbf{v}_1)_2 - (\mathbf{w}\cdot\mathbf{v}_2)_1]/\sqrt{a} \\ &= \frac{1}{\sqrt{a}}\left[\left(\mathbf{w}\cdot\frac{\mathbf{r}_{11}}{\sqrt{a_{11}}}\right)_2 - \left(\mathbf{w}\cdot\frac{\mathbf{r}_{12}}{\sqrt{a_{11}}}\right)_1\right]. \end{aligned}$$

Thus, noting (7.2) and remembering that $\mathbf{w}=\mathbf{n}\times\mathbf{v}$, we finally have

$$K = -\frac{1}{\sqrt{a}}\left[\left(\frac{\sqrt{a}}{a_{11}}\begin{Bmatrix}2\\12\end{Bmatrix}\right)_1 - \left(\frac{\sqrt{a}}{a_{11}}\begin{Bmatrix}2\\11\end{Bmatrix}\right)_2\right]. \tag{10.5}$$

By interchanging the rôles of u^1 and u^2 in the above proof we can show that

$$K = \frac{1}{\sqrt{a}}\left[\left(\frac{\sqrt{a}}{a_{22}}\begin{Bmatrix}1\\22\end{Bmatrix}\right)_1 - \left(\frac{\sqrt{a}}{a_{22}}\begin{Bmatrix}1\\21\end{Bmatrix}\right)_2\right]. \tag{10.6}$$

These equations are alternative forms of the *Gauss characteristic equation*. Although we did not prove it, the Christoffel symbols may be expressed in terms of the $a_{\alpha\beta}$ and their first derivatives; so the Gauss characteristic equation tells us that K may be expressed in terms of the $a_{\alpha\beta}$ and their first and second derivatives.

In orthogonal coördinates the last two formulas reduce to the simpler form

$$K = -\frac{1}{\sqrt{a}}\left[\left(\frac{1}{\sqrt{a_{11}}}(\sqrt{a_{22}})_1\right)_1 + \left(\frac{1}{\sqrt{a_{22}}}(\sqrt{a_{11}})_2\right)_2\right]. \tag{10.7}$$

11. Gauss-Bonnet theorem. If dS is the element of area $|\mathbf{r}_1du^1\times\mathbf{r}_2du^2| = |\mathbf{r}_1\times\mathbf{r}_2|du^1du^2=\sqrt{a}du^1du^2$, we have , noting (10.2),

$$\iint K\,dS = \iint [\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2]du^1du^2,$$

where the integral is taken over the area of our surface bounded by a closed curve C.

By (10.4) and (3.1),

$$\begin{aligned} \iint K\,dS &= \iint [(\mathbf{w}\cdot\mathbf{v}_1)_2 - (\mathbf{w}\cdot\mathbf{v}_2)_1]du^1du^2 \\ &= -\int \mathbf{w}\cdot\mathbf{v}_1du^1 + \mathbf{w}\cdot\mathbf{v}_2du^2 = -\int \mathbf{w}\cdot\dot{\mathbf{v}}ds. \end{aligned}$$

Thus, using (8.4), we have

$$\iint K\,dS + \int \kappa_g ds = \int \dot{\theta} ds. \tag{11.1}$$

If (C) has a finite number of corners such that the increase in θ at the ith of them is $\theta^{(i)}$, then

$$\int_C \dot{\theta} ds + \Sigma\theta^{(i)} = 2\pi. \tag{11.2}$$

Comparing these last two equations we have the *Gauss-Bonnet theorem*

$$\iint K\,dS + \int \kappa_g ds + \Sigma\theta^{(i)} = 2\pi. \tag{11.3}$$

A *geodesic* may be defined as a curve along which $\kappa_g = 0$. When C consists of a triangle whose sides are geodesics and whose interior angles are α, β, and γ, (11.3) reduces to

$$\iint K\,dS = \alpha + \beta + \gamma - \pi, \tag{11.4}$$

a result which was given by Gauss. The right-hand side of this equation is called the *geodesic excess* of the triangle. When our surface is a sphere of radius R, (11.4) gives the well known formula for the area, A, of a spherical triangle

$$A = R^2(\alpha + \beta + \gamma - \pi). \tag{11.5}$$

Conclusion. Much of the simplicity of the above proofs results from the use of formula (7.2) for the Christoffel symbols, which does not seem to have been noticed hitherto. The introduction of the Christoffel symbols in courses on differential geometry is highly desirable, in view of their importance in the tensor calculus. I have applied the same methods as above with considerable success in the proof of the Fundamental Theorem of differential geometry, the application of the calculus of variations to geodesics and minimal surfaces, the discussion of triply-orthogonal surfaces, and the treatment of differential invariants.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A NOTE ON TETRAHEDRAL RIEMANN SURFACE MODELS

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In the April 1939 issue of this MONTHLY there appeared an article by Dr. Sadowsky entitled "A tetrahedral Riemann surface model of a closed finite locally-Euclidean two-space." In this article the author proved that an elementary parallelogram may be folded into a *two-sheeted* Riemann tetrahedral surface model.

After a little investigation I find that this parallelogram may be folded into an *eight-sheeted* Riemann tetrahedral surface model.

Procedure:

1. In the elementary parallelogram $V(11\text{–}13\text{–}15\text{–}17)$ draw the diagonal $V(13\text{–}17)$ between the two obtuse vertices.
2. Divide the parallelogram into 32 congruent triangles as indicated.
3. Make the following cuts:

$V(14\text{–}19)$	$V(16\text{–}36)$	$V(18\text{–}25)$	$V(12\text{–}35)$
$V(32\text{–}42)$	$V(33\text{–}43)$	$V(34\text{–}44)$	$V(31\text{–}41)$

4. Fold sub-parallelogram $V(14\text{–}15\text{–}16\text{–}19)$ as explained by Dr. Sadowsky.

The index numbers assigned to each V signify that any V's containing the same first index number lie on the same vertex of the final tetrahedron and on the same vertex of the three other sub-tetrahedrons, the last number being used to prevent ambiguity in discussing the four different sub-parallelograms. Any edge formed by two neighboring V's will coincide with any other edge formed by two neighboring V's if the two first index numbers of the two sets of V's are similar, regardless of order.

5. Leaving the first tetrahedron folded, fold sub-parallelogram $V(16\text{–}17\text{–}18\text{–}19)$ so that the first sub-tetrahedron is completely enclosed by the new sub-tetrahedron.*

At this point cuts $V(32\text{–}42)$, $V(16\text{–}36)$, and $V(33\text{–}43)$ may be annihilated, thus closing the interior of the first two sub-parallelograms. Also $V(14\text{–}15)$ is joined to $V(17\text{–}18)$ by a self-intersection.

6. Now fold sub-parallelogram $V(11\text{–}12\text{–}19\text{–}18)$ so that the first two sub-tetrahedrons are completely enclosed by the new tetrahedron.

Again we may annihilate the cuts $V(18\text{–}25)$ and $V(34\text{–}44)$, thus closing the interior of the first three sub-tetrahedrons. $V(16\text{–}17)$ and $V(11\text{–}12)$ are joined by a self-intersection.

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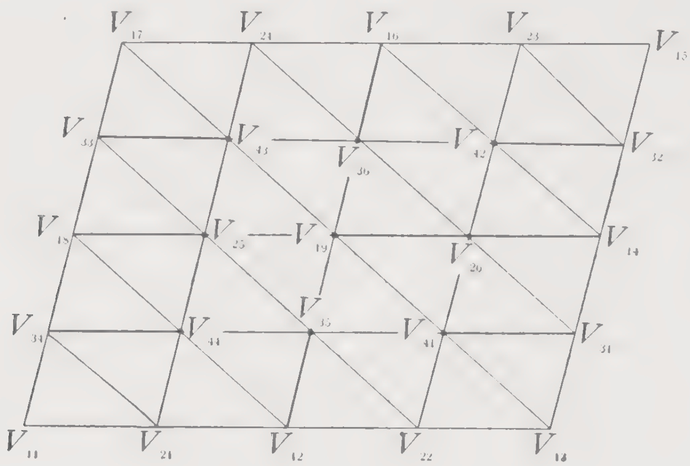
* A sub-tetrahedron, in this case, is a two-sheeted tetrahedral surface formed by a sub-parallelogram.

7. Fold the final sub-parallelogram so the three former sub-tetrahedrons are completely enclosed by the latter. We now complete the closure by annihilating the cuts $V(12–35)$ and $V(31–41)$. Now the interior of the elementary parallelogram is completely closed.* By self-intersection edge $V(11–18)$ is joined to edge $V(13–14)$, and edge $V(12–13)$ is joined to $V(15–16)$. Now our tetrahedral surface is completely closed.



It seems very evident that the possibility of the existence of an even-infinite-sheeted tetrahedral Riemann surface model derivable from an elementary parallelogram is by no means inconceivable. The difficulties of proving this by paper-doll cutting are multiplied by the folding of the large number of sheets. Personally, I should like to see this demonstrated mathematically instead of empirically.

COMMENT ON MR. BARTEN'S NOTE

Bryant Tuckerman, Princeton University

In connection with Mr. Barten's conjecture, it may be pointed out that "A tetrahedral Riemann surface model of a closed finite locally-Euclidean two-space" can be constructed with any even number of sheets. The basic parallelogram of Mr. Barten's 8-sheeted model is a 2×2 lattice of parallelograms of Mr. Sadowsky's type (each consisting of eight congruent triangles). It is shown here that a $2mn$-sheeted model (where m and n are any positive integers) can be formed by cutting and reassembling an $m\times n$ lattice of such parallelograms. Let the vertices in alternate rows be numbered 1, 2, 1, 2, $\cdots$, 1 and 3, 4, 3, 4, $\cdots$, 3 respectively, the last being 1, 2, 1, 2, $\cdots$, 1. While this parallelogram can be cut and folded as a single piece, the construction is clearer if

* Cut $V(14–19)$ is annihilated to complete the closure of the interior of the fundamental parallelogram.

it is cut into its $8mn$ elementary triangles. These triangles are assembled, according to the numbering of their vertices, into a $2mn$-layered tetrahedron. It can be seen from the numbering that the various portions of the exterior of the parallelogram which are to be identified, as well as the two lips of each cut, are superposed in proper orientation along the edges of the tetrahedron, and hence can be joined, closing the surface.

The tetrahedron is then a $2mn$-sheeted Riemann surface; and since the neighborhoods of edges and vertices are reassembled as in the original parallelogram, but with the identification of its opposite sides, the Riemann surface is everywhere locally-Euclidean as a two-space. Each neighborhood of a vertex of the parallelogram (with the identifications) has become a two-sheeted neighborhood of a vertex of the tetrahedron; access to other sheets is secured by paths around other vertices. (The metrical necessity of this pairing of sheets establishes the impossibility of an odd-sheeted model of the type under discussion.)

NOTE ON CRITERIA OF DIVISIBILITY

W. S. McCulley, A. and M. College of Texas

In several familiar text-books on number theory* and algebra† appear problems involving the establishment of criteria for divisibility of a given integer N by 7, 9, 11, 13, 73, and 137. It is possible to prove a more general theorem that yields these various criteria as corollaries. The first corollary, which was obtained independently by the author, was proved as a theorem by G. Loria‡ in 1902 in a form differing somewhat from that which follows.

Theorem. *If m denotes an integer such that $(10, m)=1$, if N denotes any integer expressed in the form $N=\sum_{i=0}^{n}a_ir^i$, where $r=10^t$, the a_i denote integers such that $0\leqq a_i<r$, and if either*

(A) $10^t\equiv 1 \pmod{m}$, *and* $S=\sum_{i=0}^{n}a_i$,

or

(B) $10^t\equiv -1 \pmod{m}$, *and* $S=\sum_{i=0}^{n}(-1)^ia_i$,

then $N\equiv S \pmod{m}$.

Proof. Since $r=10^t$, we have, by (A), $r\equiv 1 \pmod{m}$, hence to the modulus m $a_0\equiv a_0$, $a_1r\equiv a_1$, $a_2r^2\equiv a_2$, $\cdots$, $a_nr^n\equiv a_n$; and therefore $\sum_{i=0}^{n}a_ir^i\equiv \sum_{i=0}^{n}a_i \pmod{m}$, where, by hypothesis, the a_i denote residues less than r but not necessarily less than m. Therefore, if $10^t\equiv 1 \pmod{m}$, then $N\equiv S \pmod{m}$.

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* Dickson, Introduction to the Theory of Numbers, Chicago, 1929, p. 8. Carmichael, Theory of Numbers, New York, 1914, p. 46.

† Barnard and Child, Higher Algebra, London, 1936, p. 422, 427.

‡ G. Loria, Il Boll. Matematica Gior. Sc. Didat., Bologna, 1, 1902.

From (B) of the hypothesis $10^t \equiv -1 \pmod{m}$, $r = 10^t$, hence $r \equiv -1 \pmod{m}$, and as before, except that the terms are now alternately plus and minus, we obtain $\Sigma_{i=0}^{n} a_i r^i \equiv \Sigma_{i=0}^{n} (-1)^i a_i \pmod{m}$. Therefore, if $10^t \equiv -1 \pmod{m}$, then $N \equiv S \pmod{m}$. Moreover, it follows immediately that if either N or S is divisible by m, the other is also. Thus we have the following:

COROLLARY 1. *A necessary and sufficient condition that an integer N be divisible by an integer m is that $S \equiv 0 \pmod{m}$.*

Computation establishes the following table of factors:

$10^1 - 1 = 3^2$	$10^1 + 1 = 11$
$10^2 - 1 = 3^2 \cdot 11$	$10^2 + 1 = 101$
$10^3 - 1 = 3^3 \cdot 37$	$10^3 + 1 = 7 \cdot 11 \cdot 13$
$10^4 - 1 = 3^2 \cdot 11 \cdot 101$	$10^4 + 1 = 73 \cdot 137$
$10^5 - 1 = 3^2 \cdot 41 \cdot 271$	$10^5 + 1 = 11 \cdot 9091$

Several criteria of divisibility may now be stated as corollaries to the theorem and corollary just proved, to indicate their applications. We have $10^t \equiv 1 \pmod{9}$, where t denotes any positive integer, and the definition of S in the theorem leads to the following:

COROLLARY 1.1. *An integer N is congruent modulo 9 to the sum of its digits, and is divisible by 9 if the sum of its digits is so divisible.*

Application of this corollary is commonly known as "casting out the nines." The congruence $10^5 \equiv 1 \pmod{41}$ yields the following:

COROLLARY 1.2. *An integer N is congruent modulo 41 to the sum of its digits taken five at a time from the right, and is divisible by 41 if this sum is so divisible.*

It should be noted that because of the notation $N = \Sigma_{i=0}^{n} a_i r^i$, the first set of t digits on the right, as N is ordinarily written, is denoted by a_0. It will, however, be more convenient for the statement of corollaries involving congruences of the type $10^t \equiv -1 \pmod{m}$ to replace this subscript and the succeeding ones by the ordinal numbers of the sets of digits. For example, since $10 \equiv -1 \pmod{11}$, $10^2 \equiv 1 \pmod{11}$, the definition of S in the theorem leads to the following:

COROLLARY 1.3. *An integer N is congruent modulo 11 to the difference between the sum of the digits in the even-numbered places and the sum of the digits in the odd-numbered places, and N is divisible by 11 if this difference is so divisible.*

Application of this corollary is commonly known as "casting out the elevens." A similar procedure will yield corollaries for the other divisors shown in the table.

Corollaries 1.1 and 1.3 find their chief application in checking continued sums and products for errors which may arise either in writing down the numbers themselves, or in the computations, as in multiple correlation tables and in

accounting, or, more generally, in any tables involving repetition of entries for purposes of computation.

All the corollaries will, in many cases, considerably shorten the process of testing large numbers for divisors. Moreover, no other divisors besides those mentioned in the table will yield criteria which may be applied as advantageously as these.

ON EXTENDING THE DEFINITION OF A HARMONIC FUNCTION

J. H. Curtiss, Cornell University

In a recent note in this Monthly (November, 1939, pp. 587–588) the following result is stated by John Beek, Jr.:

"Let $U(x, y)$ be a real function, defined for complex values of x and y, and satisfying Laplace's equation. Then the conjugate function is the imaginary part of $2U[(z+c)/2, (z-c)/2i]$, where $z=x+iy$, and c is a constant restricted only by the condition that the function be finite."

The author then gives a brief proof, which is not entirely complete from the rigorous point of view.* The theorem itself, however, is substantially correct *if properly interpreted*, and it calls attention to the existence of an interesting and curious formal relationship between an analytic function and its real part. It seems worth while, since the subject has been opened, to indicate just what this relationship is and how it may be established.

The chief difficulty in interpreting Beek's result lies in deciding upon a suitable definition of U for complex values of x and y. Thus for example, the function $U=|x|$ is harmonic in the half-plane lying to the left of the y-axis, but it would not do for Beek's purposes to write $2U[(z+c)/2, (z-c)/2i]=2|(z+c)/2|$. Once we have arrived at a suitable extension of the definition of $U(x, y)$, Beek's proposition follows at once, and accordingly our main concern in this note will be to formulate such an extension.

To state the problem in more exact language, let us consider three complex planes, which we shall call the z-plane, the z_1-plane, and the z_2-plane. The coördinates of the point of affix z in the z-plane will be denoted by x and y; these letters will denote real numbers throughout this discussion. We shall suppose that to the arbitrary point $z=x+iy$ in the z-plane there corresponds the point $z_1=x$ on the real axis of the z_1-plane, and the point $z_2=y$ on the real axis of the z_2-plane. Let $U(x, y)$ be harmonic in a bounded, simply connected region R of the z-plane, and let $V(x, y)$ be a harmonic conjugate of U in R. The function $V(x, y)$ may be represented by the formula†

$$V(x, y) = K + \int_{x_0, y_0}^{x, y} \left(-\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy \right), \tag{1}$$

* It should also be mentioned that Beek's notation is somewhat ambiguous. By z^*, he means the conjugate imaginary of z, but by $f^*(z)$ he does not mean the conjugate imaginary of $f(z)$.

† Osgood, Lehrbuch der Funktionentheorie, fifth edition, Leipzig, 1928, volume I, p. 655.

where K is a real constant, (x_0, y_0) is an arbitrary point of R, and the path of integration is any rectifiable arc lying entirely in R with the indicated end-points. We shall denote the function $U(x, y)+iV(x, y)$ by $f(z)$; this function is an analytic function of z. Our problem is to formulate a natural and useful definition of $U(z_1, z_2)$ for values of z_1 and z_2 not on the real axes of the z_1- and z_2-planes, respectively.

The usual way to extend the definition of a real analytic function of a single real variable to complex values of the variable consists simply in replacing the real variable by a complex one in the Taylor series for the function and using complex algebra to evaluate the new series. An analogous method is readily available in the case of $U(x, y)$, for it is known that there exist sequences of harmonic polynomials in x and y which converge uniformly to U in every closed sub-region of R.* For example, if R is the interior of a circle, properly arranged sections of the Taylor expansion of U about the center of the circle will constitute such a sequence. Our method, then, will be to replace x and y by z_1 and z_2 in such a sequence of harmonic polynomials and to use the algebra of complex numbers to evaluate the resulting expressions.

But there are an infinite number of different sequences of harmonic polynomials convergent to U in R, and it is natural to inquire whether we have arrived at a unique result in choosing any one of these sequences. Also, it is inconvenient to restrict ourselves to sequences convergent throughout R, as the polynomials of such a sequence are usually difficult to obtain explicitly, and the question arises as to whether we should obtain the same definition of $U(z_1, z_2)$ for z_1 and z_2 suitably restricted if we use one of the sequences which converge only in a sub-region of R. (The sections of the Taylor series about an arbitrary point in R give an example of such a sequence.) Then, too, questions arise as to the persistence of convergence after the change has been made to complex variables. These considerations lead us to the following result, which is essentially a restatement of Beek's proposition in more exact language.

THEOREM. *Let R_1 be a sub-region of R and let $P_n(x, y)$, $(n=0, 1, 2, \cdots)$, be any sequence of harmonic polynomials of respective degrees† n in x and y which converges uniformly to $U(x, y)$ in every closed sub-region of R_1. If complex numbers z_1 and z_2 be chosen so that the points of affix z' and $\overline{z''}$ both lie in R_1, where $z'=z_1+iz_2$, $z''=z_1-iz_2$, then $\lim_{n\to\infty} P_n(z_1, z_2)=[f(z')+\overline{f(\overline{z''})}]/2$ uniformly for z' and $\overline{z''}$ in any closed sub-region of R_1.*

Thus it appears that an appropriate definition for $U(z_1, z_2)$ is $[f(z')+\overline{f(\overline{z''})}]/2$ provided that z' and $\overline{z''}$ both lie in R. It is worth noting that according to this

* Walsh, The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions, Bulletin of the American Mathematical Society, vol. 35, pp. 499–544.

† By a real polynomial of degree n in x and y, we mean an expression of the type $\Sigma a_{hk}x^hy^k$, summed over all non-negative integers h, k for which $h+k\leq n$, the a_{hk} being real numbers any or all of which may be zero. Similarly in speaking of a complex polynomial of degree n, we shall allow any of the coefficients to be zero.

definition, $U(z_1, z_2)$ is an analytic function of the two complex variables z' and z'', since $\overline{f(\bar{z})}$ is an analytic function of z.

To prove the theorem, we first observe that if $P_n(x, y)$ is a harmonic polynomial of degree n, then the conjugate

$$Q_n(x, y) = \int_{x_0, y_0}^{x, y} \left(- \frac{\partial P_n}{\partial y} dx + \frac{\partial P_n}{\partial x} dy \right) \tag{2}$$

is itself a polynomial of degree n in x and y. Furthermore, the function $f_n(z) = P_n(x, y) + iQ_n(x, y)$ is an analytic function of z, and its $(n+1)$th derivative is $\partial^{n+1}P_n/\partial x^{n+1} + i(\partial^{n+1}Q_n/\partial x^{n+1}) = 0$. Thus $f_n(z)$ is a complex polynomial of degree n. We may write

$$f_n(z) = \sum_{k=0}^{n} a_{n,k} z^k,$$

$$2P_n(x, y) = f_n(z) + \overline{f_n(z)} = \sum_{k=0}^{n} a_{n,k}(x + iy)^k + \sum_{k=0}^{n} \overline{a_{n,k}}(x - iy)^k,$$

$$2P_n(z_1, z_2) = \sum_{k=0}^{n} a_{n,k}(z')^k + \sum_{k=0}^{n} \overline{a_{n,k}}(z'')^k = f_n(z') + \overline{f_n(\overline{z''})}.$$

In (1) and (2) let (x, y) and (x_0, y_0) be points of R_1 and let the path of integration be the same in both integrals and lie in R_1. Then

$$V(x, y) - Q_n(x, y) = K + \int_{x_0, y_0}^{x, y} \left[- \left(\frac{\partial U}{\partial y} - \frac{\partial P_n}{\partial y} \right) dx + \left(\frac{\partial U}{\partial x} - \frac{\partial P_n}{\partial x} \right) dy \right]. \tag{3}$$

Now if $\lim_{n\to\infty} P_n(x, y) = U(x, y)$ uniformly in every closed sub-region of R_1, then by a well known theorem,* $\lim_{n\to\infty} \partial P_n/\partial x = \partial U/\partial x$ and $\lim_{n\to\infty} \partial P_n/\partial y = \partial U/\partial y$ uniformly in every closed sub-region of R_1. It follows from this and from (3) that $\lim_{n\to\infty} Q_n(x, y) = V(x, y) - K$ uniformly in any closed circular sub-region of R_1, and therefore (by the Heine-Borel Theorem) in any closed sub-region of R_1. That is, $\lim_{n\to\infty} f_n(z) = f(z) - iK$ and $\lim_{n\to\infty} \overline{f_n(z)} = \overline{f(z)} + iK$ uniformly in every closed sub-region of R_1. This means that

$$\lim_{n\to\infty} 2P_n(z_1, z_2) = f(z') - iK + \overline{f(\overline{z''})} + iK = f(z') + \overline{f(\overline{z''})},$$

uniformly in every closed sub-region of R_1. This completes the proof. It should be observed that the final result is independent of the choice of K and of (x_0, y_0) in (3).

If now we let $z_1 = (z+c)/2$ and $z_2 = (z-c)/2i$, so that $z' = z$, $z'' = c$, we may write

$$2U\,[(z + c)/2,\ (z - c)/2i] = f(z) + \overline{f(\bar{c})},$$

provided that z and $\bar{c}$ lie in R. This is the formal relationship between a function

* Osgood, *op. cit.*, p. 682.

and its real part mentioned in our introductory remarks. Let $c=c_1+ic_2$, c_1, c_2 real. The real part of $2U[(z+c)/2, (z-c)/2i]$ is $U(x, y)+U(c_1, -c_2)$ and the imaginary part is $V(x, y)-V(c_1, -c_2)$, in accordance with Beek's assertion.

How may this theory be used to find the conjugate of an explicitly given harmonic function? This is the problem that Beek evidently had in mind. Clearly the solution calls for a heuristic approach. If, when we substitute $(z+c)/2$, $(z-c)/2i$ for x and y in the expression for $U(x, y)$, we obtain an obviously analytic function of z and of c of which the real part differs from U by a constant, then the imaginary part is a conjugate of U. If it is not apparent that we obtain such a function, then the substitution must be made in some convenient sequence of approximating polynomials. In this case it will usually be found that (1) furnishes a more tractable solution.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department at the Mathematical Association of America, 531 West 116th St., New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

The Mathematical Theory of Huygens' Principle. By B. B. Baker and E. T. Copson. Oxford, Clarendon Press, 1939. 7+155 pages. $4.25.

Modern Machine Calculation with the Facit Calculating Machine Model Lx. By H. Sabielny. Translated and revised by L. J. Comrie and H. O. Hartley. London, Scientific Computing Service Ltd., 1939. 74 pages. 5s.

Three Copernican Treatises. The Commentariolus of Copernicus, The Letter against Werner, The Narratio Prima of Rheticus. Translated with introduction and notes by Edward Rosen. (Records of Civilization: Sources and Studies, no. 30.) New York, Columbia University Press, 1939. 10+211 pages. $3.00.

Elementary Calculus. By G. W. Caunt. Oxford, Clarendon Press, 1939. 388 pages. $2.75.

Statistical Method from the View-point of Quality Control. By W. A. Shewart and W. E. Deming. Washington, D. C., The Graduate School, Department of Agriculture, 1939. 9+155 pages.

REVIEWS

Elementary Statics. A text-book for engineers. By M. Appleby. Cambridge, The University Press; New York, The Macmillan Company, 1939. 8+164 pages. $2.25.

The author, professor of applied mathematics in the Royal University of Malta, has based this book on lectures delivered to students of engineering. As stated in the preface, no attempt has been made to develop the subject rigorously from the minimum number of fundamentals; instead, the aim has evidently been to provide the prospective engineer with a tool for the solution of specific problems.

With this end in mind, general principles and theorems receive brief or no discussion, and nearly all of this small book is devoted to the consideration of rather simple examples chosen from various fields. The mathematics used is as elementary as the mechanics involved in these problems; apart from a short section on the application of the principle of virtual work, all the rest of the book is free of calculus.

The following summary will indicate the scope and content of the book. Chapter I, Introductory, contains 2 illustrative examples and 12 exercises for the student; Chapter II, Translation and Rotation, 22 examples, 25 exercises; Chapter III; Work and Machines, 12 examples, 24 exercises; Chapter IV, Friction, 17 examples, 25 exercises; Chapter V, Centre of Gravity, 10 examples, 30 exercises; Chapter VI, Graphical Statics, 6 examples, 25 exercises; Chapter VII, Couples and Pure Rotation, 10 examples, 16 exercises; Chapter VIII, Miscellaneous (this includes Hooke's law, virtual work, the energy test of stability, hydrostatics, and the suspended cable), 12 examples, 25 exercises; 50 miscellaneous exercises on the book; answers to exercises; index.

Within the limitations of scope and size indicated above, the treatment is thorough and clear, and the book should prove satisfactory for a short course. The style is somewhat choppy, and the notation is sometimes not explained but must be inferred from the accompanying figure or discussion. There appear to be few errors or misprints, among which the following were noted. Page 11, exercise 11: *medium* is written for *median*; page 56, last line: $\sqrt{6}/2$ should be $\sqrt{3}/2$; page 103, line 8: *known* is written for *know*; pages 138–139: length is denoted by l in the statement of Example 9, but by L in the solution; page 140, Fig. 125: the point A, referred to in the solution, is not indicated; page 151, exercise 26: the answer 1/3 is obviously incorrect.

F. H. Miller

Business Mathematics. Second edition. By I. L. Miller and C. H. Richardson. New York, D. Van Nostrand Company, Inc., 1939. 12+352 pages. $3.75.

This book, which is an extension of the book *Commercial Algebra* by the same authors, is thought of as a text-book for a one-year course for students in business and finance. It does not presuppose any mathematical knowledge beyond first-year algebra.

In the first nine chapters, the authors give a detailed account of the algebraic processes that are to be used in the later chapters of the book. Although some of the details in these first nine chapters could not withstand a rigorous investigation from the modern mathematical view-point, the authors have kept the text relatively free of such inexact statements as are usually found in books of this type. These chapters cover such subjects as the four rational operations, solutions of linear equations with several unknowns as well as quadratic equations, powers, radicals, and logarithms, and arithmetical and geometrical progressions.

Chapters X to XVI deal with commercial applications. Included in these chapters are discussions of percentage, simple and compound interest and discount, annuities and related subjects. The usual formulas of the theory of interest and annuities are derived in a rather general way.

Chapters XVII to XXI give an elementary introduction into the theory of life insurance. In Chapter XVII, the authors have given a treatment of the elementary theory of probability, *i.e.*, the part of that theory based purely on combinatorial methods. In the following chapters, this theory is then applied to the most common problems in life insurance, in particular to life annuities, premiums, reserves, *etc.* The study of this part of the book cannot be considered as a sufficient background for a student who wishes to go into actuarial work, though it may be of considerable help for the layman interested in this subject.

The fact that almost every article in the book begins with a clear definition of the terms involved is a recommendable feature of the book. Many chapters contain a summary of all the formulas derived in them and thus enable the student to compare these formulas.

The book contains many illustrative examples as well as a large number of exercises and problems pertaining to the various subjects. However, the fact that the authors have not given the solutions to these problems may prove rather inconvenient to teachers and students alike.

The authors have inserted at the end of the book thirteen tables, including a table of logarithms, compound interest tables, annuity tables, a mortality table, and other life insurance tables.

FRITZ HERZOG

Bibliography of Early American Textbooks on Algebra Published in the Colonies and the United States through 1850, together with a characterization of the first edition of each work. Scripta Mathematica Studies, Number One. By Lao G. Simons. New York, Scripta Mathematica, 1936. 68 pages. $1.00.

This bibliography is in many ways a sequel to the author's *Introduction of Algebra into American Schools in the Eighteenth Century*, Washington, 1924.

Only one of the titles listed belongs to the period of the earlier study. This is the *Arithmetica of Cyffer-Konst* by Pieter Venema, Neu-York, 1730, which devotes over a third of its space to algebra. The remaining seventy titles belong to the time from the close of the American Revolution to 1850. During this period, algebra in its more elementary phases gradually shifted from being a subject for college students to that of being required for entrance to college. The dates of these requirements tell the story: Harvard 1820, Columbia 1821, Yale 1847, Princeton 1848. Professor Simons characterizes the period under consideration as follows: "During this period of time, algebra took its unquestioned place in the secondary school curriculum. The attitude of the author changed from one of apology or explanation or appeal to one of complete confidence in the acceptance of the subject among the recommended or required subjects."

The types of treatment are interesting. Between the American Revolution

and the close of the eighteenth century, algebra was included in three works, either as an integral part or as an appendix. The most important of these was the *New and Complete System of Arithmetic composed for the citizens of the United States* by Nicolas Pike, Newburyport, 1788. Pike had taken the precaution of submitting his manuscript to a number of prominent men and their letters of recommendation, which he duly published, were perhaps influential in the immediate adoption of the book as a text in the colleges in New England. In the early part of the nineteenth century, a number of compilations, translations, and reprints of foreign books appeared from American presses. It was not until 1814 that an American author produced a volume devoted entirely to algebra. From this time on, algebras of many different types appeared. Of these, Professor Simons says: "While some of the books are as much alike as peas in a pod, in spite of claims by the author to the contrary, many of them stand out through the introduction of new topics, through the independence of treatment and applications, or through the eccentricity of the author. Crank works, works covering all possible topics, or just one or two topics were put out, a rich and varied fare. There is even one work prepared especially for the blind."

But this is more than a bibliography, although it is safe to say that no work of the period has escaped Professor Simons' careful search and eagle eye. Besides giving details of the content and treatment, there are many quotations from the prefaces of these works. These alone well repay the reader. For example, Ebenezer Bailey, principal of the Young Ladies High School in Boston, writes in 1833, "This treatise is especially intended for the use of beginners. I have long wished that algebra might be introduced into common schools, as a standard branch of education; and there seems to be no good reason why the study of this most interesting and useful science should be confined to the higher seminaries of learning. The upper classes, at least, in common schools might be profitably instructed in its elements without neglecting any of those branches which they usually attend. . . . I have aimed to prepare a work, which any boy of twelve years, who is thoroughly acquainted with the fundamental rules of Arithmetic can understand, even without the aid of a teacher. . . . The subsequent Chapters on Evolution and Equations of the Second Degree, have been added with a particular reference to schools for young ladies. It is presumed that the work, in its present form, contains as much algebra as this class of learners will, in general, find time to study." Lest this seem a little conservative, consider Warren Colburn's suggestions on method in 1825 which we might well take to heart today: "The best mode, therefore, seems to be, to give examples so simple as to require little or no explanation, and let the learner reason for himself, taking care to make them more difficult as he proceeds. This method, besides giving the learner confidence, by making him rely on his own powers, is much more interesting to him, because he seems to himself to be constantly making new discoveries. Indeed, an apt scholar will frequently make original explanations much more simple than would have been given by the author."

VERA SANFORD

James Gregory Tercentenary Memorial Volume containing his correspondence with John Collins and his hitherto unpublished mathematical manuscripts, together with addresses and essays communicated to the Royal Society of Edinburgh July 4, 1938. Edited by H. W. Turnbull. London, G. Bell and Sons, 1939. 7+524 pages. 25s.

The purpose and the spirit of this book are perhaps best described by the words of the Editor: "There is an element of tragedy in the story here unfolded. In his day Gregory was held to be second only to Newton: yet within a few years his greatest work was almost forgotten. The discovery of the St. Andrews manuscripts has made it possible to restore to Gregory something of the fame which is due to the memory of a genius." And after one has finished reading this volume one must admit that these five hundred pages do restore "the fame due to the memory of a genius." It would be purposeless to enumerate all the discoveries of Gregory. It is enough to mention that he knew before Leibniz that $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, that in the pre-calculus times he was able to establish the formula $\int \sec x\,dx=\log(\sec x+\tan x)$, and that he constructed a telescope which only in details differed from that of Newton. And one cannot help wondering that this great mathematician was almost forgotten. The tercentenary volume, edited with extreme care, forms an important and a valuable contribution to the history of our science. It should be highly recommended to every one interested in the history of mathematics and to those who seek in old pages inspiration for their work.

M. KAC

Elementary Calculus. By G. W. Caunt. Oxford, Clarendon Press, 1939. 388 pages. $2.75.

The author states in his preface, "This book has been written in response to numerous requests for a book on Calculus smaller than my *Introduction to the Infinitesimal Calculus*. The treatment of the subject is on the same lines as in that book, but many of the articles have been rewritten. Many of the applications to Physics and Mechanics have been omitted, and it has also been necessary, from considerations of space, to leave out the chapters on Partial Differentiation and Differential Equations."

The American student would find in this text little that is strange to him. There is the usual continental "misplacement" of the decimal point, references to pounds instead of the almighty dollar, the term "differential coefficient" instead of "derivative" and, on occasion, the most commendable "D" instead of "d/dx."

A priori, one might think that an English text would expect greater sophistication on the part of its readers than the U. S. texts but this seems not to be the case. There are, to be sure, two instances in which a little more firmly grounded experience is assumed: first, in the use of trigonometric identities without a quaver or previous list; second, it is "taken for granted that the student is al-

ready familiar with as much of the theory and construction of graphs as is usually included in text-books on elementary algebra, including the graphs of the functions $y=ax+b$, $y=ax^2+bx+c$, $x^2+y^2=a^2$, together with their simpler properties, and also with the graphs of the circular functions sin x, cos x, tan x, *etc.*" But in the new material dealt with the author is unusually painstaking. There is, at times, a kind of postponed motivation for steps taken; that is, the reasons for taking the steps are often given afterward. Also, though applications are given, it is assumed that the student is interested enough in the subject for its own sake to take time and thought for its theory. If that be sophistication, make the most of it!

The exposition and development is clear, slow, careful and accurate except for the usual fallacious "proof" for the asymptotes of an hyperbola—the "proof" that would show that $y=mx+b$ has $y=mx$ as an asymptote since y/x approaches m as x becomes infinite. With this rather surprising exception, the author's treatment of limits is very careful and could be understood by the average U. S. student, though the reviewer wishes in several cases the author had taken the one additional step necessary to make the theory completely precise. One of the best features of the book is his application of this theory to the examples needed in the calculus. In particular, the derivation of the formula for the derivative of x^n is refreshing.

The topics covered are the ones usual to our texts except that there is more stress laid on limits, convergence, and applications to mathematics (such as envelopes) than is often the custom. There is an "appendix" of three pages on the conic sections.

This book is especially to be commended for its insistence on a clear understanding of some of the fundamentals of calculus and should be a very salutary text for those students really interested in mathematics whether or not their ability in the subject is above average.

B. W. JONES

A Short History of Science. By W. T. Sedgwick and H. W. Tyler. Revised by H. W. Tyler and R. P. Bigelow. New York, The Macmillan Company, 1939. 21+512 pages. $3.75.

The revised edition of Sedgwick and Tyler's *History of Science* (1917) attempts *to trace briefly the history of the foundations* [of science] *upon which recent, as well as earlier, advances were made; to correlate the steps of progress with the spirit of the time; and to increase the emphasis on the evolution of scientific methods. . . . Like the original work it is the outgrowth of a course of lectures given for a number of years to undergraduates of Massachusetts Institute of Technology. . . . In comparison with the parent volume the space devoted to mathematical science has been considerably reduced by the omission of proofs and relatively technical material, as well as of quotations*, and moreover forty pages of appendices containing source material have been removed. In consequence, new material has been added making the present volume larger and more satisfactory.

Within recent years considerable work has been done in the history of science. On the one hand an active periodical literature has been announcing discoveries to specialists; and on the other efforts have been made to select from this, material suited for presentation to the layman. It is being generally realized that the history of ideas and, specifically, that influential group of ideas known as science is a subject worthy of serious study.

In the opinion of the reviewer, if the history of science were taught to upper-class science majors, the present volume would be an excellent text-book. The point of view adopted is that of the scientist rather than that of the antiquarian or philosopher. The book is obviously designed to be taught from. It is enriched by numerous well-selected illustrations and quotations, and also an excellent bibliography in English. This last point is important, since undergraduate science majors rarely have sufficient command of foreign languages to cope with the references usually found in other histories.

SEYMOUR SHERMAN

Higher Mathematics. With applications to science and engineering. By R. S. Burington and C. C. Torrance. New York and London, McGraw-Hill Book Co., 1939. 13+844 pages.

This book is a text in advanced calculus with supplementary material in higher algebra and partial differential equations growing out of course work at Case School for Applied Science, and intended to meet the needs of students interested in the applications of mathematics to physics and engineering. There are, in general, three types of texts in this field; those which aim at a bird's-eye view of the subject without much attention to the more complicated parts of the subject which would arise from attempts to state the most general hypotheses under which the theorems are applicable; those which narrow the fields, usually omitting much of the more complicated use of differential equations, especially partial differential equations, the theory of integration, and which omit numerical computations; and those books which are almost compendiums. This book is definitely of the third type.

To cover the book completely would require a two-year course with a thorough course in the elementary calculus as a prerequisite. To this reviewer, however, a valuable feature of the book is that it may be used in a year course in advanced calculus to acquaint the student with the style and notation of a book which he can continue to consult when he wishes to learn more in the field than can be covered in class. A good student should be able to learn a great deal working alone with the book, and it should be valuable as a reference work in a small college where courses in advanced mathematics are necessarily few. The outstanding contribution of the book, however, is its collection of problems. No teacher who wishes to touch upon the subject-matter in the large number of fields where mathematical analysis is used can afford to overlook it as a source of material.

The text is concerned with the differential calculus, integral calculus, ordi-

nary differential equations, infinite series and sequences, partial differential equations, calculus of variations and dynamics, and at the end a brief introduction to real variable theory. There follows a bibliography of texts which may be consulted with profit for collateral reading, and a reasonably complete index. For so large a book the number of errors in proof-reading is quite small, most of those noticed being merely lost subscripts.

There is an excellent level of rigor maintained throughout, great care having been taken to make matters clear and accurate; the only lapse noted is in Chapter 2, sec. 13, where the proof of the fundamental theorem about the existence of Riemann's Integral (13.1) uses the theorem of the mean (differential) without having in the hypothesis of the theorem the necessary conditions about the derivative of this function existing, which is however done later (part c, chapter 2). It seems, however, to this reviewer that the old Darboux Theorem is needed at this point to give a rigorous proof.

The various symbols introduced to separate the antiderivative, the limit of the sum, and the various other concepts which lead to integrals make this section look strange although it is, perhaps, a good idea to keep these concepts separate by means of a distinctive notation.

Noteworthy inclusions in the text not usually found in older books of this type are: the section on algebra concerned with matrices and determinants; an extended section on tensor analysis; one on the solution of the partial differential equations of mathematical physics by separating the dependent variables into a product of functions of one of the independent variables, and then summing the solutions of this type to obtain a series solution in terms of the various orthogonal functions; a section on the application of matrices to electrical networks (which seems misplaced before the algebra section despite its connection with systems of differential equations); a more extended treatment of conformal mapping; and an amplified discussion of elliptic functions.

There is one omission which seems unusual in so comprehensive a book, in that no treatment of integral equations appears, a subject of increasing importance in the advanced fields of engineering and physics.

On the whole this is an excellent book and deserves great popularity both as a reference work and as a text.

T. C. Benton

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.

NOTICE TO ALL CLUBS

We wish to remind each club that its final activity should be to close the year's program with a report to this department. In the annual spring letter to each club we will stress again that we not only welcome a report on the topic discussed at each meeting and the speakers, but short critical summaries of these papers. Copies of any mimeographed material used at club meetings, magazines or publications, and bibliographies of topics are useful material for the department files.

MATHEMATICAL GAMES, STUNTS, AND RECREATIONS

1. *Mathematico.* In answer to requests of several clubs for games which may be played by either a large or a small group of persons, we recommend the game Mathematico which has often proved successful. The materials required are paper and pencil for each member of the group, and an ordinary deck of playing cards for the announcer. Before the start of the game, the paper is ruled off into 5 by 5 (25 cells) squares. (These may be done beforehand on mimeographed sheets and distributed at the time of play.)

The procedure: the announcer shuffles the cards, then turns up the uppermost one, and announces the number (not the suit) appearing on the face of the card. Each player then writes this number in any one of the 25 cells of his choosing. The position of a number, once written, may not be changed. The next card is turned up and the players write this number in one of the remaining cells. The purpose of the game is to place as many like numbers as possible in any row, column, or diagonal. The cards—ace, 2, 3, $\cdots$, jack, queen, and king —may be given the numbers 1, $\cdots$, 13, respectively, or single digits may be used, *i.e.*, A, 2, 3, $\cdots$, 9, T, J, Q, K. After the twenty-five cells have been filled, each player adds up his score based on the following table of points:

2 numbers alike in a row or column	10 points
2 pairs of two numbers alike in a row or column	20 points
3 numbers alike in a row or column	40 points
3 numbers alike and two other numbers alike	80 points
4 numbers alike in a row or column	160 points
5 numbers of a sequence, not necessarily in order	50 points
3 A's and 2 K's in a row or column	100 points
Sequence of A, K, Q, J, T in a row or column	150 points
4 A's in a row or column	200 points

For scores made diagonally, add 10 points to each of the above.

The person with the highest number of points in three or more games wins.

An example of the game and the method of scoring is shown in the illustration given below (Fig. 1). The total of points scored in this case is 570.

	1	1	7	1	7	(80)
	2	T	2	K	2	(40)
	5	Q	K	5	7	(10)
	3	3	3	J	3	(160)
	4	Q	4	K	Q	(20)
(20)	(50)	(10)		(10)	(10)	(160)

FIG. 1

2. *Cross-Sums.* Cross-word puzzle fans and those delighting in the solution of magic squares may find a new interest in "Cross-Sums," which is explained by a recent book of that title.* The solving of Cross-Sums can be a pastime for one person or can be turned into a competitive contest. In order to play the game, one must employ only the numerals given with each problem and each numeral must be used at least once. The object of the game is to arrange the numerals so that both down and across the totals will be the same. In competition, two winners are possible, one being the competitor who arrives first at a solution in the time limit set at the beginning of the game, the second being the competitor whose 25 squares give the highest total.

The following problems, two with solutions and a third to be solved by the reader, are taken from the book *Cross-Sums.*

I. Problem: To arrange the 25 numerals so that both down and across the totals shall be the same. The numerals to be used are 0 1 2 3 4 5 6 7 8 and 9. Each of these numerals is to be used *at least once.* It will be noted that in the solution given (Fig. 2), each line down and across totals the same, *i.e.*, 30.

6	8	7	0	9
6	5	8	9	2
9	2	7	9	3
5	7	7	4	7
4	8	1	8	9

FIG. 2

II. Problem: Arrange numerals so that sums down, across, and of the four

* Whitelaw, David, Cross-Sums. Geoffrey Bles, Two Manchester Square, W. 1., London. 64 pages. 2 shillings.

shaded squares shall each be the same. The numerals to be used at least once are 1 2 3 4 5 6 and 7 (Fig. 3).

7	3	4	2
3	4	4	5
5	4	4	3
1	5	4	6

FIG. 3

III. Problem: Arrange numerals so that sums down and across shall be the same, and the numerals in the shaded portion are to have the same sum. The numerals to be used at least once are 0 1 2 3 4 5 6 7 and 8. The numerals 0 and 6 are to be used once only (Fig. 4).

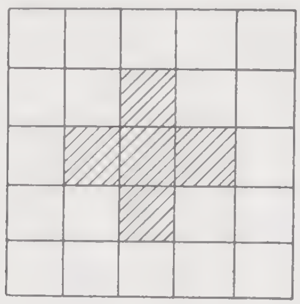

FIG. 4

3. *Squaring Four Points.* An example of the use that can be made of a series of related problems appearing in Problems and Solutions departments of mathematics magazines appears in an article *Squaring Four Points* by the Reverend Edward C. Phillips, S.J., of Woodstock College, Maryland.*

One of the first problems is to construct a square so that each side shall pass through a given point. It is suggested that interest in this problem can be promoted in the form of a recreation or game. At one club meeting, two persons unacquainted with the geometrical solution proceed as follows: one player marks four points on a piece of paper or blackboard and the second player is then required to draw a rectangle through these four points, making it as nearly as possible into a square. Measuring two adjacent sides of the rectangle and taking the ratio of the longer to the shorter side, the ratio will be of the form $1+e$. The second player then selects four points "to be squared" by the first player, and the ratio of these sides is found to be $1+f$. The player who has the smaller remainder or excess over unity, wins the point for the trial. The play may con-

* Phillips, Edward C., Squaring four points, Bulletin of the American Association of Jesuit Scientists, Oct. and Dec. 1939.

tinue until one player has won 2 out of 3, or 5 out of 9 "innings." At a subsequent meeting the geometrical solution of this problem may be given, along with solutions of the following related problems:

I. Construct a square, given its four intersections with a line in its plane.*

II. Given four points in a plane, to find the slope of the side of a square passing through the four points, and to determine the corners of the square and its area.†

III. Show how to construct a square with one corner on each of four generally placed straight lines in a plane. How many solutions are there in general? What constitute special cases?‡

4. *Stunt: Solving for a Partner. The Mathematics Club* of *New Jersey College for Women* at New Brunswick used the novel form of a mathematical intrigue to acquaint the visiting members of the *Mathematics Clubs* of *Rutgers University* and *Montclair State Teachers College* during a recent joint meeting on their campus. In place of name identification cards, each person had pinned upon him a slip of paper containing either an equation or the solutions to an equation. During the social hour which followed the lecture, mathematical knowledge was tested by the ability of each club member to link his individual problem with the correct counterpart. Following is a group of equations and solutions, which are *not* listed in correct pairs:

1. $x^{-1/4} = 2$	1) 2
2. $4x^{2/3} = 9$	2) $-1 \leqq x \leqq 1$
3. $x^3 = 1$	3) $\pm i$
4. ${}_5C_x = 10$	4) .30103
5. ${}_4P_x = 12$	5) $1, \omega, \omega^2$
6. $\sin x = \cos x$	6) $1/16$
7. $\tan 4x = \sqrt{3}$	7) πi
8. $\cot^{-1} x = \pi/6$	8) $45°, 225°$
9. $e^{\tan x} = 1$	9) $0, 0, 0, 0, 1$
10. $\cos(\cos^{-1} x) = x$	10) 5
11. $e^x + 1 = 0$	11) e
12. $10^x = 2$	12) $\pm 27/8$
13. $\cos(\log x) = 1$	13) $0°, 180°$
14. $e^{\log_e x} = e$	14) 2, 3
15. $x^x = 3125$	15) -6
16. $5^{x+5} = 1/5$	16) 1
17. $\log_{x^2} 729 = 3$	17) 7
18. $5^{\log_7 x} = 5$	18) $15°, 60°$
19. $x^2 + \cos 2\pi = 0$	19) ± 3
20. $x^5 - x^4 = 0$	20) $\sqrt{3}$

* Revista Mathematica Hispano-Americana, Sept. 1920, pages 228–29. See also Archibald R. C., this MONTHLY, vol. 28, p. 185.

† School Science and Mathematics, vol. 38, Feb. 1938, pages 222–223.

‡ Clarke, W. B., this MONTHLY, vol. 46, May 1939, problem E381, p. 297.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate consideration of solutions, they should be submitted on separate, signed sheets within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 416. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that

$$3\cdot 2^{p-1}-2\equiv \pm\binom{p-1}{[p/4]}\pmod{p^2},$$

where p is any odd prime.

E 417. *Proposed by J. F. Kenney, Northwestern University.*

Let E be any point outside a circle, ABE the diameter through E, and CDE any chord through E. In the triangle BCE, show that the side CE is divided by D into segments (CD and DE) whose ratio is less than the ratio of the angles E and C.

E 418. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Find triangles with rational sides and angle-bisectors.

E 419. *Proposed by V. Thébault, Le Mans, France.*

In what direction must a billiard ball be hit in order to return to its starting point after a given (even) number of rebounds? Neglect spun of the ball.

E 420. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Let M be the point of intersection of the diagonals of a quadrangle inscribed in a circle with center O. Let parallels through M to the four sides meet the respective opposite sides at P, Q, R, S. Prove that these four points are collinear, that their line is perpendicular to OM, and that analogous results hold for a cyclic hexagon whose three main diagonals concur at a point M.

SOLUTIONS

E 380 [1939, 297]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

If the radius of a circle is any odd prime, p, there are just two different primitive Pythagorean triangles circumscriptible about that circle. Show that, for each such pair of triangles:

(A) their shortest sides differ by one;

(B) their hypothenuses exceed their corresponding longer legs by one and by two, respectively;

(C) the sum of their perimeters is six times a perfect square;

(D) as p increases without limit, the ratio of their least angles approaches the limit 2;

(E) as p increases without limit, the ratio of their areas approaches the limit 2; and finally,

(F) the smaller triangle can always be placed inside the larger, so as not to touch it.

Solution by C. W. Trigg, Los Angeles City College.

The sides of a primitive Pythagorean triangle are of the form $a=m^2-n^2$, $b=2mn$, $c=m^2+n^2$, where m is greater than n, one is odd and the other even, and m and n are relatively prime. If p is the in-radius, then $c=a+b-2p$, so $p=n(m-n)$. In the present case p is an odd prime, so there are just two possibilities: either $n=1$, $m=p+1$, and the sides are

$$a_1 = p(p+2), \qquad b_1 = 2(p+1), \qquad c_1 = p^2+2p+2;$$

or $n=p$, $m=p+1$, and the sides are

$$a_2 = 2p+1, \qquad b_2 = 2p(p+1), \qquad c_2 = 2p^2+2p+1.$$

(A) Since $a_1-b_1=p^2-2$ and $b_2-a_2=2p^2-1$, the shortest sides are b_1 and a_2, which differ by 1.

(B) $c_1-a_1=2$, and $c_2-b_2=1$.

(C) $a_1+b_1+c_1+a_2+b_2+c_2=6(p+1)^2$.

(D) The least angles, *i.e.*, the angles arc tan $2(p+1)/p(p+2)$ and arc tan $(2p+1)/2p(p+1)$, tend to zero; hence their limiting ratio is equal to the limiting ratio of their tangents: $\lim_{p\to\infty}2(p+1)^2/[(p+\frac{1}{2})(p+2)]=2$.

(E) The ratio of the areas is

$$\frac{\frac{1}{2}a_2b_2}{\frac{1}{2}a_1b_1} = \frac{2p+1}{p+2} = 2 - \frac{3}{p+2}.$$

(F) The smaller triangle can never be placed inside the larger so as not to touch it, for they have equal inscribed circles.

Also solved by W. B. Clarke and W. R. Talbot.

E 382 [1939, 297]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

A, B, and C are three fixed points in space. (S) is a sphere fixed in space, bearing A and B, but not C, on its surface. P is a point which moves over the surface of (S). (Q) is the sphere determined by A, B, C, and P. T is the plane tangent to (Q) at C. M is the plane ABP. Planes M and T intersect in the line l. Determine the locus of l.

Solution by Donald Boyd, University of Toronto.

Consider the fixed point C as a null sphere (C). Then T is the radical plane of (Q) and (C), while M is the radical plane of (Q) and (S). Hence l is the radical axis of (Q), (S), and (C), and its locus is the radical plane of (S) and (C).

Also solved by Melvin Preston, A. E. Roffey, V. Thébault, J. B. Walker, and C. G. White.

E 383 [1939, 361]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

The diameters from the vertices of the triangle ABC, in the circumscribed circle, cut the opposite sides in E, F, and G, respectively. L, M, and N are the respective midpoints of AE, BF, and CG. Show that triangle LMN is homologous to triangle ABC, and that the axis of homology is the orthic axis of the triangle.

Solution by L. M. Kelly, Boston University.

The triangle LMN is obviously homologous to ABC, by perspective from the circumcenter. The only question, therefore, is whether the point P, where LM intersects AB, lies on the orthic axis, *i.e.*, the polar of the orthocenter with respect to ABC. To answer this, we calculate the ratio of PA to PB. Let R, S, T be the midpoints of BC, CA, AB. Suppose for definiteness that $BC<CA$, and let LM meet RS (produced) at Q. By similar triangles, we have

$$\frac{RS}{PT}=\frac{QR}{PT}-\frac{QS}{PT}=\frac{MR}{MT}-\frac{LS}{LT}=\frac{FC}{FA}-\frac{EC}{EB}=\frac{\sin 2A}{\sin 2C}-\frac{\sin 2B}{\sin 2C}$$

$$=\frac{2\sin(A-B)\cos(A+B)}{2\sin C\cos C}=\frac{-\sin(A-B)}{\sin(A+B)}=\frac{-\tan A+\tan B}{\tan A+\tan B},$$

and therefore

$$\frac{PA}{PB}=\frac{PT+RS}{PT-RS}=\frac{\tan B}{\tan A}.$$

Thus P divides the side AB externally in the ratio $\tan B:\tan A$. Since the altitude from C divides AB internally in the same ratio, this shows that P lies on the orthic axis.

Editorial Note. This result can be obtained without trigonometry by taking A, B, C to be the points $(1/l, 0, 0)$, $(0, 1/m, 0)$, $(0, 0, 1/n)$ in the plane $lx+my+nz=1$, with the unit of measurement adjusted so as to make $l^2+m^2+n^2=1$. Then we easily find that the orthocenter is (l, m, n), and the circumcenter $[(1-l^2)/2l,\ (1-m^2)/2m,\ (1-n^2)/2n]$, that E is $[0,\ (1-m^2)/m(1+l^2),\ (1-n^2)/n(1+l^2)]$, and consequently L is $[1/2l,\ (1-m^2)/2m(1+l^2),\ (1-n^2)/2n(1+l^2)]$. From here on, it is convenient to regard x, y, z as homogeneous coördinates (by projection from the origin). Then L is $(l^{-1}+l,\ m^{-1}-m,\ n^{-1}-n)$, and similarly M is $(l^{-1}-l,\ m^{-1}+m,\ n^{-1}-n)$. These points are clearly collinear with the point $(l, -m, 0)$, in which the line $z=0$ (which is AB) meets the orthic axis $x/l+y/m+z/n=0$.

E 384 [1939, 361]. *Proposed by V. Thébault, Le Mans, France.*

Determine the envelope generated by the polar line w of the fixed point W, with respect to the circle Q, which is of constant size, but rolls around a fixed circle C. What happens if circle C degenerates into a straight line L?

Solution by L. M. Kelly, Boston University.

Let us first consider the envelope of the radical axis w' of the circle Q and the null circle W. This radical axis lies mid-way between W and w, so its envelope is homothetic to the envelope of w. From O, the center of C, draw OK perpendicular to w'. Take a point X on OK, so as to make $OMWX$ a parallelogram, M being the center of Q. Then the product $2\,OK \cdot MW$ is constant, being equal to the difference of the powers of O with respect to the circles W and Q. Since $OK \cdot OX$ is constant, the envelope of w' is the polar reciprocal of the locus of X. But X is the symmetric of M with respect to the midpoint of OW (which is a fixed point). Since M describes a circle with center O, X must describe a circle with center W. Thus the envelope of w' is the polar reciprocal of a circle, *i.e.*, a conic. Finally, the envelope of w is a similar conic.

When C degenerates to a straight line, it is simplest to use coördinates. The polar of the point $(a, 0)$ with respect to the circle $x^2+(y-t)^2=b^2$ is $ax-ty+t^2=b^2$. Differentiating with respect to t, $-y+2t=0$. Eliminating t, we obtain

$$y^2 = 4(ax - b^2).$$

Thus the envelope is a parabola.

In C. Michel's *Complements de Géométrie Moderne*, p. 298, the following more general theorem is proved: "L'enveloppe des polaires d'un point donné W par rapport aux cercles (d'une même famille) tangents à deux cercles donnés est une conique."

E 385 [1939, 361]. *Proposed by C. W. Trigg, Los Angeles City College.*

What is the largest prime whose square contains no duplicate digits?

Solution by G. W. Wishard, Norwood, Ohio.

Any number of ten digits, all different, is divisible by 9, and so cannot be the square of a prime. We therefore seek such a square with nine digits. Since this is congruent to 1 modulo 3, the omitted digit must be 2, 5, or 8. Let p be the prime sought. Since $p^2<987,654,310$, we have $p<31,427$. It appears that p has to be hunted down by testing the squares of various primes. In doing this we can make use of various devices to save labor; for instance, if $p>30,000$, it cannot end in 3 or 7. Barlow's Table of Squares shows eventually that $p=21,397$ and $p^2=457,831,609$.

Also solved by the proposer, who points out that 13,147 and 20,089 are the only other primes whose squares have nine digits, all different.

E 386 [1939, 361]. *Proposed by W. O. Pennell, Exeter, N. H.*

A certain irregular polygon has its n sides skewed in space. From each of its n vertices vectors are drawn to the midpoints of each of the $n-2$ non-adjacent sides. Show that the vector sum of these n^2-2n vectors is zero.

Solution by C. W. Trigg, Los Angeles, City College.

Start at any vertex, O, and consecutively indicate the sides of the polygon as vectors, $a_1, a_2, \cdots, a_n$, so that $\sum_{i=1}^n a_i=0$. Join the midpoint of each side to the midpoints of its adjacent sides, thus forming another n-gon. Indicate the join of the midpoints of a_1 and a_2 as vector b_1, and continue the vector representation of the sides consecutively as $b_2, b_3, \cdots, b_n$. Then $\sum_{i=1}^n b_i=0$.

The sum of the vectors from O to the midpoints of the non-adjacent sides is

$$\begin{aligned}(\tfrac{1}{2}a_1 + b_1) + (\tfrac{1}{2}a_1 + b_1 + b_2) + \cdots + (\tfrac{1}{2}a_1 + b_1 + b_2 + \cdots + b_{n-2})\\ = \tfrac{1}{2}(n-2)a_1 + (n-2)b_1 + (n-3)b_2 + \cdots + b_{n-2}.\end{aligned}$$

Likewise the sums of the vectors from the successive vertices are

$$\begin{gathered}\tfrac{1}{2}(n-2)a_2 + (n-2)b_2 + (n-3)b_3 + \cdots + b_{n-1},\\ \tfrac{1}{2}(n-2)a_3 + (n-2)b_3 + (n-3)b_4 + \cdots + b_n,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ \tfrac{1}{2}(n-2)a_n + (n-2)b_n + (n-3)b_1 + \cdots + b_{n-3}.\end{gathered}$$

When these sums are added, we have the vector sum of these $n(n-2)$ vectors, namely $\frac{1}{2}(n-2)\sum_{i=1}^n a_i+[1+2+\cdots+(n-2)]\sum_{i=1}^n b_i=0$.

Also solved by F. M. Fletcher, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3952. *Proposed by N. A. Court, University of Oklahoma.*

The spheres of a coaxal net which meet a given straight line in pairs of points in involution form a coaxal pencil.

3953. *Proposed by N. A. Court, University of Oklahoma.*

If L is a point on the radical axis r of a coaxal net of spheres, and q is the polar line of r with respect to a variable sphere (S) of the net, the locus of the circle of intersection of (S) with the plane Lq is a sphere.

3954. *Proposed by Oystein Ore, Yale University.*

From three elements a, b, c in given order one can form two products, namely, $(ab)c$ and $a(bc)$ when the associative law is not assumed. Similarly four elements a, b, c, d give $N_4=5$ products; $[(ab)c]d$, $[a(bc)]d$, $a[(bc)d]$, $a[b(cd)]$, $(ab)(cd)$. Find the general expression for the number N_i of products with i factors.

3955. *Proposed by V. Thébault, Le Mans, France.*

A triangle with unequal sides has one angle of 60°, or 120°, and a side adjacent to that angle of length m, a prime. (1) Determine the lengths of the other two sides so that they are integers. (2) Show that to each value of m there correspond two triangles such that the difference of their perimeters is a perfect square in one case, and in the other case the sum of the perimeters increased by unity is the sum of squares of two consecutive integers.

3956. *Proposed by V. Thébault, Le Mans, France.*

An arbitrary diameter Δ of the circumcircle of an equilateral triangle cuts the sides BC, CA, AB in the points α, β, γ. Show that the Euler lines of triangles $A\beta\gamma$, $B\gamma\alpha$, $C\alpha\beta$ determine a triangle T symmetrically equal to ABC with the center of symmetry on Δ.

3957. *Proposed by Otto Dunkel, Washington University.*

Given a triangle ABC with angles $A<B<C$, show that there are precisely one, two, three straight line segments which bisect both its perimeter and area according as

$$1-\frac{\sin A}{\sin B} \gtreqless 2\tan^2(A/2)\tan^2(B/2),$$

where we may replace B by C. If $B=C$, there are one, two, three such segments, according as $A \lesseqgtr A_0$, where $\sin(A_0/2)=\sqrt{2}-1$.

SOLUTIONS

3865 [1938, 190]. *Proposed by H. D. Grossman, New York, N. Y.*

Prove that the general solution in integers of $ab=cd$, $a+b=c-d$, where a, b, c, d are integers whose G. C. D. is unity, is $a=r(r-s)$, $b=s(r+s)$, $c=r(r+s)$, $d=s(r-s)$, where r and s are relatively prime integers. It will be observed that $a-d$, $b+c$, $ac-bd$, $bc-ad$ are each perfect squares. If a is unity in the preceding problem, the roots of the given two polynomials are the a, b, c, d in this second problem.

Remarks by E. T. Bell, California Institute of Technology, Pasadena, California.

The theorem is inexact, as shown by the example $r=3$, $s=1$, when $a=6$, $b=4$, $c=12$, $d=2$, and these have 2 as their G. C. D. In order that the stated formulas shall give the general solution in integers a, b, c, d with G. C. D. unity, it is necessary and sufficient that r, s be relatively prime integers one of which is

even and the other odd. This follows simply from: (1) The general integral solution of $xy=zw$ is $x=eg$, $y=fh$, $z=eh$, $w=fg$, where e, f, g, h are integers and h, g are relatively prime; (2) If r, s are relatively prime and of opposite parity, then $r+s$, $r-s$ are relatively prime. The essential argument for (1) is given in L. E. Dickson's *Introduction to the Theory of Numbers*, p. 43, Exercise 5.

Solved also by E. P. Starke.

Editorial Note. Starke, by a different analysis, obtained the result in the second sentence of the above solution. He also showed that: If the roots of $x^2+Bx+C=0$ are called a and b, and the roots of $x^2+Bx-C=0$ are c and $-d$ (*not* d); then $a+b=-B$, $c+(-d)=-B$, $ab=C$, $c(-d)=-C$. Thus we have $a+b=c-d$ and $ab=cd$, the hypothesis of the present problem. (See problem 3864 [1940, 187].) A further property of the numbers $a, b, c, -d$ is the following: their sum, the sum of their squares, and the sum of their cubes form a geometrical progression; if the four numbers be divided by $(r^2+s^2)/2$, the resulting numbers have these three sums each equal to 2. (See problem E 313 [1938, 630].)

3866 [1938, 190]. *Proposed by J. M. Feld, New York, N. Y.*

Show that the equation of the circumcircle of the triangle whose sides have the equations $L_i \equiv a_ix+b_iy+c_i=0$, $i=1, 2, 3$, can be written in the form

$$(a_1^2+b_1^2)(a_2b_3)L_2L_3+(a_2^2+b_2^2)(a_3b_1)L_3L_1+(a_3^2+b_3^2)(a_1b_2)L_1L_2=0,$$

where $(a_ib_j)=a_ib_j-a_jb_i$.

I. *Solution by M. W. Aylor, University of Virginia.*

For K_1, K_2, and K_3 arbitrary constants, the equation

$$K_1L_2L_3+K_2L_3L_1+K_3L_1L_2=0 \tag{1}$$

represents a locus through the vertices of the triangle formed by $L_1=0$, $L_2=0$, $L_3=0$. This equation is of the second degree in x and y, and will represent a circle if the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero. These conditions give the equations

$$K_1(a_2a_3-b_2b_3)+K_2(a_1a_3-b_1b_3)+K_3(a_1a_2-b_1b_2)=0, \tag{2}$$

$$K_1(a_3b_2+a_2b_3)+K_2(a_1b_3+a_3b_1)+K_3(a_1b_2+a_2b_1)=0. \tag{3}$$

In solving (2) and (3), we get

$$K_1=(a_1^2+b_1^2)(a_2b_3), \qquad K_2=(a_2^2+b_2^2)(a_3b_1), \qquad K_3=(a_3^2+b_3^2)(a_1b_2).$$

The substitution of these values in (1) leads to the required equation.

II. *Solution by the Proposer.*

The conics through the vertices of the triangle are given by

$$k_1L_2L_3+k_2L_3L_1+k_3L_1L_2=0. \tag{1}$$

In order that the conic be a circle it must pass through the circular points at infinity. By writing $L_i=0$ in homogeneous form in the variables x, y, t, and replacing x, y, and t, respectively, by 1, i, 0, we obtain from (1)

$$k_1(a_2+ib_2)(a_3+ib_3)+k_2(a_3+ib_3)(a_1+ib_1)+k_3(a_1+ib_1)(a_2+ib_2)=0,$$

or, since $a_i^2+b_i^2\neq 0$,

$$\Sigma k_i/(a_i+ib_i)=0.$$

Therefore

$$\Sigma k_i(a_i-ib_i)/(a_i^2+b_i^2)=0,$$

and since the a_i, b_i, and k_i are real,

$$\Sigma k_i a_i/(a_i^2+b_i^2)=0, \qquad \Sigma k_i b_i/(a_i^2+b_i^2)=0.$$

Solving for the k_i we obtain

$$k_1\colon k_2\colon k_3=(a_1^2+b_1^2)(a_2b_3)\colon(a_2^2+b_2^2)(a_3b_1)\colon(a_3^2+b_3^2)(a_1b_2),$$

from which the required equation follows.

Solved also by E. F. Allen, W. V. Parker, O. J. Ramler, F. C. Smith, E. P. Starke, R. Stephens, M. J. Turner, F. Underwood, Maud Willey, and S. Zuckerman.

Editorial Note. Stephens stated that a solution of this problem was given by P. A. Caris in this MONTHLY, 1927, p. 254. Ramler used the equation of the circle in absolute normal coördinates, which leads easily to the desired result.

3867 [1938, 190]. *Proposed by V. Thébault, Le Mans, France.*

Given a hexagon $A_1A_2A_3A_4A_5A_6$ whose consecutive sides are perpendicular, show that: (1) The diagonals A_1A_4, A_2A_5, A_3A_6 meet in a point M. Let Δ_1 and Δ_4, Δ_2 and Δ_5, Δ_3 and Δ_6 be the perpendiculars to A_1A_4, A_2A_5, A_3A_6 at their corresponding extremities; and let B_1, B_2, B_3, B_4, B_5, B_6 be the intersections of (Δ_6, Δ_1), (Δ_1, Δ_2), (Δ_2, Δ_3), (Δ_3, Δ_4), (Δ_4, Δ_5), (Δ_5, Δ_6). The hexagon $B_1B_2B_3B_4B_5B_6$ is inscriptible in a circle through M. (3) The areas of the A and B hexagons are equal.

Editorial Note. The proposer gave no proof, but stated that the proof offered no difficulty and that the theorems of the problem are special cases of more general theorems which will appear in *Mathesis.*

In the following proof the given hexagon is denoted by (B) with the vertices B_i and the derived hexagon is denoted by (A) with the vertices A_i in order to conform with the notation of 3861 [1940, 118]. Suppose first that the sides B_1B_2, B_3B_4, B_5B_6 are parallel to one direction, and that the remaining three sides are parallel to another direction. We shall prove that the diagonals B_1B_4, B_2B_5, B_3B_6 meet in a point M. Let x_4x_5 be a variable segment parallel to B_4B_5 with x_4 moving on the straight line of B_3B_4 and x_5 on the straight line of

B_5B_6. The two pencils of rays $B_2(x_5)$ and $B_1(x_4)$ are projective and have a self-corresponding ray through B_2 and B_1. Hence the locus of P, the intersection of corresponding rays, is a straight line. When x_4 is at B_3 so also is P, and when x_5 is at B_6 so also is P. Hence the locus of P is the straight line of B_3B_6; and the proof of this simple generalization of (1) is complete. There are obviously two types of (B) when no side has a zero or infinite length: in one type, no two sides have a point in common within each of their segments; in the other, two opposite sides intersect in a point within the segment of each side. In this second type M may be at infinity, and this happens when the directed area of (B) is zero. Thus if B_3B_4 and B_6B_1 intersect in K within each of these two opposite sides and if $B_3K \cdot KB_1 = B_6K \cdot KB_4$, the point M is at infinity and the vector area of (B) is zero; and conversely.

In what follows the two directions above are perpendicular, that is, consecutive sides of (B) are perpendicular. We shall prove the following theorem:

The two given perpendicular segments B_3B_4 and B_6B_1 with distinct finite end-points are such that the intersection of B_1B_4 and B_3B_6 is a finite point M distinct from the B_i points; the two points A_1 and A_4 are determined so that the orthogonal projections of A_1 on MB_1 and MB_6 are B_1 and B_6, and the orthogonal projections of A_4 on MB_3 and MB_4 are B_3 and B_4. Then A_1A_4 subtends a right angle at M.

The points A_1, B_1, B_6, M are concyclic and determine the circle (O_1) with the diameter MA_1, and similarly B_3, B_4, A_4, M determine the circle (O_4) with a diameter MA_4. The tangent to (O_1) at M is antiparallel to B_6B_1 with respect to the angle formed by B_1B_4 and B_3B_6, and the tangent to (O_4) at M is antiparallel to B_3B_4 with respect to the same angle. The two tangents at M must be perpendicular, since B_3B_4 and B_6B_1 are perpendicular; and therefore the two diameters MA_4 and MA_1 must be perpendicular.

The four points B_3, B_4, B_6, B_1 determine a hexagon (B) of one or the other type of the problem, and the point M; and (B) determines an (A) hexagon. From the above theorem the diagonals A_1A_4, A_2A_5, A_3A_6 of (A) subtend a right angle at M. We now show that (A) is inscribed in a circle (O) which passes through M. Considering first the sides B_3B_4 and B_6B_1, and the circle (O) through M with diameter A_1A_4, the straight lines of B_6B_1 and of B_3B_4 cut this circle (O) again in A_6' and A_3'. We shall show that $\angle A_6' B_5 M = \pi/2$, and therefore $A_6' \equiv A_6$, and similarly $A_3' \equiv A_3$.

We have two rectangles $B_3A_4A_6'B_6$ and $KB_4B_5B_6$, where B_3B_4 and B_6B_1, or their extensions, intersect in K. Let B_4B_5 and A_4A_6' intersect in B_{45}; then the circle (B_3B_{45}) on B_3B_{45} as diameter passes through B_4 and A_4. The circle (MA_4) on MA_4 as diameter passes through B_3 and B_4: these two circles must coincide since they have B_3, B_4, A_4 in common. Then $\angle MB_{45}A_4 = \pi/2$, since MA_4 is a diameter, and $MB_{45}A_6'B_6$ is therefore a rectangle. The circle (B_6B_{45}) on B_6B_{45} as diameter passes through A_6', B_5, M, and it has MA_6' as a diameter. Therefore $\angle A_6' B_5 M = \pi/2$ and $A_6' \equiv A_6$. Let B_1B_2 cut MB_{45} in B_{12}, then circle (B_6B_{12}) passes through B_1 and M. Also circle (MA_1) passes through B_1 and B_6, and the

two circles must coincide, having M, B_1, B_6 in common. Hence $\angle A_1B_{12}M=\pi/2$ and B_{12} lies also on A_1A_3'. The circle (B_3B_{12}) passes through A_3', M, B_2; hence $\angle MB_2A_3'=\pi/2$ and $A_3'\equiv A_3$.

It remains to prove that A_2 and A_5 lie on (O). The argument is obtained by adding 2 to each subscript and reducing mod 6 in the above reasoning. This gives the two perpendicular sides B_3B_2, B_5B_6 and the diameter A_3A_6. This completes the proof of (2) for all cases of (B) of either type. It was shown in the solution of 3861 [1940, 118] that the areas of (A) and (B) are equal, and also that the converse theorem of this problem is true.

The following facts regarding the configuration of (B) with its associated (A) may be proved. If (B) is of the first type it must have a re-entering angle, say $B_5B_6B_1$; and M must lie within the right triangle $B_5B_6B_1$. A necessary and sufficient condition that (A) be convex with distinct vertices is that M lies within the oval overlapping area of circles (B_5B_6) and (B_6B_1), where the segments within the parentheses are diameters. The only vertices of (B) lying within the segments of corresponding sides of (A) are B_3 and B_6. If M lies on the (B_5B_6) boundary of the oval, the exterior angle at A_1 is $\pi/2$, and (A) consists of a rectangle and the two tangents at the ends of a diagonal. There is a similar result if M lies on the other boundary. If (B) is of the second type, (A) is not convex and each B_i vertex lies within the segment of its corresponding side of (A), A_iA_{i+1}; and thus (B) lies within (O).

3868 [1938, 253]. *Proposed by Arnold Dresden, Swarthmore College.*

Prove that, if $0\leqq\alpha\leqq x_1\leqq x_2$ and n is a positive integer, then

$$x_2^{1/n} - x_1^{1/n} \leqq (x_2-\alpha)^{1/n} - (x_1-\alpha)^{1/n}.$$

I. *Solution by Margaret Gurney, Washington, Conn.*

Consider the function $y(x)=x^{1/n}$. Since its derivative decreases as x increases, the increment of y corresponding to a given interval on the x-axis decreases as that interval moves to the right. In the problem the interval (x_1, x_2) is obtained from the interval $(x_1-\alpha, x_2-\alpha)$ by shifting the latter through a distance α to the right. Hence we have $y(x_2)-y(x_1)\leqq y(x_2-\alpha)-y(x_1-\alpha)$, and the required proof follows.

This discussion is applicable whenever n is any positive number greater than or equal to unity.

II. *Solution by E. S. Pondiczery, Princeton, N. J.*

This is a special case of

$$\phi(x_2)-\phi(x_1)\leqq\phi(x_2-\alpha)-\phi(x_1-\alpha), \tag{1}$$

where $0\leqq\alpha\leqq x_1\leqq x_2$ and $\phi(x)$ is continuous and concave on $x_1-\alpha\leqq x\leqq x_2$. (In particular, the requirement that n is an integer is redundant.) To prove (1), we use the relation

$$\frac{\phi(y_4) - \phi(y_3)}{y_4 - y_3} \leqq \frac{\phi(y_2) - \phi(y_1)}{y_2 - y_1}, \tag{2}$$

which is known to hold for a concave ϕ if $y_1 < y_2 < y_3 < y_4$ (see Hardy, Littlewood, and Pólya, *Inequalities*, p. 94, where the reversed inequality for a convex ϕ is given). If $y_4 - y_3 = y_2 - y_1$, (2) gives

$$\phi(y_4) - \phi(y_3) - \phi(y_2) + \phi(y_1) \leqq 0, \tag{3}$$

which is clearly valid (by continuity) if some of the y's coincide. We have either $x_1 - \alpha \leqq x_1 \leqq x_2 - \alpha \leqq x_2$ or $x_1 - \alpha \leqq x_2 - \alpha \leqq x_1 \leqq x_2$; using these points for y_1, y_2, y_3, y_4, we have in either case $y_4 - y_3 = y_2 - y_1$, and, by (3),

$$\phi(x_2) - \phi(x_2 - \alpha) - \phi(x_1) + \phi(x_1 - \alpha) \leqq 0,$$

which is (1).

It is interesting to consider when equality can actually occur in (1). There is clearly equality if $\alpha = 0$, if $x_1 = x_2$, or if ϕ is linear on $(x_1 - \alpha, x_2)$ (this is the case $n = 1$ of the original inequality). It is easy to verify analytically that there is equality in no other case. It is, however, perhaps more illuminating to reason geometrically. The concave curve $y = \phi(x)$ is characterized by the property that no point of any chord lies above its arc. Suppose that we have equality in (1), with $x_1 < x_2$, $\alpha > 0$. Then

$$\phi(x_2) - \phi(x_2 - \alpha) = \phi(x_1) - \phi(x_1 - \alpha)$$

states that the chord whose end-points have abscissas $x_2 - \alpha$, x_2, is parallel to the chord whose end-points have abscissas $x_1 - \alpha$, x_1. These chords must lie in the same line, since otherwise we could clearly draw a chord which would be above its arc at some point; but if the chords are in the same line, the curve must coincide with the line, since otherwise we could again draw a chord having some point above its arc (this is obvious if one draws a figure).

We could also have reasoned geometrically in establishing (1).

Solved also by Harry Gershenson, Michael Goldberg, V. W. Graham, Roy MacKay, C. E. Melville, E. G. Olds, H. D. Ruderman, E. P. Starke, F. Underwood, Morgan Ward, C. W. Williams, R. H. Wilson, Jr., and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A conference in topology will be held at the University of Michigan from June 24 to July 6, 1940.

Professor Julian L. Coolidge, senior member of the department of mathematics at Harvard University will retire on September 1, 1940, with the titles professor of mathematics emeritus and master of Lowell House emeritus. Professor Coolidge has been a member of the department of mathematics for forty years and master of Lowell House for ten years.

Assistant Professor Mina S. Rees of Hunter College has been promoted to an associate professorship.

Dr. Samuel Saslaw has been appointed to an assistant professorship at the University of Miami.

A. R. Turquette of Florida Southern College has been promoted to an associate professorship.

Dr. S. E. Warschawski of Brown University has been appointed visiting assistant professor at Washington University.

C. W. Williams of Washington and Lee University has been appointed to a professorship at Armstrong Junior College, Savannah, Georgia.

Professor John von Neumann of the Institute for Advanced Study has been appointed to a Walker-Ames professorship in mathematics for the first term of the summer quarter 1940 at the University of Washington.

U. S. Hanna, professor emeritus of mathematics at Indiana University, for forty-two years a member of the department, died on February 18, 1940. He was a charter member of the Mathematical Association.

Miss Beulah Russell, associate professor of mathematics at the College of William and Mary, died February 23, 1940. She had been a member of the Association since 1923.

Dr. J. H. Tanner, professor of mathematics at Cornell University from 1904 until his retirement in 1926, died on March 11, 1940.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1940.

University of California at Los Angeles. July 1 to August 9. In addition to the usual courses in algebra, trigonometry and calculus, the following advanced and graduate courses will be offered: By Dr. P. G. Hoel: Statistics. By Professor Glenn James: Selected topics in groups and number theory. By Professor G. T. Whyburn of the University of Virginia: Seminar in analytic topology.

Catholic University of America. By Dr. Joseph Daly: Fundamental concepts. By Professor Finan: Modern algebraic theories. By Professor Rice: Analytic geometry of three dimensions. By Professor Ramler: Differential equations, Synthetic projective geometry.

University of Chicago. June 19 to August 23. By Professor Bliss: Partial differential equations. By Professor Lane: Modern theories of differential geometry, Projective differential geometry. By Professor Logsdon: Elliptic functions, Solid analytic geometry. By Professor Barnard: Theory of functions of real variables, Infinite series and definite integrals. By Professor Albert: Theory of groups, Introduction to algebraic theories. By Professor Bartky: Theory of sampling with applications, Celestial mechanics. By Professor MacLane: Linear algebras. By Professor Myers: Metric differential geometry. By Dr. Schilling: Theory of algebraic numbers, Differential equations.

University of Colorado. By Professor Kempner: Advanced calculus (both terms), Group theory and Galois theory (both terms), Selected topics in elementary modern geometry (first term only). By Professor Hutchinson: Advanced teachers course (each term), Fundamental differential equations of mathematical physics (both terms).

Duke University. First term, June 12 to July 23. By Professor Carlitz: Modern algebra, Modern geometry, Thesis seminar. By Professor Rankin: The teaching of mathematics. By Professor Roberts: Modern developments in mathematics, Introductory topology, Thesis seminar. By Professor Ward of the California Institute of Technology: Complex variable, Number theory, Thesis seminar. *Second term, July 25 to September 3.* By Professor Elliott: Differential equations. By Professor Gergen: Thesis seminar. By Professor Miles: Complex variable, Fourier series and spherical harmonics. By Professor Thomas: Modern algebra, Thesis seminar.

University of Illinois. In addition to the usual elementary courses, the following advanced courses will be offered: By Dr. Bower: Elementary statistics. By Dr. Moore: Fundamental concepts. By Dr. Mendel: Theory of equations. By Professor Bourgin: Advanced calculus, Complex variable. By Professor Levy: Projective geometry, Theory of matrices. By Dr. Chanler: Geometry. By Professor Crathorne: Calculus of variations, Theory of statistics. In addition, a seminar and thesis course will be arranged.

The State University of Iowa. By Professor Chittenden: Differential equations, Topology. By Professor Reilly: Advanced algebra. By Professor Craig: Statistics, Analytical methods of mathematical statistics. By Professor Woods: Projective geometry, Studies in secondary algebra. By Dr. Oberg: Vector analysis.

University of Michigan. June 24 to August 16. A Conference in Topology will be held at the University of Michigan from June 24 to July 6, 1940. In addition

to elementary courses and the standard courses in differential equations, theory of equations and determinants, advanced solid analytic geometry, and advanced calculus, the following courses will be offered: By Professor Anning: Modern geometry, Teacher's seminar in algebra, History of geometry and trigonometry. By Professor Ayres: Transformations in topology. By Dr. Bartels: Fourier series. By Professor Carver: Finite differences, Theory of statistics I. By Professor Coe: Vector analysis. By Professor Craig: Theory of statistics II, Advanced theory of statistics I. By Professor Dwyer: Social statistics. By Dr. Elder: Theory of numbers. By Professor Field: Synthetic projective geometry, Differential geometry. By Dr. Greville: Theory of probability. By Professor Hildebrandt: Theory of functions of a complex variable, Calculus of variations. By Dr. Nesbitt: Modern algebra. By Dr. Rainville: Functions defined by infinite processes. By Professor Rouse: Empirical formulas. By Professor Wilder: Introduction to the foundations of mathematics, Unified topology. In addition there will be an orientation seminar, a seminar in pure mathematics conducted by Professor Hildebrandt and other members of the staff, and a seminar in statistics by Professor Craig.

University of Minnesota. In addition to the usual elementary courses, the following courses will be offered: *First term, June 17 to July 26.* By Professors Underhill and Carlson, Dr. Koehler: Selected topics in senior college mathematics. By Professor Underhill: Differential equations. By Professors Jackson, Underhill and Carlson: Selected topics in advanced mathematics. By Professor Jackson: Advanced algebraic theory, Limits and series. By Professor Carlson: Projective geometry. *Second term, July 29 to August 30.* By Professor Gibbens and Dr. Campaigne: Selected topics in senior college mathematics. By Professor Gibbens: Selected topics in advanced mathematics.

University of North Carolina. In addition to the regular courses through the calculus, the following courses will be offered: *First term, June 13 to July 20.* By Professor Mackie: College geometry, Advanced calculus. By Professor Linker: Differential equations. By Professor Browne: Theory of equations. By Professor Henderson: Special relativity. *Second term, July 22 to August 28.* By Professor Cameron: Differential equations (continued). By Professor Hill: Theory of equations (continued). By Professor Lasley: Higher plane curves.

Northwestern University. In addition to courses in algebra, analytic geometry, trigonometry and the calculus, the following advanced courses will be offered: By Professor Curtiss: Infinite series and differential equations, Differential equations of mathematical physics. By Professor Simmons: Number theory. By Mr. Kenney: Theory of statistics. By Dr. Wescott: Algebra for teachers. By Dr. Hellinger: Higher algebra.

Ohio State University. June 17 to August 30. In addition to the usual courses in trigonometry, analytic geometry and the calculus, the following advanced courses will be offered: By Professor Bamforth: Solid analytical geometry, Differential equations. By Professor LaPaz: Advanced calculus, Tensor analy-

sis. By Dr. Wylie: Fundamental ideas in algebra and geometry. By Dr. Hendrickson: Probability.

University of Pennsylvania. June 24 to August 6. In addition to the regular undergraduate courses in mathematics, the following advanced courses will be offered: By Dr. Clarkson: Analytic geometry of three dimensions. By Professor Kline: Differential equations. By Professor Beal: Infinite series and products. By Professor Hallett: Galois theory of equations. By Professor Caris: Diophantine analysis.

University of Southern California. Six weeks session, June 29 to August 8. The following advanced courses will be offered: By Professor Gurney, Mathematical astronomy. By Professor Butter: Calculus III, Theory of numbers. By Professor Steed: Vector analysis, Differential geometry. *Four weeks session, August 8 to August 31.* By Professor Ames: Theory of probability and statistics, History of mathematics.

Teachers College, Columbia University. July 8 to August 16. By Professor Upton: Teaching arithmetic in primary grades, first three grades (July 8–26), Curriculum problems in elementary arithmetic (July 8–26). By Professor Breslich: Teaching and supervision of mathematics in the junior high school, Teaching and supervision of mathematics in the senior high school. By Professor Clark: Teaching geometry in secondary schools, Demonstration class in plane geometry. By Professor Shuster: Modern business arithmetic, Field work in mathematics. By Miss Sutherland: Teaching arithmetic in intermediate grades, fourth, fifth and sixth (July 8–26). By Dr. Swenson: Professionalized subject matter in senior high school mathematics, Demonstration class in high school calculus. By Dr. Lazar: Teaching algebra in junior high school, Logic for teachers of mathematics. By Dr. R. R. Smith: Teaching algebra in secondary schools, Teaching of trigonometry, solid geometry and advanced algebra.

University of Texas. In addition to the regular elementary courses, the following advanced courses will be offered: *First term, June 4 to July 15.* By Professor R. L. Moore: Introduction to the foundations of geometry, Theory of sets. By Professor E. L. Dodd: Actuarial mathematics, Infinite processes. By Professor H. S. Vandiver: Introduction to the foundations of algebra, Theory of linear associative algebras. By Professor H. J. Ettlinger: Ruler and compass constructions, Research in differential equations and applications. By Mr. R. H. Sorgenfrey: Advanced calculus. By Professor A. E. Cooper: Group theory of differential equations. By Professor R. N. Haskell: Dynamics. *Second term, July 15 to August 26.* By Professor P. M. Batchelder: Difference equations. By Professor R. G. Lubben: Non-Euclidean geometry. By Dr. F. B. Jones: Advanced calculus. By Professor E. G. Keller: Advanced applied mathematics.

Texas Technological College. First term, June 6 to July 15. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor Michie: Differential equations. By Dr. Ollmann: Vector analysis. By

Dr. Hazelwood: Research and reading course for Master's thesis. *Second term, July 16 to August 23.* By Professor Heineman: Complex variable. By Professor Michie: Advanced calculus, Research and reading course for Master's thesis.

University of Virginia. First term, June 17 to July 27. By Professor McShane: Functions of a complex variable, Functions of a real variable. *Second term, July 29 to August 31.* By Professor W. M. Whyburn of the University of California at Los Angeles: Foundations of geometry, Functions of real variables.

University of Washington (Seattle). By Professor McFarlan: Differential equations. By Professor Jerbert: Selected topics in mathematics. By Professor Cramlet: Vector analysis. By Professor van Neumann: Selected topics in operator theory, Theory of games (evening lectures).

University of Wisconsin. By Professor Langer: Lie theory of differential equations, Topics in the history of mathematics, Differential equations. By Professor MacDuffee: Higher algebra, Topics in the theory of projective geometry, Theory of equations. By Professor March: Fourier series, Topics in dimensional analysis, Theoretical mechanics. By Professor Trump: College geometry, The teaching of mathematics. By Dr. Kershner: Advanced calculus.

ENROLLMENT IN JUNIOR COLLEGES

Enrollment in junior colleges in the United States has doubled in the last seven years, according to the 1940 Junior College Directory, just issued by the American Association of Junior Colleges.

Enrollment has increased from 155,588 to 196,510 in the last year. This 41,122 increase, which is 26.4 per cent, is the greatest ever reported, according to Walter C. Eells, secretary of the association. There are now 575 junior colleges, as compared with 556 reported a year ago.

For eleven leading states the number of junior colleges (shown in parentheses) and the number of enrolled students are as follows: California (64) 73,669; Texas (40) 12,804; Iowa (36) 3,409; Oklahoma (29) 5,394; Missouri (24) 7,831; Kansas (24) 5,398; Illinois (23) 14,711; Pennsylvania (23) 3,246; Mississippi (21) 4,645; North Carolina (21) 4,592; Georgia (20) 5,925.

A junior college, it is explained, is one which does work of college or university grade for two years beyond high school. Twenty-seven of these institutions include also, however, the last two years of high school.

The largest junior college is the San Bernardino Valley Junior College in California, which has 8,317 students. This number includes 7,499 special students, most of whom are adults. An extensive adult education program is offered by eight California junior colleges. Los Angeles City College, with 6,687 full-time students, has the largest full-time enrollment of all junior colleges in the country.

There are 33 junior colleges in the country with enrollments of more than 1,000. The size which is most general is between 100 and 200, in which group there are 153 reported. There are 212 junior colleges with enrollments between 200 and 1,000. The average for all is 349.

EXAMINATION QUESTIONS FOR THE THIRD WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, MARCH 2, 1940

MORNING SESSION: 9:00 A.M. to 12:00 NOON. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books permitted. Synthetic proofs are admissible in the geometric problems.*)

1. Prove that if $f(x)$ is a polynomial with integral coefficients, and there exists an integer k such that none of the integers $f(1), f(2), \cdots, f(k)$ is divisible by k, then $f(x)$ has no integral root.

2. Let A and B be two fixed points on the curve $y=f(x)$, where $f(x)$ is continuous and has a continuous derivative, and the arc AB is concave to the chord AB. If P is a point of the arc AB for which $AP+PB$ is a maximum, prove that PA and PB are equally inclined to the tangent to the curve $y=f(x)$ at the point P.

3. Find $f(x)$ such that

$$\int [f(x)]^n dx = \left[\int f(x) dx \right]^n,$$

when constants of integration are suitably chosen.

4. The parabola $y^2=-4px$ rolls without slipping around the parabola $y^2=4px$. Find the equation of the locus of the vertex of the rolling parabola.

5. Prove that the simultaneous equations

$$x^4 - x^2 = y^4 - y^2 = z^4 - z^2$$

are satisfied by the points of four straight lines and six ellipses, and by no other points.

6. $f(x)$ is a polynomial of degree n, such that a power of $f(x)$ is divisible by a power of its derivative $f'(x)$ i.e. $[f(x)]^p$ is divisible by $[f'(x)]^q$; p, q, positive integers. Prove that $f(x)$ is divisible by $f'(x)$, and that $f(x)$ has a single root of multiplicity n.

7. If $u_1^2+u_2^2+\cdots$ and $v_1^2+v_2^2+\cdots$ are convergent series of real constants, prove that

$$(u_1 - v_1)^p + (u_2 - v_2)^p + \cdots, \quad p \text{ an integer} \geqq 2,$$

is convergent.

8. A triangle is bounded by the lines

$$A_1x + B_1y + C_1 = 0, \qquad A_2x + B_2y + C_2 = 0, \qquad A_3x + B_3y + C_3 = 0.$$

Show that the area, disregarding sign, is

$$\frac{\begin{vmatrix} A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \\ A_3, & B_3, & C_3 \end{vmatrix}^2}{2\begin{vmatrix} A_2, & B_2 \\ A_3, & B_3 \end{vmatrix} \cdot \begin{vmatrix} A_3, & B_3 \\ A_1, & B_1 \end{vmatrix} \cdot \begin{vmatrix} A_1, & B_1 \\ A_2, & B_2 \end{vmatrix}}.$$

AFTERNOON SESSION: 2:00 P.M. to 5:00 P.M. (*Answer the questions in any order and by any method. Show all your work, and indicate your answers clearly. No tables or other books permitted.*)

9. A projectile, thrown with initial velocity v_0 in a direction making angle α with the horizontal, is acted on by no force except gravity. Find the length of its path until it strikes a horizontal plane through the starting point. Show that the flight is longest when

$$\sin \alpha \log (\sec \alpha + \tan \alpha) = 1.$$

10. A cylindrical hole of radius r is bored through a cylinder of radius $R(r \leqq R)$ so that the axes intersect at right angles. (*a*) Show that the area of the larger cylinder which is inside the smaller can be expressed in the form

$$S = 8r^2 \int_0^1 \frac{1 - v^2}{\sqrt{(1 - v^2)(1 - m^2v^2)}} dv, \text{ where } m = \frac{r}{R} \cdot$$

(*b*) If

$$K = \int_0^1 \frac{dv}{\sqrt{(1 - v^2)(1 - m^2v^2)}} \quad \text{and} \quad E = \int_0^1 \sqrt{\frac{1 - m^2v^2}{1 - v^2}} dv,$$

show that $S = 8[R^2E - (R^2 - r^2)K]$.

11. From any point (a, b) in the Cartesian plane, show that (*a*) three normals, real or imaginary, can be drawn to the parabola $y^2 = 4px$; (*b*) these are real and distinct if $4(2p - a)^3 + 27pb^2 < 0$; (*c*) two of them coincide if (a, b) lies on the curve $27py^2 = 4(x - 2p)^3$; (*d*) all three coincide only if (a, b) is the point $(2p, 0)$.

12. Prove that the locus of the point of intersection of three mutually perpendicular planes tangent to the surface

$$ax^2 + by^2 + cz^2 = 1 \qquad (abc \neq 0)$$

is the sphere

$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \cdot$$

13. Determine all rational values for which a, b, c are the roots of

$$x^3 + ax^2 + bx + c = 0.$$

14. Prove that

$$\begin{vmatrix} a_1^2 + k, & a_1a_2, & a_1a_3, & \cdots, & a_1a_n \\ a_2a_1, & a_2^2 + k, & a_2a_3, & \cdots, & a_2a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_na_1, & a_na_2, & a_na_3, & \cdots, & a_n^2 + k \end{vmatrix}$$

is divisible by k^{n-1} and find its other factor.

15. Which is greater

$$(\sqrt{n})^{\sqrt{n+1}} \quad \text{or} \quad (\sqrt{n+1})^{\sqrt{n}},$$

where $n > 8$?

NOTE. Chairmen of mathematics departments may obtain copies of the examination questions for the Putnam Competition for 1938, for 1939, and for 1940 by writing for them to Professor W. D. Cairns, 97 Elm Street, Oberlin, Ohio.

THE SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-third summer meeting of the Mathematical Association of America will be held at Dartmouth College, Hanover, New Hampshire, Monday, September 9, 1940, in conjunction with the meetings of the American Mathematical Society and the Institute of Mathematical Statistics. Because the meetings are held a week later than usual, the Association holds its first session at ten o'clock Monday morning, in order to facilitate the week's schedule and to accommodate those who must leave for their college teaching Thursday afternoon or Friday morning. It will be entirely feasible, with careful planning, for those attending the meetings to reach Hanover Sunday afternoon or early Monday morning in time for the first session.

The meetings of the Society will begin Tuesday morning and will continue through Thursday. A series of four colloquium lectures will be given by Professor G. T. Whyburn of the University of Virginia. The full program, with further details of the meetings of the Society and the Institute, will be sent to the members of the Association in July.

The dormitories of Dartmouth College will be available to the visitors at a reasonable price and may be occupied from one P.M. Sunday until Friday noon. Rooms are also available at the Hanover Inn for those who prefer hotel accommodations. Meals will be served in the college restaurant.

A reception Monday evening, the joint dinner Tuesday evening and an excursion to Franconia Notch in the White Mountains Wednesday afternoon will be attractive features of the week.

W. D. CAIRNS, *Secretary-Treasurer*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Morgantown, W. Va., April 20.
ILLINOIS, Bloomington, May 10–11.
INDIANA, Richmond.
IOWA, Mt. Vernon, April 19–20.
KANSAS, Wichita, March 30.
KENTUCKY, Lexington, April 27.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 8–9.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 11.
MICHIGAN, Ann Arbor, April 26–27.
MINNESOTA
MISSOURI, Warrensburg, April 19–20.
NEBRASKA, Omaha, May 9–11.
NORTHERN CALIFORNIA, Berkeley, January 27.
OHIO, Columbus, April 4.
OKLAHOMA
PHILADELPHIA, November 23 or 30.
ROCKY MOUNTAIN, Fort Collins, Colo., April 19.
SOUTHEASTERN, Athens, Ga., March 29–30.
SOUTHERN CALIFORNIA, Compton, March 2.
SOUTHWESTERN, Tucson, Ariz., April 22–23.
TEXAS, Dallas, March 29–30.
WISCONSIN, Milwaukee.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

MODULAR FIELDS*

SAUNDERS MAC LANE, Harvard University

1. Introduction. The general theory of modular fields, though elementary in its presuppositions, offers an instructive cross-section of modern algebraic methods. These fields exhibit the generality of subject-matter inherent in abstract algebra, and at the same time illustrate the intimate connection between algebraic and arithmetic problems.

Modular fields arise first in number theory in the consideration of congruences with a prime modulus p. For integers a and b the ordinary definition states that

$$a \equiv b \pmod{p} \qquad \text{means that} \qquad p \text{ divides } (a-b).$$

Any integer a on division by p yields a quotient q and a remainder r,

$$a = qp + r, \qquad 0 \leqq r < p;$$

hence $a \equiv r \pmod{p}$, where the remainder r is one of the integers

$$(1) \qquad F_p: \; 0, 1, 2, \cdots, p-2, p-1.$$

Any integer is congruent to one of those in this set of p numbers.

With these numbers alone one can still carry out algebraic operations, provided one adds and multiplies these numbers in the ordinary fashion, and then reduces the answer by congruence to one of the numbers (1). For example, if $p=5$, the product $2\cdot 3=6$ should really be $2\cdot 3\equiv 6-5=1$. In this fashion one can make multiplication and addition tables for $p=5$, as shown. It is strange

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

that this idea has not appeared more† in texts on number theory, for the idea is an essentially simple one. One can introduce it by the intuitively natural algebra of the words "even" and "odd," as

$$\text{even} \cdot \text{even} = \text{even}, \qquad \text{even} \cdot \text{odd} = \text{even}, \qquad \text{odd} \cdot \text{odd} = \text{odd},$$
$$\text{even} + \text{even} = \text{even}, \qquad \text{even} + \text{odd} = \text{odd}, \qquad \text{odd} + \text{odd} = \text{even}.$$

This is just the algebra of integers modulo $p=2$.

* An address delivered before the Mathematical Association of America at Columbus, Ohio, December 30, 1939.

† *Cf.* remarks in Weiss [**26**].

A congruence modulo p has all the properties of an equation; congruences can be added and multiplied term by term, and the relation of congruence is reflexive, symmetric, and transitive. If the modulus p is fixed, one might just as well dub congruence "equality." Every integer is then "equal" to one of the p symbols, $0, \cdots, p-1$, and the sums and products of these symbols, so identified, give exactly the algebra of the integers modulo p, as described above.

If one objects to rebaptizing "congruence" by fiat, one may adopt the more sophisticated procedure* of replacing each remainder r modulo p by the *class* r_p of all integers $r, r+p, r+2p, \cdots$ congruent to it. Such "congruence classes" are then added and multiplied according to the rules

$$r_p + s_p = (r + s)_p, \qquad r_p \cdot s_p = (rs)_p. \tag{2}$$

Furthermore the congruence classes r_p and s_p will be equal (*i.e.*, will contain the same elements) if and only if the integers r and s are congruent, so the desired "equality" has now been properly introduced. In any event the *integers modulo p* form a finite set of objects (1) satisfying all rules of algebra.

The presence of such arithmetic objects, which are certainly not ordinary numbers but which still obey ordinary algebra, is the reason why modern algebra is abstract. To separately discuss the algebra of numbers, then the algebra of congruence classes, then the algebra of functions, and so on would be most inefficient. Instead, theorems are better proved for any (abstractly conceived) system of objects whatever to which the basic rules of algebra apply.

These laws of algebra for a set F of objects, such as the integers modulo p, are codified as follows: For a and b in F there is uniquely defined a *sum* $a+b$ and a *product* $a \cdot b$. This product is *commutative* $[ab=ba]$ and *associative* $[a(bc)=(ab)c]$, as is also the sum. The *distributive* law $a(b+c)=ab+ac$ holds for all a, b, and c. The set F contains a zero 0 and a unit 1, with the characteristic properties

$$a + 0 = a = 0 + a, \qquad 1 \cdot a = a = a \cdot 1,$$

respectively. Finally, *subtraction* and *division* are possible, which is to say that the equations $a+x=0$ and $b \cdot y=1$ have solutions x and y in F, except when $b=0$. Any set F of elements with all these properties is called a *field*. One may say that a field is any system of elements within which addition, subtraction, multiplication, and division (excluding division by zero) can be carried out in the usual fashion.

Well known fields are: (a) the set of all rational numbers; (b) the set of all real numbers; (c) the set of all complex numbers. The field (1) composed of the integers modulo p is often called the *Galois field* $GF[p]$. A *modular field* is any field containing such a $GF[p]$.

These fields $GF[p]$ are not the only finite fields. One may construct larger fields by simply adjoining to a $GF[p]$ the roots of certain algebraic equations. The process resembles the construction of the complex numbers from the field R

* *Cf.* Albert [1, p. 7]; van der Waerden [27, p. 13]; or Mac Lane [17, Chapter I].

of real numbers. Here one adjoins to R a symbol i representing a root of the equation $x^2+1=0$; the field C of all complex numbers $a+bi$ then contains everything which can be expressed rationally in terms of i and real numbers. The fact that C is generated over R by adjoining i is symbolized by $C=R(i)$. Note in particular that the polynomial x^2+1 used to generate this extension is *irreducible* over R, because it cannot be factored into polynomials of smaller degree with coefficients in R.

In similar vein consider the polynomial $f(x)=x^2+x+1$ over the field F_2 with two elements (the integers modulo 2). Neither $f(1)$ nor $f(0)$ is zero, so this polynomial $f(x)$ has no roots in F_2, hence has no linear factors, hence is irreducible over F_2. Invent a symbol u to denote a root of $f(x)=0$, so that

$$u^2+u+1=0, \qquad u^2=-u-1=u+1.$$

(Recall that $-1=+1$, modulo 2.) All higher powers of u can thereby be successively reduced to linear expressions in u. Reciprocals can be similarly reduced, so that the field generated by u contains all told just four linear expressions: 0, 1, u, $u+1$. These combine under addition and multiplication

$+$	0	1	u	$u+1$
0	0	1	u	$u+1$
1	1	0	$u+1$	u
u	u	$u+1$	0	1
$u+1$	$u+1$	u	1	0

$\cdot$	0	1	u	$u+1$
0	0	0	0	0
1	0	1	u	$u+1$
u	0	u	$u+1$	1
$u+1$	0	$u+1$	1	u

as shown in the tables. The process of obtaining this field by adjoining to the original F_2 a root u of x^2+x+1 is known as *algebraic extension* of F_2, and the resulting field $F_2(u)$ is called a Galois field of 4 elements.

For each prime p and each integral exponent n one may analogously extend the field of integers modulo p to a field consisting of exactly* p^n elements. As E. H. Moore first showed, *any* two fields with p^n elements each are algebraically indistinguishable (isomorphic). The arithmetic origin of all these finite fields is the study of algebraic integers. If $\mathfrak{p}$ is a prime ideal in a field K of algebraic numbers, then the congruences modulo this ideal behave as do ordinary congruences, and yield like them a finite field with p^n elements, where p^n is the so-called "norm" of the ideal $\mathfrak{p}$. The properties of the resulting finite fields play an essential rôle in the class field theory and in the study of rational division algebras (Albert [**2**, ch. 9]).

2. Characteristics. The integers modulo p have one peculiar property. The unit 1, added p times to itself, yields $p\equiv 0 \pmod p$ as answer; hence

$$1+1+\cdots+1=0, \qquad (p \text{ summands}). \tag{3}$$

* See detailed discussion of finite fields in van der Waerden [**27**, §31]; or Albert [**1**, p. 166].

On multiplying this equation by any integer a, one has

$$a + a + \cdots + a = 0, \qquad (p \text{ summands}), \tag{4}$$

in the Galois field F_p. Any field F, all of whose elements a have the property (4), is called a field of *characteristic* p, or a *modular field*. It can be shown* that any non-modular field has an infinite characteristic, in the sense that $a \neq 0$ entails $a+a+\cdots+a \neq 0$, for any number of summands. Any finite field of p^n elements essentially contains the integers modulo p, hence satisfies (3) and therefore (4). Thus any finite field is modular.

Watch the effect of (4) on the binomial expansion,

$$(a + b)^p = a^p + pa^{p-1}b + (p(p-1)/2)a^{p-2}b^2 + \cdots + pab^{p-1} + b^p.$$

According to the genesis of this expansion, the term $pa^{p-1}b$ second on the right really represents a sum of p products $a^{p-1}b+a^{p-1}b+\cdots+a^{p-1}b$. In a field of characteristic p this sum is zero. The other intermediate terms of the binomial expansion suffer the same fate, for each binomial coefficient $p(p-1)/2, \cdots, p$ is a multiple of the characteristic p. One has left only

$$(a + b)^p = a^p + b^p, \qquad (a, b \text{ in } F \text{ of characteristic } p). \tag{5}$$

As S. C. Kleene has remarked, a knowledge of the case $p=2$ of this equation would corrupt freshman students of algebra!

The pth power of a product is always a product of pth powers, so the rules

$$(a \pm b)^p = a^p \pm b^p, \qquad (ab)^p = a^p b^p, \qquad (a/b)^p = a^p/b^p \tag{6}$$

hold in any field of characteristic p. These rules state that the process of raising to a pth power leaves the operations of addition, division, *etc.*, unchanged. This process yields a correspondence

$$a \longleftrightarrow a^p, \qquad (\text{from } F \text{ to } F^p), \tag{7}$$

which carries the field F into the field F^p composed of all pth powers from F. The correspondence is one-to-one, for the equality of two pth powers $a^p = b^p$ would entail $0 = b^p - a^p = (b-a)^p$, and hence $b = a$. To summarize, the correspondence $a \longleftrightarrow a^p$ is an isomorphism, where an *isomorphism* between two fields is defined to be any one-to-one correspondence which preserves sums and products.

Repeated application of the rules in (6) shows that the pth power of any rational expression can be computed by applying the exponent p to each term or factor in the expression. In particular,

$$(1 + 1 + \cdots + 1)^p = 1^p + 1^p + \cdots + 1^p = 1 + 1 + \cdots + 1 \tag{8}$$

holds in the field of integers modulo p. If we use m summands here, this is $m^p = m$. In terms of congruences this is $m^p \equiv m \pmod{p}$, which is the little Fermat Theorem!

* *Cf.* Albert [**1**, p. 30]; Mac Lane [**17**, §21]; van der Waerden [**27**, §25].

3. Algebraic and transcendental extensions. Our major concern is the structure of the general modular field, finite or infinite. In the analogous case of fields of numbers it is customary to distinguish the algebraic numbers, such as $\sqrt{3}$, which satisfy some polynomial equation with rational coefficients, from the transcendental numbers (e, π), which satisfy no such equation. In general, let a given field F be contained in any larger field K. An element u of K is *algebraic* over F if u is a root of a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0 \tag{9}$$

with coefficients a_i in F. If this equation $f(x)=0$ for u be chosen with a degree n as small as possible, the polynomial $f(x)$ is *irreducible* over F. For, a reducible $f(x)$ would have factors $f(x)=f_1(x)f_2(x)$ with coefficients in F, and u would satisfy one of the equations $f_1(x)=0$, $f_2(x)=0$, of degree smaller than n. An element u in K not algebraic over F is called *transcendental*; for u transcendental, $f(u)=0$ implies that all the coefficients in $f(x)$ are zero.

Important is not the element u in K by itself, but the field $F(u)$ which it generates. The field consists of all rational combinations of u with coefficients in F, and is called a *simple extension* of F, "algebraic" or "transcendental" according as u is algebraic or transcendental over F. This dichotomy is the root of one of the basic results found by Steinitz in his pioneering investigations of fields (Steinitz [**23**]): *Any modular field can be obtained by successive transcendental and algebraic extensions of a field (isomorphic to the field) of integers modulo p.*

Such extensions can be used not only to build up a given field K from a subfield F, but also to manufacture new fields from old. Given a polynomial $f(x)$ irreducible over a field F, one can concoct a symbol u for a root of this polynomial and construct therewith an algebraic extension $F(u)$ generated by the root u. In point of fact, $F(u)$ consists of elements expressible as polynomials $b_0+b_1u+\cdots+b_{n-1}u^{n-1}$, with coefficients in F and of degree less than the degree n of the given $f(x)$.

Alternatively, a variable t over a modular field gives rise to rational functions

$$\frac{g(t)}{h(t)} = \frac{b_0 + b_1 t + \cdots + b_r t^r}{c_0 + c_1 t + \cdots + c_m t^m}, \qquad (c_i,\ b_j \text{ in } F, \text{ not all } c_i=0). \tag{10}$$

Under the usual rules for adding and multiplying such expressions, the totality of these rational functions is a field $F(t)$ which is a simple transcendental extension of F. If F is a finite field, the resulting field $F(t)$ is the simplest instance of an infinite modular field.

4. Inseparable equations. Over the transcendental extension $F(t)$ there are in turn algebraic extensions, such as that generated by a root of the polynomial $f(x)=x^p-t$. This $f(x)$ is irreducible over $F(t)$, for if it could be factored, the denominators in t could be eliminated, and we could write $x^p-t=g(x, t)h(x, t)$, with factors which are polynomials in x and t. Since the product of these two polynomials is linear in t, one of them must be linear in t, while the other cannot

involve t at all! This is absurd unless one of the factors is a constant; hence $f(x)$ is indeed irreducible.

But trouble arises with the introduction of a root u for this equation $x^p-t=0$. Since this u is a pth root of t, we have a factorization

$$x^p - t = x^p - u^p = (x - u)^p, \tag{11}$$

according to the rule (6) for the pth power of a difference. This means that u is a p-fold root of x^p-t, so this irreducible polynomial has all its roots equal, and t has only one pth root.

This differs drastically from the usual situation with ordinary complex nth roots, for an irreducible polynomial $f(x)$ with *rational* coefficients can never have a multiple root. Let us trace the proof of this fact. If $f(x)$ has a complex number r as m-fold root, then $f(x)=(x-r)^m g(x)$, with $m>1$. The derivative is

$$f'(x) = (x - r)^{m-1}[mg(x) + (x - r)g'(x)]. \tag{12}$$

Since $m>1$, this insures that $f(x)$ and $f'(x)$ have a common factor $(x-r)^{m-1}$, not a constant. But the highest common factor of $f(x)$ and $f'(x)$ can be found by the euclidean algorithm, using only rational operations. This highest common factor then has rational coefficients, and its degree is at most that of $f'(x)$. It must divide $f(x)$, counter to the assumed irreducibility of that polynomial.

Can this contradiction be deduced for a polynomial $f(x)$, irreducible not over the rationals but over some modular field, and having a multiple root r in a larger field? The derivative $f'(x)$ of calculus is no longer available, but for any polynomial $f(x)$ as in (9) a "formal" derivative can still be defined as

$$f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + (n-2)a_{n-2}x^{n-3} + \cdots + a_1. \tag{13}$$

Here the coefficient ia_i of the term x^{i-1} is to denote the sum

$$ia_i = a_i + a_i + \cdots + a_i, \qquad (i \text{ summands}). \tag{14}$$

Apply this derivative to the troublesome polynomial x^p-t of (11). We find

$$(x^p - t)' = px^{p-1} = x^{p-1} + \cdots + x^{p-1} = 0, \qquad (p \text{ summands}).$$

No wonder that an argument on the H. C. F. of x^p-t and 0 runs aground! Looking back, one sees that the argument following (12) about multiple roots will work, except in such cases when $f'(x)$ vanishes.

When do all coefficients ia_i of $f'(x)$ vanish? In a modular field $ia_i=0$ means either that a_i itself is zero, or that the number i of summands, in (14), is a multiple of the characteristic p. A coefficient a_i can thus differ from zero only for terms $a_i x^i$ with exponent $i\equiv 0 \pmod p$. The vanishing of $f'(x)$ means therefore that $f(x)$ can involve x only as powers of x^p, so that $f(x)$ has the form

$$g(x) = b_m x^{mp} + b_{m-1}x^{(m-1)p} + \cdots + b_1 x^p + b_0. \tag{15}$$

An irreducible polynomial $g(x)$ of this form must always have p-fold roots. Such a polynomial is called *inseparable* (its roots cannot be "separated" into distinct roots). Many properties of ordinary equations fail for inseparable equations.

An element u algebraic over a modular field F is called *separable* over F if the irreducible equation for u is separable (*i.e.*, has no multiple roots). Of the inseparable algebraic elements the simplest examples are pth roots which satisfy inseparable equations $x^p=a$. Consider an arbitrary inseparable element u, root of an inseparable polynomial (15) of degree mp. This polynomial involves only pth powers of its variable, so u^p is a root of an equation

$$h(y) = b_m y^m + b_{m-1}y^{m-1} + \cdots + b_1 y + b_0, \tag{16}$$

of smaller degree m. The adjunction of the root u to our field F can then be effected in two stages

$$F \to F(u^p) \to F(u^p, \sqrt[p]{u^p}) = F(u).$$

The element u^p first adjoined may still belong to an inseparable equation $h(y)=0$; in that event the process can be reapplied to get u^{p^2} satisfying an equation of still smaller degree. The adjunction of an inseparable algebraic element to a modular field can be accomplished by adjoining successive pth roots of a suitable separable algebraic element (Steinitz [**23**]). This reduction of algebraic extensions to separable extensions followed by extensions by pth roots, indicates that the novel properties are concerned chiefly with the latter type of extension.*

5. Perfect fields. There are no inseparable algebraic elements over the field of integers modulo p, for this field already contains the pth roots of all of its elements—indeed, the Fermat Theorem, $a^p=a$, asserts that every element is its own pth root. A *perfect* field F of characteristic p is a field in which each element a has a pth root. Over such a field each pth root equation $x^p=a$ is reducible, as $x^p-a=(x-\sqrt[p]{a})^p$. More generally *any inseparable polynomial* $g(x)$ *involving only pth powers of* x *must be reducible over a perfect field.* For, each coefficient b_i of the polynomial $g(x)$ in (15) has in F a pth root $b_i^{1/p}$; according to the simple behavior of pth powers this gives a factorization

$$g(x) = (b_m^{1/p}x^m + b_{m-1}^{1/p}x^{m-1} + \cdots + b_1^{1/p}x + b_0^{1/p})^p.$$

Every finite field F is perfect, hence has no inseparable algebraic extensions. To prove this, recall the correspondence $a \longleftrightarrow a^p$ of (7), which is a one-to-one correspondence between *all* elements of F and those elements a^p which are pth powers. Since there are but a finite number of elements in F, there must be the same number of pth powers. This means that every element is a pth power.

A simple transcendental extension $F(t)$ of a modular field can never be perfect. To verify this we need only produce an element with no pth root in the field. The variable t itself is such an element, for if t had as pth root some rational function $g(t)/h(t)$ in the field, t would equal $[g(t)/h(t)]^p$, a pth power

* Technically, the least power $q=p^e$ such that u^q is separable over F is known as the *exponent* of u over F. The *degree* of u over F is the degree of its irreducible equation, while the degree of u^q is known as the *reduced degree* of u.

which can be calculated by the rule (6). In the notation of (10), the result is

$$(c_0^p + c_1^p t^p + \cdots + c_m^p t^{mp})t = b_0^p + b_1^p t^p + \cdots + b_r^p t^{rp},$$

an identity which clearly cannot hold good. For similar reasons a multiple transcendental extension $F(t_1, t_2, \cdots, t_n)$, consisting of all rational functions of n independent variables t_i, cannot be a perfect field.

6. Galois theory. To what extent can one generalize to modular fields the ordinary properties of fields of rational and algebraic numbers? A major topic is the Galois theory, which analyzes the solvability of a polynomial equation $f(x)=0$ over a field F. The roots $r_1, \cdots, r_n$ of this equation generate over F a *root field*

$$K = F(r_1, r_2, \cdots, r_n), \qquad \text{where} \qquad f(x) = (x - r_1)(x - r_2) \cdots (x - r_n); \tag{17}$$

the Galois Theory studies K in terms of its group of automorphisms, each of which is an isomorphism of the field K with itself, induced by a permutation of the roots r_i. Should these roots all be equal, the only such permutation is the identity, and the theory breaks down. Only if one assumes that the roots are all distinct, *i.e.*, that $f(x)$ is separable, does the standard theory of root fields hold* over a modular F.

This straightforward generalization does not suffice for irreducible *in*separable polynomials. The first process to fail is the construction of a "Galois resolvent," which is an equation with a root u in K such that all the roots r_i can be rationally expressed in terms of this single quantity u. In terms of fields, this means that the multiple algebraic extension $K = F(r_1, \cdots, r_n)$ can be represented as a simple extension $F(u)$. Over an imperfect field F there may be multiple algebraic extensions which cannot be so represented. Consider for instance the rational function field,

$$F_0 = P(t_1, t_2), \qquad P \text{ perfect}, \tag{18}$$

in two independent variables t_1 and t_2. An adjunction of pth roots will yield an extended field

$$K_0 = F_0(u_1, u_2); \qquad u_1^p = t_1, \qquad u_2^p = t_2, \tag{19}$$

which consists of all elements expressible as polynomials

$$w = \sum_{i,j} a_{ij} u_1^i u_2^j = h(u_1, u_2), \qquad (i, j = 0, \cdots, p - 1), \tag{20}$$

with coefficients a_{ij} in F_0. This field K_0 is not a simple extension $K_0 = F_0(w)$ for any w. For, if there were a generator w, then by the rule for pth powers,

$$w^p = \sum_{i,j} a_{ij}^p u_1^{ip} u_2^{jp} = \sum_{i,j} a_{ij}^p t_1^i t_2^j$$

* *Cf.* Albert [1, ch. VIII]; van der Waerden [27, ch. 7]; Mac Lane [17, §68].

is in F_0, so w is a pth root of an element of F_0. That such a single pth root could generate the field K_0 containing two independent pth roots u_1 and u_2 is unreasonable. This hunch can be substantiated by an argument on the degree* of the extension K_0 of F_0.

If a multiple extension does not have one generator, what is then the *minimum* number of generators? Miriam Becker [6] has recently found the answer. Over the particular field $P(t_1, t_2)$ of (18) it appears that *any* multiple algebraic extension can be expressed by two generators, just as in the case of the special extension K_0 of (19). The underlying reason is the presence of just two independent pth roots, $\sqrt[p]{t_1}$ and $\sqrt[p]{t_2}$, not in the field $P(t_1, t_2)$; the pth root of any other rational function $g(t_1, t_2)$ in the field can be expressed by the rule (6) in terms of these two pth roots, together with pth roots of coefficients which already lie in the perfect base field P.

Over any modular field F one calls the r pth roots $a_1^{1/p}, a_2^{1/p}, \cdots, a_r^{1/p}$ *p-independent* if no one of them can be rationally expressed in terms of F and the others. Becker proves that *any multiple algebraic extension of an imperfect field F can be generated by m elements, where m is the maximum number of independent pth roots over F.* If $m=0$, F is perfect: if $m=1$, any multiple algebraic extension is simple, as shown by Steinitz.

7. Derivatives. The solution of an ordinary equation $f(x)=0$ by radicals (if possible) proceeds in successive stages which correspond to successive fields lying between the coefficient field F and the root field K. For a separable equation the whole array of possible intermediate fields is finite—but not so for some inseparable extensions. Between the fields F_0 and K_0 of (19) lie infinitely many distinct fields $F_0((t_1+t_2^m)^{1/p})$, with $m=1, p+1, 2p+1, \cdots$. For a separable equation the fields intermediate between K and F can be put into one-to-one correspondence with the sub-groups of the Galois group of automorphisms of K over F. This certainly fails for an inseparable extension like (19), for in that case the Galois group of K_0 over F_0 consists of the identity alone and so has no proper sub-groups to correspond to intermediate fields. Specifically, the Galois group consists of all isomorphisms of K_0 with itself which leave fixed each element in the base field F_0; but an isomorphism leaving fixed the elements t_1 and t_2 of F_0 must likewise leave fixed their *unique* pth roots u_1 and u_2 and hence must leave all elements of K_0 fixed.

For this description of intermediate fields by the Galois group Jacobson has found a substitute, in the special case of extensions K obtained by adjoining any number of pth roots to a modular field F, as

$$K = F(a_1^{1/p}, a_2^{1/p}, \cdots, a_n^{1/p}), \qquad \text{each } a_i \text{ in } F. \tag{21}$$

By a piece of poetic justice, his solution depends on exploiting the very formal

* This degree is the maximum number of elements of K_0 "linearly independent" over F_0. This maximum is p^2, for any w is linearly dependent on the p^2 elements $u_1^iu_2^j$ of (20). For a simple extension $F_0(w)$ the degree would be only p. Hence $F_0(w)$ cannot equal K_0.

derivatives whose misbehavior (*cf.* §4) is at the root of inseparability. For example, in the field K_0 of (19) one has two "derivative" operators D_1 and D_2, defined for the arbitrary element $w=h(u_1, u_2)$ of (20) by

(22) $$h(u_1, u_2)D_1 = \partial h(u_1, u_2)/\partial u_1, \qquad h(u_1, u_2)D_2 = \partial h(u_1, u_2)/\partial u_2.$$

This time the properties of pth powers are fortunate, for $u_1^pD_1 = pu_1^{p-1}=0$, as it ought to be, for $u_1^p=t_1$ is in the base field and so should have derivative 0 according to the definition (22). These derivatives can be used to characterize sub-fields of K_0; for example, the sub-field $F_0(u_1)$ consists of everything annihilated by the operator D_2 (*i.e.*, of all w with $wD_2=0$).

In general, Jacobson considers **[12]** all *formal differentiation operators* D which map K into itself by a correspondence $w \to wD$ which carries elements of F into zero and which obeys the usual formal rules for differentiation:

$$(v + w)D = vD + wD, \qquad (vw)\, D = v(wD) + (vD)w.$$

From any two such operators D_1 and D_2 one may construct new differentiations $D_1 \pm D_2$, D_1^p, and D_1c, for c in F. Furthermore, the commutator $[D_1, D_2]=D_1D_2 - D_2D_1$ is again a formal differentiation. This commutator satisfies the identity

$$[[D_1, D_2], D_3] + [[D_2, D_3], D_1] + [[D_3, D_1], D_2] = 0,$$

which is one of the essential postulates for a Lie algebra. The set $\mathfrak{L}$ of all differentiations is in fact a Lie algebra over the base field F. This algebra acts as a substitute for the Galois group of a field K of type (21), in the sense that *there is a one-to-one correspondence between the fields intermediate between K and F and the restricted Lie sub-algebras of the algebra $\mathfrak{L}$ of all formal differentiations of K over F.* For this purpose a *restricted* sub-algebra of $\mathfrak{L}$ is a sub-set $\mathfrak{L}'$ of $\mathfrak{L}$ which is itself a Lie algebra and which is restricted to contain D^p for each D of $\mathfrak{L}'$.

8. Algebraic geometry. A skew curve can be represented as the intersection of two surfaces, which may often be taken as cylinders

(23) $$f(x, y) = 0, \qquad g(x, z) = 0$$

with axes parallel to the z and y coördinate axes, respectively. If f and g are polynomials, the intersection of these cylinders is an algebraic curve. Alternatively, x may be viewed as a quantity transcendental over the field C of complex numbers; the polynomial equations then make the quantities y and z algebraic over the field $C(x)$ of rational functions of x. All told they give a field $C(x, y, z)$ generated by "algebraic functions" y and z of x. This field is the algebraic invariant of the curve (23). The ordinary analytic theory of these algebraic function fields can be developed, without using the geometry of the Riemann surface, if the base field C of complex numbers is replaced by a perfect modular field P or even by an imperfect one.*

* *Cf.* general discussion of these abstract algebraic functions in Mac Lane-Nilson **[19]** or Schilling **[21]**. Especially interesting is the introduction of a Riemann Zeta function when P is finite (Hasse **[9]**), the peculiar behavior of the Weierstrass points whenever P is modular (Schmidt **[22]**), and the generalizations of Abelian functions (Schilling **[20]**).

In an n-dimensional euclidean space an r-dimensional algebraic manifold can be described as the set of points common to $n-r$ suitable algebraic hypersurfaces. These hypersurfaces may be taken, as in (23), in the form of "cylinders"

$$(24)\quad f_1(y_1, \cdots, y_r, y_{r+1}) = f_2(y_1, \cdots, y_r, y_{r+2}) = \cdots = f_{n-r}(y_1, \cdots, y_r, y_n) = 0,$$

where each f_i is an irreducible polynomial actually containing y_{r+i}. As coefficients in (24) we use not complex numbers but elements from a perfect modular field P. If this field P is finite, this means that we are considering a manifold in some finite affine (or projective) geometry, consisting of a finite number of "points" specified by coördinates in P. Algebraically, the symbols $y_1, \cdots, y_n$ related by (24) generate a field $K=P(y_1, \cdots, y_r, y_{r+1}, \cdots, y_n)$, consisting of all rational functions of these quantities, subject only to the rules of algebra and the special conditions (24). This field is obtained from the base field P by r successive simple extensions by the transcendentals $y_1, \cdots, y_r$, followed by $n-r$ successive algebraic extensions by the roots $y_{r+1}, \cdots, y_n$ of the polynomial equations (24). In a sense, the geometry of the manifold depends on the structure of this field.

What of the presence of inseparable equations in the definition (24) of such a manifold? Suppose, for instance, that the equation $f_1=0$ is inseparable in y_{r+1}, so that this variable appears only as a pth power. Certainly this could not simultaneously be the case for all the variables $y_1, \cdots, y_r, y_{r+1}$ in f_1, for in that event we could extract the pth root of every term in the equation $f_1=0$, thus making $f_1=(g_1)^p$, counter to the assumed irreducibility of f_1 over the perfect field P. Suppose then that y_1 is one of the variables which does not appear in $f_1(y_1, \cdots, y_r, y_{r+1})$ only as a pth power. The equation $f_1(y_1, \cdots, y_{r+1})$, which originally defined y_{r+1} inseparably over the field $P(y_1, \cdots, y_r)$, can be turned about and viewed as a definition of y_1 as a quantity *separable* and algebraic over the field $P(y_2, \cdots, y_r, y_{r+1})$, generated by the r independent transcendentals $y_2, \cdots, y_{r+1}$. A further juggling of the independent variables can then be applied to any subsequent equations of (24) which may be inseparable. Hence the result: *If a field $K=P(y_1, \cdots, y_n)$ is obtained from a perfect field P by adjoining a finite number of elements $y_1, \cdots, y_n$, one can find for K a generation $K=P(t_1, \cdots, t_r; u_1, \cdots, u_{n-r})$ involving r simple transcendental extensions by variables t_i, followed by $n-r$ separable algebraic extensions.* Whenever independent transcendents t_i in K have this property, that every element in K is *separable* and algebraic over $P(t_1, \cdots, t_r)$, we say that the $t_1, \cdots, t_r$ form a *separating transcendence basis* for K over P.

This construction of separating transcendence bases was discovered independently for different purposes: by the author, in connection with Albert's theory of pure forms (Albert [4]); by van der Waerden [28], for a new proof of the theorem that two distinct irreducible algebraic manifolds M_r and M_{n-r} in projective n-space intersect in a finite number of points, and, moreover, that the "number" of points, properly counted, is the product of the degrees of M_r and M_{n-r}.

9. Preservation of independence. The troubles of inseparable equations can be avoided whenever we find a separating transcendence basis for the field under consideration. Unfortunately this cannot always be done. Suppose, for instance, that the base field is the field $F_0 = P(t_1, t_2)$ of all rational functions of two transcendents t_1 and t_2 over a perfect field P, and construct a larger field L by adjoining first a new transcendent z and then an algebraic element u, with

$$u^p = t_1 + t_2 z^p, \qquad L = F_0(z, u). \tag{25}$$

Since the pth root u is inseparable over $F_0(z)$, this z is surely not a *separating* transcendence basis for L over F_0. The order of adjunction might have been inverted, adding u first as a transcendent to L and then z, but the equation (25) indicates that z would then be a pth root. The same trouble would always arise: one can prove that L has over F_0 *no* separating transcendence basis.* The same troublesome example arises in Krull's general ideal theory [**13**].

To find the reason for this absence of separability one must look at the possible independent pth roots in the base field F_0. In §6 we saw that the pth roots $\sqrt[p]{t_1}$ and $\sqrt[p]{t_2}$ were p-independent there, because neither can be expressed in terms of F_0 and the other. These pth roots are no longer p-independent in the top field L, for the defining equation (25) of that field gives an expression $\sqrt[p]{t_1} = u - z\sqrt[p]{t_2}$. This suggests that we restrict attention to those extensions L over F which *preserve p-independence*, in the sense that any set of p-independent pth roots over F remains p-independent over L. The relevance of this concept is indicated by the following alternative description: *a field L preserves p-independence over F if and only if the adjunction to F of any finite set of elements $y_1, \cdots, y_n$ from L yields a field $F(y_1, \cdots, y_n)$ which has over F a separating transcendence basis.*

This concept also makes it possible to find explicit conditions that given extensions have separating transcendence bases (Mac Lane [**16**]). One simply stated result is this: *If a field K has a finite separating transcendence basis over a sub-field M, then any field L between K and M also has a finite separating transcendence basis over M.* In other words, one can find a set S of independent transcendents in L, such that every element of L satisfies over $M(S)$ an algebraic irreducible equation without multiple roots.

10. General field towers. What can be said of the structure of arbitrarily complicated modular fields? The fields $P(y_1, \cdots, y_n)$ associated with algebraic manifolds had separating transcendence bases over a perfect field P. Does every modular field have a separating transcendence basis T over a suitable perfect sub-field?

The answer is no. A simple counterexample may be built from the extension $P(t)$ of a finite field P by a transcendental t. We saw in §5 that $P(t)$ is imperfect because t has in it no pth root. If we try to embed $P(t)$ in a larger field P' which

* Even though, according to the Theorem of §8, L has over the *original* perfect field P a separating transcendence basis consisting of u, z, and t_1.

will be perfect, we must have in P' a pth root $t^{1/p}$ and hence the whole rational function field $P(t^{1/p})$ generated by this root. In this field $t^{1/p}$ has no pth root, so we add $t^{p^{-2}}$, and so on, till we have the "tower"

$$(26) \qquad P(t) \subset P(t^{p^{-1}}) \subset P(t^{p^{-2}}) \subset P(t^{p^{-3}}) \subset \cdots .$$

The field enveloping everything in this tower may be called $P(t^{p^{-\infty}})$; it consists of all elements lying in any one of the fields (26). Furthermore this sum field $P(t^{p^{-\infty}})$ is perfect, for an element in any one of the fields of (26) does have a pth root in the next field of the tower.

This perfect field $P(t^{p^{-\infty}})$ can have over P no separating transcendence basis. Any such basis would consist of a single transcendent t', which must lie in some one of the fields $P(t^{p^{-e}})$ of the tower (26). The generating element $t^{p^{-(e+1)}}$ of the next field is then a quantity inseparable over $P(t')$, so t' cannot have been the desired separating basis.

The tower (26) as written shows $P(t^{p^{-\infty}})$ generated by a transcendental extension followed by successive (inseparable) extractions of pth roots. Nevertheless each field of this tower, considered by itself, is a simple transcendental extension of P by $t^{p^{-e}}$. The whole field is thereby approximated by a tower of fields, each of which has a separating transcendence basis over the base field P, and each of which consists of pth powers of elements in the next field. F. K. Schmidt has shown that any perfect field P' has a similar "separating tower" over any one of its perfect sub-fields. He also stated without proof an analogous tower theorem for an imperfect field, but it was later shown by examples* that this general theorem could not hold. Recently F. K. Schmidt and the author have jointly [**18**] found a modified tower theorem: *If a modular field K is generated from a perfect sub-field P by a denumerable number of elements, then there is a sub-field L with a separating transcendence basis over P and a tower of fields $L \subset M_0 \subset M_1 \subset \cdots$ which collectively exhaust K, such that each M_i has over L a separating transcendence basis and is generated over L by pth powers from M_{i+1}.* The non-denumerable cases can then be broken down into a transfinite sequence of denumerable steps, each of which "preserves p-independence" in the sense discussed in §9.

The separability of these field towers is essential to get polynomials with distinct roots, in order to apply an implicit function theorem.† This is used in the proof of the structure theorem for p-adic fields (*cf.* Hasse-Schmidt [**10**]). These p-adic fields are fields topologically complete with respect to a suitable norm (or "absolute value"), obtained by extending the norm for the p-adic numbers of Hensel.‡ These p-adic fields are not themselves modular fields, but they determine a congruence relation $a \equiv b \pmod{p}$ from which modular fields can be obtained by the standard arithmetic device.

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* *Cf.* Mac Lane [**15**]. Curiously enough, these examples involve a use of the modular law of lattice theory!

† The so-called Hensel-Rychlik theorem; *cf.* Albert [**1**] or Mac Lane-Nilson [**19**, §11].

‡ See the description in C. C. MacDuffee [**14**].

11. Troublesome examples. The extent of our ignorance of general modular fields can be forcibly illustrated by various startling examples. The field $P(t^{p^{-\infty}})$ used to illustrate §10 was still manageable, for though it had no separating transcendence basis, it at least was itself perfect. But can there be an imperfect field K which has no separating transcendence basis over some perfect sub-field P? There is indeed such a K, for which P may even be chosen as the maximum perfect sub-field. Over a finite field P choose a countable set of indeterminates $t_1, t_2, \cdots$, and then introduce additional algebraic elements in accord with the inseparable relations

$$y_1^p = t_1 + t_2 t_3^p, \qquad y_2^p = t_2 + t_3 t_4^p, \qquad y_3^p = t_3 + t_4 t_5^p, \cdots. \tag{27}$$

Our example is the field $K = P(t_1, t_2, \cdots; y_1, y_2, \cdots)$. Since the y's are pth roots, the t's clearly cannot form a separating transcendence basis. One might try to invert the equations (27) to define everything in terms of the basis $t_1, t_2, y_1, y_2, y_3, \cdots$, but that still leaves the pth roots such as $t_3^p = (y_1^p - t_1)/t_2$. It can be shown that no method of picking a transcendence basis for K over P will yield a basis which is separating, and this example is but a taste of the trouble possible (*cf.* [**15**], [**16**]).

12. p-Algebras. The relevance of the study of inseparable extensions to other algebraic questions is clearly illustrated by the p-algebras, which are defined* as linear algebras over a field F of characteristic p which have as degree some power of the characteristic. The theory of these algebras, which culminates in the theorem that every such algebra is "similar" to a cyclic algebra, depends essentially on the construction of inseparable fields contained in the algebra (in technical parlance, every p-algebra has a purely inseparable splitting field). To illustrate this, choose as the base field the field $P(t)$ of all rational functions of t with coefficients in a perfect field P. Introduce a pth root u, with $u^p = t$, and a a quantity v with $v^p = v + t$. The set of all sums

$$w = \sum_{i,j} a_{ij} u^i v^j, \qquad (i = 0, \cdots, p-1;\ j = 0, \cdots, p-1;\ a_{ij} \text{ in } F),$$

then forms a linear algebra of degree p over F, if one uses the multiplication table

$$u^p = t, \qquad v^p = v + t, \qquad vu = u(v+1).$$

The essential point for the theory is that this algebra contains both the inseparable extension $F(u)$ and the cyclic separable extension $F(v)$ of the base field F.

There are many further ways in which modular fields can arise in other algebraic investigations. We mention here only the use of fields of characteristic 2 in discussing Boolean algebras (Stone [**24**]), the theory of matrices over a modular field (Albert [**5**]), the definition of modular fields by special polynomials (Carlitz [**7**]), and the quasi-algebraic closure of finite fields (Chevalley [**8**]).

* *Cf.* Albert [**2**, ch. 7]; and also Jacobson [**11**], Teichmüller [**25**].

13. Summary. Modular fields include finite fields, Galois extensions of fields, algebraic function fields, and fields for algebraic manifolds, as well as for more bizarre types. The study of such fields is suggested by their origin in arithmetic questions about congruences, p-adic numbers, and ideal theory. On the other hand, an independent survey of their structure is indicated by the program of abstract algebra: first the development of the abstract concept ("field") in order to cover the variegated known examples, then the derivation of general theorems touching this concept, and lastly a classification of the types of systems which fall under the concept. We have seen that the straightforward generalization of the known properties of number fields is but one phase of our structure theory. There is also the investigation of characteristic new phenomena, of inseparability, of p-independence and the like, which distinguish the modular fields from the non-modular. The presence of curious examples of fields, which must at present still be given individual treatment, indicates that the present situation abounds in new questions, and that abstract algebra can very well give rise to concrete conundrums.

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PROPER CONTINUED FRACTIONS

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This paper generalizes the so-called "regular" continued fraction expansion of a real number. The treatment includes as a special case the "continued cotangent" expansion of Lehmer [2].

1. The expansion of a real number into a proper continued fraction. Let y_0 be any real number and $a_1, a_2, a_3, \cdots$ an arbitrary sequence of positive integers. If y_0 is an integer we shall say that its expansion into a *proper* continued fraction *terminates* and is given by

$$y_0 \sim b_0,$$

where $b_0 = y_0$. If y_0 is not an integer, let b_0 be the greatest integer $\leq y_0$ (in symbols $b_0 = [y_0]$) and define real numbers $y_1, y_2, y_3, \cdots$ and positive integers $b_1, b_2, b_3, \cdots$ by the relations

$$y_n = \frac{a_n}{y_{n-1} - b_{n-1}}, \qquad b_n = [y_n], \tag{1.1}$$

successively for $n = 1, 2, 3, \cdots$. It is clear from (1.1) that each $b_n \geq a_n$, $(n = 1, 2, 3, \cdots)$. If, eventually, some y_n, say y_k, is itself an integer, we shall say the expansion terminates, and that the proper expansion of y_0 into a continued fraction is given by

$$y_0 \sim b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_{k-1}}{b_{k-1}} + \frac{a_k}{b_k}, \qquad (b_k = y_k). \tag{1.2}$$

We note that $b_k > a_k$. If no y_n is an integer, the expansion will not terminate and the proper continued fraction expansion of y_0 will be given by

$$y_0 \sim b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots. \tag{1.3}$$

It is clear from (1.1) that a simple induction proves that

$$y_0 = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{y_n}, \qquad (n = 1, 2, 3, \cdots). \tag{1.4}$$

As usual we shall define the nth approximant A_n/B_n of (1.2), for $n \leqq k$, and of (1.3) by the recursion relations

$$\begin{aligned} A_0 &= b_0, & A_1 &= b_0 b_1 + a_1, & A_n &= b_n A_{n-1} + a_n A_{n-2}, \\ B_0 &= 1, & B_1 &= b_1, & B_n &= b_n B_{n-1} + a_n B_{n-2}, \quad (n = 2, 3, 4, \cdots). \end{aligned} \tag{1.5}$$

THEOREM 1.1. *A terminating expansion is the rational number y_0 which generated it.*

This theorem follows at once from (1.4).

THEOREM 1.2. *If y_0 is rational, its expansion terminates.*

The theorem follows from the definition, if y_0 is an integer. If y_0 is not an integer, write it as p/q, where p and $q>0$ are relatively prime integers. We observe that y_1 is either an integer >0 (in which case the expansion terminates) or it is a positive rational number p_1/q_1, where

$$p_1 = a_1 q, \qquad q_1 = p - b_0 q.$$

Since b_0 is the largest integer $<p/q$, we have

$$0 < p - b_0 q < q.$$

Continue the process. Each $y_n = p_n/q_n$, where p_n and q_n are positive integers and

$$q > q_1 > q_2 > \cdots > 0.$$

Some q_n must therefore be unity and the corresponding y_n an integer. The proof is complete.

We turn now to a consideration of the non-terminating proper continued fractions.

THEOREM 1.3. *Every proper non-terminating continued fraction converges to the real number y_0 which generated it.*

To prove the theorem we observe that by (1.4) and (1.5) we have

$$y_0 = \frac{y_n A_{n-1} + a_n A_{n-2}}{y_n B_{n-1} + a_n B_{n-2}}. \tag{1.6}$$

It will be sufficient to show that

$$\lim_{n\to\infty} \left(y_0 - \frac{A_{n-2}}{B_{n-2}} \right) = 0.$$

But, as is easily seen,

$$y_0 - \frac{A_{n-2}}{B_{n-2}} = (-1)^n \frac{a_1 a_2 \cdots a_{n-2}}{B_{n-2}} \cdot \frac{a_{n-1} y_n}{y_n B_{n-1} + a_n B_{n-2}}, \tag{1.7}$$

where we have used the well known formula

$$A_{n-1}B_{n-2} - A_{n-2}B_{n-1} = (-1)^n a_1 a_2 \cdots a_{n-1}.$$

Using the fact that $b_n \geqq a_n$ one can show readily by induction that the first fraction in the right-hand member of (1.7) remains less than one in absolute value. To prove that the second fraction tends to zero we write it in the form

$$\frac{a_{n-1}}{B_{n-1} + \dfrac{a_n}{y_n} B_{n-2}}.$$

Since every term in the expression is positive, it is sufficient to prove that $\lim a_{n-1}/B_{n-1} = 0$. The recursion relations (1.5) insure that the integers B_n are strictly increasing. If the integers a_n remain bounded, the proof is immediate; otherwise, $\lim a_1 a_2 \cdots a_{n-1} = +\infty$ and the proof is completed by observing that

$$\frac{a_{n-1}}{B_{n-1}} < \frac{a_{n-1}}{a_1 a_2 \cdots a_{n-2} a_{n-1}}.$$

THEOREM 1.4. *The even approximants A_{2r}/B_{2r} in the proper expansion of y_0 are less than y_0 and the odd approximants A_{2r+1}/B_{2r+1} are greater than y_0.*

The proof follows at once from (1.7) by replacing n by $2r+2$ first, and then by $2r+3$.

We proceed with the following result.

THEOREM 1.5. *Each approximant of the proper continued fraction is a closer approximation to y_0 than the preceding.*

To prove the theorem we use (1.6) and observe that

$$y_0 - \frac{A_{n-1}}{B_{n-1}} = \frac{-a_n}{y_n} \frac{B_{n-2}}{B_{n-1}} \left(y_0 - \frac{A_{n-2}}{B_{n-2}} \right). \tag{1.8}$$

The coefficient of $y_0 - A_{n-2}/B_{n-2}$ in the last member is absolutely less than unity since $y_n \geqq b_n \geqq a_n$ and the continued equality is impossible.

2. Conditions that a given continued fraction be proper. In this section we establish necessary and sufficient conditions that a continued fraction

$$b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots, \tag{2.1}$$

where the numbers b_n and a_n, $(n = 1, 2, 3, \cdots)$, are positive integers, be a proper expansion of some real number y_0.

From (1.1) we see that a necessary condition for this is that $b_n \geqq a_n$, $(n=1, 2, 3, \cdots)$.

THEOREM 2.1. *If* (2.1) *is non-terminating, a necessary and sufficient condition that* (2.1) *be a proper expansion of a real number* y_0 *into a continued fraction is that*

$$b_n \geqq a_n, \qquad (n = 1, 2, 3, \cdots). \tag{2.2}$$

If (2.1) *terminates, a necessary and sufficient condition that* (2.1) *be proper is that conditions* (2.2) *hold and in addition that the final* b *be greater than the final* a.

It remains to prove the sufficiency of the conditions. Suppose first that (2.1) is non-terminating. If A_n/B_n is the nth approximant of (2.1), we have [**1**, pp. 16, 17]

$$\begin{aligned} \frac{A_{n+1}}{B_{n+1}} - \frac{A_n}{B_n} &= (-1)^n \frac{a_1 a_2 \cdots a_{n+1}}{B_n B_{n+1}}, \\ \frac{A_{n+1}}{B_{n+1}} - \frac{A_{n-1}}{B_{n-1}} &= (-1)^{n-1} \frac{a_1 a_2 \cdots a_n b_{n+1}}{B_{n-1} B_{n+1}}. \end{aligned} \tag{2.3}$$

It follows that

$$\frac{A_0}{B_0} < \frac{A_2}{B_2} < \frac{A_4}{B_4} < \cdots,$$

$$\frac{A_1}{B_1} > \frac{A_3}{B_3} > \frac{A_5}{B_5} > \cdots > b_0.$$

Thus

$$\lim_{n\to\infty} \frac{A_{2n+1}}{B_{2n+1}} = \lambda \geqq b_0$$

and

$$\lim_{n\to\infty} \frac{A_{2n}}{B_{2n}} = \mu,$$

where λ is finite and μ may possibly be infinite. However, by the first line of (2.3) and an argument strictly analogous to that used in the proof of Theorem 1.3,

$$\lim_{n\to\infty} \left(\frac{A_{2n+1}}{B_{2n+1}} - \frac{A_{2n}}{B_{2n}} \right) = 0,$$

and hence $\mu = \lambda$ (finite). The continued fraction (2.1) subject to (2.2) therefore converges. It is clear, further, that

$$y_0 > b_0,$$

where $y_0 = \mu = \lambda$. Similarly, the continued fractions

$$b_k + \frac{a_{k+1}}{b_{k+1}} + \cdots, \qquad (k = 1, 2, 3, \cdots),$$

converge to numbers y_k in similar fashion and

$$y_k > b_k, \qquad (k = 1, 2, 3, \cdots).$$

Let the proper expansion of y_0 associated with the set of positive integers $a_1, a_2, a_3, \cdots$ of (2.1) be

$$\beta_0 + \frac{a_1}{\beta_1} + \frac{a_2}{\beta_2} + \cdots.$$

We shall show that $\beta_n = b_n$, $(n = 0, 1, 2, \cdots)$.

To that end we note that by a well known principle [1, p. 22]

$$y_0 = b_0 + \frac{a_1}{y_1}.$$

Since $y_1 > b_1 \geqq a_1$, it follows that $b_0 = [y_0]$. Hence $\beta_0 = b_0$. Similarly, $y_2 > b_2 \geqq a_2$ and

$$y_1 = b_1 + \frac{a_2}{y_2}.$$

Hence $b_1 = [y_1] = \beta_1$. A simple induction completes the proof of the theorem for the case that (2.1) does not terminate. The proof for the case that (2.1) terminates is immediate. Thus the theorem is proved.

Conditions (2.2) suggest the term "proper" for continued fraction expansions of the type considered in this paper.

If one sets $a_1 = a_2 = a_3 = \cdots = 1$, the proper expansion of y_0 becomes the usual regular continued fraction. The conditions of Theorem 2.1 become familiar ones.

If we set

$$a_n = r_{n-1}^2 + 1, \qquad (n = 1, 2, 3, \cdots),$$

where the sequence $r_0, r_1, r_2, \cdots$ is defined recursively by the relations

$$r_0 = b_0,$$

$$r_n - r_{n-1} = b_n, \qquad (n = 1, 2, 3, \cdots),$$

the proper continued fraction expansion reduces to the "continued cotangent" expansion of Lehmer [2]. The conditions of Theorem 2.1 are then precisely his regularity conditions.

3. Rapidity of convergence of a proper continued fraction. In this section we shall investigate the question of the rapidity of convergence of a proper continued fraction. It is natural to consider the differences

$$\Delta_n = \left| \frac{A_n}{B_n} - \frac{A_{n-1}}{B_{n-1}} \right| = \frac{a_1 a_2 \cdots a_n}{B_{n-1} B_n}, \qquad (n = 1, 2, 3, \cdots),$$

inasmuch as the number y_0 lies between each two successive approximants, and, as is well known,

$$\begin{aligned} y_0 &= \frac{A_0}{B_0} + \left(\frac{A_1}{B_1} - \frac{A_0}{B_0}\right) + \left(\frac{A_2}{B_2} - \frac{A_1}{B_1}\right) + \cdots \\ &= b_0 + \sum_{1} (-1)^{n-1} \Delta_n . \end{aligned}$$

We note that

$$\frac{\Delta_{n+1}}{\Delta_n} = a_{n+1} \frac{B_{n-1}}{B_{n+1}} = \frac{1}{1 + \dfrac{b_{n+1} B_n}{a_{n+1} B_{n-1}}} < \frac{1}{1 + a_n}$$

inasmuch as

$$B_n > b_n B_{n-1}, \qquad b_n \geqq a_n .$$

It is clear that Δ_{n+1}/Δ_n can be made arbitrarily small by choosing the integer a_n sufficiently large. Further, every proper continued fraction converges more rapidly than a geometric series with ratio 1/2.

4. Examples. It is clear that all the examples of regular continued fractions and those of Lehmer's "continued cotangent" now become examples of this theory. A few others are given below. Each is a periodic representation of a quadratic surd. The general question of periodic proper continued fractions will be discussed in a later paper.

The expansion of $\sqrt{3}$ corresponding to the set of partial numerators $a_{3n+1} = 1$, $a_{3n+2} = 2$, $a_{3n+3} = 3$, $(n = 0, 1, 2, \cdots)$, is given by

$$\sqrt{3} = 1 + \frac{1}{1+} \frac{2}{5+} \frac{\dot{3}}{6+} \frac{1}{2+} \frac{2}{12+} \frac{3}{3+} \frac{1}{4+} \frac{\dot{2}}{6+} \frac{3}{6+} \cdots .$$

The period in a periodic proper continued fraction may be longer or shorter than that of the regular continued fraction for a given quadratic surd, as the following examples demonstrate. First

$$\begin{aligned} \frac{\sqrt{5}+1}{2} &= 1 + \frac{1}{1+} \frac{1}{1+} \frac{1}{1+} \cdots & (a = 1), \\ &= 1 + \frac{2}{3+} \frac{2}{8+} \frac{2}{4+} \frac{2}{8+} \frac{2}{4+} \cdots & (a = 2), \end{aligned}$$

while on the other hand

$$\frac{\sqrt{17}+3}{2} = 3 + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \cdots \qquad (a = 2),$$

$$= 3 + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{1} + \frac{1}{3} + \cdots \qquad (a = 1).$$

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ON THE DIOPHANTINE EQUATION $x(x+1)\cdots(x+n-1)=y^k$*

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Let the product of n consecutive integers be

$$P_n = x(x+1)\cdots(x+n-1), \qquad x > 0, \qquad n > 1.$$

The equation

$$(1) \qquad P_n = y^k, \qquad y \text{ and } k \text{ integral}, \qquad k > 1,$$

has been studied by many mathematicians. Their results give particular values of n and k for which (1) is impossible, or restrict the conditions under which (1) may occur.

One of the earliest and most important results is the following theorem of Liouville, published in 1857 [1]:

THEOREM L. *If at least one of the integers $x, x+1, \cdots, x+n-1$ is prime, or if $x<n+5$, then $P_n\neq y^k$.*

The demonstration depends upon Bertrand's postulate (first proved by Tschebyscheff [2]) that between x and $2x-2$, $x>3$, there is always a prime.

Between this date and 1920 numerous papers were published [3], the results of which may be summarized as follows: $P_2\neq y^k$; $P_3\neq y^k$; $P_k\neq y^k$; $P_n\neq y^2$ if $n\leqq 203$; $P_n\neq y^3$ if $n=4$, 5, 6, or 9. It will be noted that none of these theorems, except that $P_k\neq y^k$, gives any information about the equation $P_n=y^k$, $n>3$, for $k>3$, except for those cases settled by the fact that if P_n is not a kth power, it cannot be an mkth power.

A paper published by R. Oblàth in 1933 [4] considerably extends these results. He shows that $P_4=y^k$ is impossible for a certain infinite set of prime values of k, which includes all primes $\leqq 107$ except 31, 59, 89; that $P_n\neq y^k$, $5\leqq n\leqq 13$,

* Part of a thesis submitted to the faculty of the University of Colorado, 1938, in partial fulfillment of the requirements for the Ph.D. degree.

Read before the Rocky Mountain Section of the Mathematical Association of America, April, 1939.

$k=3$ or $k=5$; and that $P_5=y^7$ and $P_6=y^7$ are impossible for x less than certain very large numbers. Oblàth proves also the following extensions of Theorem L: if one of $x, x+1, \cdots, x+n-1$ is ap, p prime, $a\leqq 8$, or if $x<n^2$, then $P_n\neq y^k$. The last mentioned result is a consequence of the following theorem of Sylvester [5] and Schur [6]:

THEOREM S. *At least one of the n consecutive integers $x, x+1, \cdots, x+n-1$, $x\geqq n$, is divisible by a prime $>n$.*

The theorem of Liouville and its extensions by Oblàth state certain conditions on x, n, and k under which $P_n=y^k$ cannot hold. This paper contributes a group of more strongly restrictive theorems (I–IV below) of a similar type, and proves that $P_{2k}\neq y^k$, and that $P_n=y^k$ has at most one solution for $n=4, 5, 6$, or 7, k a prime $\geqq 7$.

THEOREM I. *If $x\leqq(n+1)^k-n$, then $P_n\neq y^k$.*

Proof. Suppose $P_n=y^k$. By Theorem L, $x>n$. Then, by Theorem S, P_n is divisible by a prime $p>n$. Only one of the n consecutive integers $x+i$, $(i=0, 1, \cdots, n-1)$, can be divisible by a number $p>n$, and if $P_n=y^k$, then p^k must divide this one. Hence

$$x+n-1\geqq p^k\geqq(n+1)^k,$$

or

$$x>(n+1)^k-n,$$

from which the theorem follows.

COROLLARY. *If $x\leqq 5^k-4$, then $P_n\neq y^k$.*

For $P_n=y^k$, $n\geqq 4$. Hence, by Theorem I, $x>5^k-4$.

THEOREM II. *If any one of $x+i$, $(i=0, 1, \cdots, n-1)$, is divisible by a product of distinct primes $p_1, p_2, \cdots, p_\mu$, each of which is greater than n, and if, moreover, $x\leqq(n+1)^{k\mu}-n$, then $P_n\neq y^k$.*

Proof. Suppose that $P_n=y^k$ and that one of the $x+i$ is divisible by $p_1p_2\cdots p_\mu$, where $p_r>n$, $(r=1, 2, \cdots, \mu)$. Only one of the n consecutive integers $x+i$ can be divisible by p_r; hence $p_1^kp_2^k\cdots p_\mu^k$ must divide this one. Then we have in succession,

$$\begin{aligned} x+n-1&\geqq p_1^kp_2^k\cdots p_\mu^k\geqq(n+1)^{k\mu},\\ x&\geqq(n+1)^{k\mu}-n+1,\\ x&>(n+1)^{k\mu}-n. \end{aligned}$$

THEOREM III. *If any one of $x+i$, $(i=0, 1, \cdots, n-1)$, is expressible as a product of powers of μ distinct primes, $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\mu^{\alpha_\mu}$, with $\sum_{r=1}^{\mu}\alpha_r<k$, then $P_n\neq y^k$.*

Proof. Suppose $P_n=y^k$. Suppose one of $x+i$, where $i=0, 1, \cdots, n-1$, is $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\mu^{\alpha_\mu}$, where p_r is prime and $\alpha_r>0$ for $r=1, 2, \cdots, \mu$. Then either

(a) at least one $\alpha_r\geqq k$, and then $\sum_{r=1}^{\mu}\alpha_r\geqq k$, or

(b) every $\alpha_r<k$.

Now p_r^k must divide P_n; hence if $\alpha_r<k$, p_r must divide another of the $x+i$, and this is possible only if $p_r<n$. Hence if every $\alpha_r<k$, then every $p_r<n$, and

$$x \leqq p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\mu^{\alpha_\mu} \leqq (n-1)^S, \qquad \text{where} \qquad S=\sum_{r=1}^{\mu}\alpha_r.$$

But by Theorem I, $x>(n+1)^k-n$. Hence

$$(n-1)^S > (n+1)^k - n. \tag{2}$$

Now

$$\begin{aligned}(n+1)^k - n &= [(n-1)+2]^k - n\\ &= (n-1)^k + k(n-1)^{k-1}\cdot 2 + \cdots + 2^k - n\\ &\geqq (n-1)^k + 4(n-1) + 2^k - n \qquad \text{(since } k \geqq 2)\\ &> (n-1)^k + 2^k \qquad \text{(since } n > 4/3)\\ &> (n-1)^\beta \text{ for } \beta \leqq k.\end{aligned}$$

Hence (2) can be true only for $S>k$. This completes the proof.

For the conditions of part (b), it is seen that the following slightly stronger theorem has been proved:

THEOREM III A. *If any one of $x+i$, $(i=0, 1, \cdots, n-1)$, is expressible as a product of powers of distinct primes, $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_\mu^{\alpha_\mu}$, with every $\alpha_r<k$ and $\sum_{r=1}^{\mu}\alpha_r<k+1$, then $P_n\neq y^k$.*

It may be of interest to examine a few special cases of Theorems III and III A. For $\mu=1$, $\alpha_1=1$, we have the original Liouville Theorem L. If any one of the consecutive integers in P_n is a power of a prime, then P_n is not a higher power of an integer. If any one of the consecutive integers is a product of μ distinct primes, then P_n is not a μth or higher power.

THEOREM IV. *If one of $x+i$, $(i=0, 1, \cdots, n-1)$, is a^k, $a<p$, where p is the greatest prime which divides P_n, then $P_n\neq y^k$.*

Proof. Suppose $P_n=y^k$. Let p be the largest prime which divides P_n. By Theorem S, $p>n$. Then one of the $x+i$ must be mp^k. Suppose another of them is a^k, $a<p$. Then $p\geqq a+1$, and

$$\begin{aligned}n > mp^k - a^k &\geqq m(a+1)^k - a^k\\ &= (m-1)a^k + m(ka^{k-1}+\cdots+1) > ka^{k-1}.\end{aligned}$$

Then $a<(n/k)^{1/(k-1)}$ and

$$x \leqq a^k < (n/k)^{k/(k-1)}. \tag{3}$$

But by Theorem I,

$$x > (n+1)^k - n, \quad \text{and}$$

$$(n+1)^k - n = n^k + kn^{k-1} + \cdots + (k-1)n + 1 \tag{4}$$
$$> n > (n/k)^k > (n/k)^{k/(k-1)}.$$

Hence (3) and (4) can never both hold, and the theorem is proved.

THEOREM V. *The equation $P_{2k}=y^k$ is impossible.*

Proof. Let

$$\begin{aligned} P_{2k} &= x(x+1)\cdots(x+2k-1) \\ &= \prod_{i=1}^{k}\left[x + (2k-1)x + (i-1)(2k-i)\right] \\ &= \prod_{i=1}^{k}(z+\alpha_i) \end{aligned}$$

where $z=x^2+(2k-1)x$ and $\alpha_i=(i-1)(2k-i)$, $(i=1, 2, \cdots, k)$. Then

$$P_{2k} = z^k + c_1 z^{k-1} + c_2 z^{k-2} + \cdots + c_r z^{k-r} + \cdots + c_k = F(z),$$

where $c_r=\sum\alpha_1\alpha_2\cdots\alpha_r$, $(r=1, 2, \cdots, k)$, are the elementary symmetric functions of the α's.

Let A, G, H be respectively the arithmetic, geometric, and harmonic means of the $z+\alpha_i$. Now $G=\sqrt[k]{F(z)}=\sqrt[k]{P_{2k}}$. Hence to prove that G is not integral is to prove the theorem. But $A>G>H$. Then if

$$A - H \leqq 1, \tag{5}$$

there is at most one possible integral value for G. Now

$$A = \frac{\sum_{i=1}^{k}(z+\alpha_i)}{k} = z + \frac{\sum_{i=1}^{k}\alpha_i}{k} = z + \frac{c_1}{k}.$$

and

$$\begin{aligned} H &= \frac{k}{\sum_{i=1}^{k} 1/(z+\alpha_i)} = \frac{k\prod_{i=1}^{k}(z+\alpha_i)}{\sum (z+\alpha_1)\cdots(z+\alpha_{k-1})} \\ &= \frac{kF(z)}{F'(z)} = z + \frac{c_1}{k} - \frac{\phi(z)}{F'(z)}, \end{aligned}$$

where

$$\phi(z) = \sum_{r=2}^{k}\left[(k-r+1)\frac{c_1c_{r-1}}{k} - rc_r\right]z^{k-r}.$$

Then $A-H=\phi(z)/F'(z)$, and (5) holds if

$$F'(z) - \phi(z) \geqq 0. \tag{6}$$

Computing coefficients and combining terms in (6) gives the condition in the form

$$a_1z^{k-1} + a_2z^{k-2} + \cdots + a_{k-r}z^r + \cdots + a_{k-1}z + a_k \geqq 0, \tag{7}$$

where

$$a_1 = k, \qquad a_r = (k-r+1)c_{r-1}(1-c_1/k) + rc_r, \qquad (r = 2, \cdots, k).$$

In order to get an expression for an upper bound for the zeros of the polynomial of (7), it will next be shown that a_k is the negative coefficient of greatest absolute value in (7). Remembering that $\alpha_1=0$, we have

$$c_k = \alpha_1\alpha_2\cdots\alpha_k = 0,$$
$$c_{k-1} = \alpha_2\cdots\alpha_k = (2k-2)!,$$

and hence

$$a_k = (2k-2)!(1-c_1/k).$$

Now $c_1=\sum\alpha_1\geqq k$ and hence $(1-c_1/k)\leqq 0$; therefore $a_k\leqq 0$. In the expression for a_r, the second term, rc_r, is never negative and the first term, which has always the factor $(1-c_1/k)$, is never positive. Hence a_k will be the negative a_r of greatest numerical value if

$$(k-r+1)c_{r-1} \leqq (2k-2)!, \qquad (r = 2, \cdots, k-1).$$

Now $c_{r-1}=\sum\alpha_1\alpha_2\cdots\alpha_{r-1}$ has $(k-1)!/[(r-1)!(k-r)!]$ terms, since there are $(k-1)$ non-zero α's; since $\alpha_1<\alpha_2<\cdots<\alpha_k$, the largest of these terms is $\alpha_{k-r+2}\cdots\alpha_k=(k+r-2)!/(k-r)!$. Therefore

$$\begin{aligned}(k-r+1)c_{r-1} &\leqq (k-r+1)\frac{(k-1)!}{(r-1)!(k-r)!}\cdot\frac{(k+r-2)!}{(k-r)!}\\ &= (k-r+1)\frac{r(r+1)\cdots(k-2)(k-1)}{[(k-r)!]^2}\\ &\quad\cdot\frac{(2k-2)!}{(k+r-1)(k+r)\cdots(2k-2)}\\ &\leqq \frac{(k-r+1)}{2[(k-r)!]^2}(2k-2)!\\ &\leqq (2k-2)!, \text{ as was to be shown.}\end{aligned}$$

It follows that $|a_k|/a_1+1$ is an upper bound for the zeros of the polynomial of (7).

Our next step is to show that this upper bound is less than $(2k)^{2k-2}$. Now

$$\frac{|a_k|}{a_1}+1=\frac{(2k-2)!}{k}\left(\frac{c_1}{k}-1\right)+1.$$

Moreover $[(2k-2)!]^{1/(2k-2)}$ is the geometric mean of the integers $1, 2, \cdots, 2k-2$, and their arithmetic mean is $\frac{1}{2}(2k-1)$. Hence

$$(2k-2)!<[\tfrac{1}{2}(2k-1)]^{2k-2}.$$

Since the largest α_i is $\alpha_k=(k-1)k$, we see that

$$c_1/k=\sum\alpha_1/k\leqq k(k-1),$$

$$c_1/k-1\leqq k^2-k-1.$$

Hence

$$\begin{aligned}\frac{|a_k|}{a_1}+1&<\frac{1}{k}\left(\frac{2k-1}{2}\right)^{2k-2}(k^2-k-1)+1\\&=(2k-1)^{2k-2}\frac{(k^2-k-1)}{2^{2k-2}k}+1\\&<(2k-1)^{2k-2}<(2k)^{2k-2}.\end{aligned}$$

It follows that (7) holds and hence (5) holds if $z\geqq(2k)^{2k-2}$. Now

$$A=z+\sum\alpha_1/k,$$

$$\begin{aligned}\sum\alpha_1&=\sum_{i=1}^{k}(i-1)(2k-i)=-\sum_{i=1}^{k}i^2+(2k+1)\sum_{i=1}^{k}i-2k^2\\&=-k(k+1)(2k+1)/6+(2k+1)k(k+1)/2-2k^2\\&=k(k-1)(2k-1)/3.\end{aligned}$$

Then

$$A=z+(k-1)(2k-1)/3.$$

If $k\equiv1$ or $2 \bmod 3$, $(k-1)(2k-1)\equiv0 \bmod 3$ and A is integral. Hence in this case G is not integral if $z\geqq(2k)^{2k-2}$. If $k\equiv0 \bmod 3$, $(k-1)(2k-1)\equiv1 \bmod 3$, and the only integer between A and H is

$$\begin{aligned}A-1/3&=z+(k-1)(2k-1)/3-1/3\\&=z+(2k^2-3k)/3.\end{aligned}$$

Hence, in this case, G is integral only if

$$[z+(2k^2-3k)/3]^k-F(z)=0.$$

The constant term of this equation is $[(2k^23-k)/3]^k$, since $F(z)$ has no constant

term, and all of its coefficients are integers. Hence for the last equation there is certainly no integral root greater than $[(2k^2-3k)/3]^k$. But

$$\left(\frac{2k^2-3k}{3}\right)^k < \frac{(2k^2)^k}{3^k} = \frac{(2k)^{2k}}{2^k 3^k} < \frac{(2k)^{2k}}{(2k)^2} = (2k)^{2k-2}.$$

Therefore G is not integral, which is to say P_{2k} is not a kth power, if

$$z \geqq (2k)^{2k-2}. \tag{8}$$

But by Theorem I, P_n is not a kth power if $x \leqq (n+1)^k - n$. Then *a fortiori* P_n is not a kth power if $x \leqq n^k$, and P_{2k} is not a kth power if $x \leqq (2k)^k$, or

$$z \leqq (2k)^{2k} + (2k-1)(2k)^k. \tag{9}$$

Always, at least one of the inequalities (8) and (9) holds. Therefore P_{2k} is never a kth power.

THEOREM VI. *The equation $P_n = y^k$, for $n=4$, 5, 6, or 7, k prime, $k \geqq 7$, has at most one solution.*

Proof. Suppose $P_n = y^k$. Any common divisor of two of the $x+i$, $(i=0, 1, \cdots, n-1)$, must divide their difference and hence be less than n. Any divisor of only one of the $x+i$ must appear in that one to a power mk, $m \geqq 1$. Therefore every $x+i$ can be written in the form

$$x + i = 2^{\alpha_i} 3^{\beta_i} \cdots p^{\nu_i} x_i^k, \tag{10}$$

where 2, 3, $\cdots$, p are all the primes less than n with

$$0 \leqq \alpha_i \leqq k-1,\ 0 \leqq \beta_i \leqq k-1, \cdots,\ 0 \leqq \nu_i \leqq k-1,$$

and

$$\sum_{i=0}^{n-1} \alpha_i \equiv \sum_{i=0}^{n-1} \beta_i \equiv \cdots \equiv \sum_{i=0}^{n-1} \nu_i \equiv 0 \bmod k. \tag{11}$$

(A) Let $n=4$. Then, since the only primes less than 4 are 2 and 3, (10) becomes

$$x + i = 2^{\alpha_i} 3^{\beta_i} x_i^k, \qquad (i = 0, 1, 2, 3).$$

Two of the α_i are 0, say $\alpha_{r+2} = \alpha_r = 0$, with $r=0$ or 1. Since of two consecutive even integers one is divisible by 2 and one by a higher power of 2, one of the other two α's is 1 and, by (11), the other must be $k-1$. For the β's, either

(a) all are 0, since if only one of the $x+i$ is divisible by 3, it must be divisible by 3^{mk}, or

(b) $\beta_1 = \beta_2 = 0$; since of two consecutive multiples of 3 at least one is divisible by 3 but not by 3^2, one of the other β's is 1, and then by (11), the other is $k-1$. If (a) holds, we would have $x_{r+2}^k - x_r^k = 2$, $(r=0$ or $1)$, which is impossible. If

(b) holds, it is possible to choose a pair of consecutive integers such that their difference gives an equation of the form

$$ax_{r+1}^k - bx_r^k = 1, \quad \text{with} \quad |ab| \geqq 2\cdot 3^{k-1}, \tag{12}$$

and x_{r+1} and x_r relatively prime, since they are factors of consecutive integers.

(B) Let $n=5$. Again

$$x + i = 2^{\alpha_i}3^{\beta_i}x_i^k, \qquad (i = 0, 1, \cdots, 4).$$

(1) For the α's, either
  (a) two are 0 and the others 2, 1, and $k-3$, or else 1, $k-2$, and 1; or
  (b) three are 0 and the others 1 and $k-1$.
(2) For the β's, either
  (a) all are 0; or
  (b) three are 0 and the others 1 and $k-1$.

If (2a) holds, we would have $x_{r+2}^k - x_r^k = 2$, $(r=0$ or $1)$, which is impossible. If (2b) holds, it is again possible to choose a pair of consecutive integers of which the difference gives an equation of the form (12).

(C) Let $n=6$. Then (10) becomes

$$x + i = 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}x_i^k, \qquad (i = 0, \cdots, 5).$$

Three of the α's are 0 and the other three are 1, 1, $k-2$, or 1, 2, $k-3$. Four of the β's are 0 and the other two are 1 and $k-1$. For the γ's, either
  (a) all are 0; or
  (b) $\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$ and the others are 1 and $k-1$.

One of the $x+i$ is divisible by 6; hence only four are divisible by 2 or 3. If (a) holds we would have $x_r^k - x_s^k < 6$, with $x_s > 5$, by Theorem IV, which is impossible. If (b) holds, it is possible to choose a pair of consecutive integers such that their difference gives

$$ax_{r+1}^k - bx_r^k = 1, \quad \text{with} \quad |ab| \geqq 2\cdot 5^{k-1}. \tag{13}$$

(D) Let $n=7$. Then (10) becomes

$$x + i = 2^{\alpha_i}3^{\beta_i}5^{\gamma_i}x_i^k, \qquad (i = 0, \cdots, 6).$$

(1) Either
  (a) $\alpha_0=\alpha_2=\alpha_4=\alpha_6=0$, and the other three α's are 1, 1, and $k-2$ or 1, 2, and $k-3$; or
  (b) $\alpha_1=\alpha_3=\alpha_5=0$, and the other four are 1, 1, 2, $k-4$.
(2) Either
  (a) four of the β's are 0 and the other three are 1, 1, $k-2$; or
  (b) five are 0 and the others 1 and $k-1$.
(3) For the γ's, either
  (a) all are 0; or
  (b) five are 0 and the others 1 and $k-1$.

Either one or two of the $x+i$ are divisible by 6. In the case when four of the $x+i$ are divisible by 2 and three by 3, both x and $x+6$ must be divisible by both 2 and 3, and hence two of the $x+i$ are divisible by neither. Any other combination from (1) and (2) gives at most six numbers divisible by 2 or 3 even if they are all distinct. But since 6 must divide at least one, they cannot be all distinct. Therefore at most five of the $x+i$ are divisible by 2 or 3. Hence if (3a) holds, we would have

$$x_r^k - x_s^k < 7,$$

with $x_s > 7$ by Theorem IV, which is impossible. If (3b) holds, a pair of consecutive integers can again be chosen such that their difference gives (13). Hence for $n=4$, 5, 6, or 7, if $P_n = y^k$, an equation of the form

$$ax_{r+1}^k - bx_r^k = 1, \qquad |ab| \geqq 2\cdot 3^{k-1}, \tag{14}$$

with x_{r+1}, x_r relatively prime, must be satisfied.

C. L. Siegel gives the following theorem [7]:

THEOREM. *The equation* $|ax^n - by^n| \leqq c$, a, b, c *integral*, $c>0$, $n\geqq 3$, *has at most one solution in relatively prime positive integers if*

$$|ab|^{n/2-1} \geqq \lambda_n c^{2n-2}, \tag{15}$$

where

$$\lambda_n = 4\left(n\prod_{p|n} p^{1/(p-1)}\right)^n, \tag{16}$$

where p runs over all prime divisors of n.

In (14), suppose k prime, $k\geqq 3$. Applying Siegel's theorem, (16) becomes

$$\lambda_k = 4(k\cdot k^{1/(k-1)})^k = 4k^{k^2/(k-1)},$$

and (15) becomes

$$|ab|^{k/2-1} \geqq 4k^{k^2/(k-1)}.$$

Then (14) has at most one solution if

$$(2\cdot 3^{k-1})^{k/2-1} \geqq 4k^{k^2/(k-1)},$$

or, in succession,

$$2^{(k-6)/2}3^{(k-1)(k-2)/2} \geqq k^{k^2/(k-1)},$$

$$2^{(k-6)(k-1)/2}3^{(k-1)^2(k-2)/k^2} \geqq k^2,$$

$$\frac{(k-6)(k-1)}{k^2}\log 2 + \frac{(k-1)^2(k-2)}{k^2}\log 3 \geqq 2\log k. \tag{17}$$

Let the function on the left of (17) be designated by $f_1(k)$, the one on the right by $f_2(k)$. Then

$$f_1(k) = (1 - 7k^{-1} + 6k^{-2})\log 2 + (k - 4 + 5k^{-1} - 2k^{-2})\log 3,$$
$$f_1'(k) = (7k^{-2} - 12k^{-3})\log 2 + (1 - 5k^{-2} + 4k^{-3})\log 3,$$
$$f_2'(k) = 2/k > 0.$$

Then we will have

$$f_1'(k) \geqq f_2'(k) \tag{18}$$

if

$$(7k - 12)\log 2 + (k^3 - 5k + 4)\log 3 \geqq 2k^2,$$

or

$$k^3 \log 3 - 2k^2 + (7\log 2 - 5\log 3)k + 4\log 3 - 12\log 2 \geqq 0. \tag{19}$$

Now $|7\log 2 - 5\log 3| < 1$ and $3 < |4\log 3 - 12\log 2| < 4$. Therefore an upper bound for the zeros of the polynomial in (19) is $4/(\log 3)+1 < 4+1 = 5$. Therefore (18) is satisfied for $k \geqq 5$. Now $f_1(7) > f_2(7)$, and we have shown $f_1'(k) > f_2'(k) > 0$ for $k \geqq 7$. Hence (17) is satisfied for $k \geqq 7$. Therefore (14) has at most one solution for k prime, $k \geqq 7$. Therefore $P_n = y^k$, $n = 4$, 5, 6, or 7, k prime, $k \geqq 7$, has at most one solution.

Added, February 22, 1940. Since this paper was written, Dr. P. Erdös has published proofs that P_n is never a square [8], and that for $k > 2$, there exists an $n_0 = n_0(k)$, such that for $n \geqq n_0$, $P_n = y^k$ has no solutions [9]. Erdös has also an unpublished proof that $P_{2k} \neq y^k$, and reports that G. Szekeres has proved that $P_n \neq y^k$ for $n \leqq 9$, which of course includes Theorem VI above.

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MATHEMATICAL EDUCATION

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This department of the MONTHLY *affords a place for the discussions of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

REPORT OF THE COMMITTEE ON TESTS*

1. Introduction. The Committee on Educational Testing of the American Council of Education proposed that the Mathematical Association of America collaborate with this sub-committee of the Central Committee on Personnel in the construction of tests in first and second-year college mathematics. A committee of the Association was appointed to consider this proposal. The members were H. S. Everett (Chairman), W. L. Hart, C. J. Jaeger, C. N. Moore, and Clara E. Smith. This committee reported on December 31, 1934, at the Pittsburgh meeting, recommending that the Association appoint a committee on test construction to coöperate with the Committee on Educational Testing, stipulating that the Association's committee be free to adopt its own procedures.

Following the recommendation of the Everett Committee and authorization by the Board of Trustees of the Association, President Curtiss appointed a Committee on Tests with the following members: Ralph Beatley, E. W. Chittenden (Chairman), A. R. Crathorne, L. L. Dines, and H. S. Everett. Marie M. Johnson was added to the committee in the fall of 1935 to assist in the secretarial work of the committee and in the construction of tests. H. S. Everett withdrew from the committee because of the press of other duties in 1937 and was replaced by R. G. Sanger. The Coöperative Test Service of the American Council on Education,† directed by Dr. Ben D. Wood, undertook to publish, distribute, and analyze the tests prepared by the committee.

The committee met in Chicago on the week-end of April 18, 1935, in conjunction with a meeting of the American Mathematical Society and prepared a program of test construction.

The committee prepared a pre-test in three forms which was administered in the fall of 1935 to 2625 students of first-year college mathematics in six institutions. From the budget of tested questions thus obtained, a revised test in two forms was constructed and tried out in the fall of 1936. The tests were administered to 1750 students in five coöperating institutions. The revised forms are now listed in the Catalog of the Coöperative Test Service. The 4375 papers obtained contain a mass of hitherto inaccessible information regarding the relative capacities and training of entering college students in institutions in different parts of the country. During the years 1936–1939 the analyses of the Coöperative Test Service were supplemented extensively by studies supported by NYA. Part of the results of these statistical investigations is presented below.

The committee has prepared and offered for general use a comprehensive ex-

* A condensation of the report presented to the Trustees of the Association at Madison, Wisconsin, September 8, 1939, by a Committee on Tests.

† 500 West 116th Street, New York City.

amination in college algebra. Two forms have been constructed, of which one only has been printed. Two forms of a comprehensive examination in trigonometry have been constructed but not printed. Copies of these can be obtained by application to Secretary Cairns. The committee has accumulated a budget of five hundred tested questions on Junior College Mathematics. It is hoped that this collection will be increased and become useful to college teachers who wish to compare the work of their students with that done at other institutions.

2. Objectives and administration of pre-tests. The pre-tests were designed to give information regarding the preparation of entering college students in fundamental algebraic skills. Certain items were introduced to test the effect on difficulty of various small changes in wording and form. The tests were limited to algebra because it was found impracticable to devote more than a single class period to their administration and it is generally recognized that success or failure in college mathematics is largely dependent on preparation in algebra. It was expected that the tests would be useful for sectioning or placement, diagnosis, prediction of subsequent performance, and provide a basis for the measurement of growth.

The committee decided to allow two weeks for review, so the pre-tests were administered uniformly on the ninth class day. No norms are available for the results of administration prior to review. Because of the difficulty of these tests there is reason to believe that scores obtained without preliminary review would be low and of less predictive value. The easier Iowa Placement Examinations* have proved valuable for prediction and placement under these conditions.

3. Distribution of scores on pre-tests. The following table shows the distribution of the scores obtained from the administration of these forms. Forms A(1935) and C(1935) contain 44 items. Form B(1935) has 42 items. Many of the items in the 1935 forms had several parts. The item was not scored correct unless all the parts were answered correctly. Several studies were made in which the parts of the items were used as units. The resulting scores will be referred to as partial scores. It is believed that the 1935 tests are too difficult. Each of the revised forms A(1936) and B(1936) contains fifty items, nearly all from the preceding tests. Some items were added to fill gaps in the order of difficulty.

TABLE I

Distribution of Scores in Pre-Tests

Form	Total number of students	Number of students having given scores									
		0–4	5–9	10–14	15–19	20–24	25–29	30–34	35–39	40–44	45–50
A(1935)	1499	15	148	334	348	289	227	105	28	5	0
B(1935)	554	9	67	140	150	105	54	20	9	0	0
C(1935)	572	1	32	93	164	148	78	37	17	2	0
A(1936)	835	3	11	35	68	120	147	168	164	83	36
B(1936)	915	2	23	71	123	183	183	168	89	60	13

* Published by the Bureau of Educational Research, University of Iowa, Iowa City, Iowa.

4. Pre-test scores and grades. The coöperating institutions supplied first semester grades for all students taking the pre-tests. The correlation between semester grades and test scores was computed by the Coöperative Test Service from samples for each institution and each of the five forms. These correlations vary from 0.42 to 0.74. The mean is 5.9 and the median 5.8. The correlation between test scores and grades is highest for groups with a high average score on the pre-test. Studies of homogeneous groups with three or more semesters of preparation in high school algebra show correlations with grades consistently over 0.6.

5. Sampling of pre-test scores. The forms administered in the years 1935, 1936, were arranged in fifteen groups with respect to three grades of preparation in high school algebra, (P_1) one year, (P_2) one and one-half years, and (P_3) two years, and the five forms. The papers in each form were arranged by institutions and representative samples were selected in as random a manner as possible preserving the relative proportions of the students in the different institutions. The total number of papers in the various samples was 1100. The very laborious task of sorting and tabulating the responses item by item was accomplished by students under NYA.

6. Comparison of Part I and Part II, 1935 forms. The 1935 pre-tests were divided into two parts. Part I was composed entirely of formal questions dealing with routine skills, and the emphasis was on manipulation. Part II was intended to contribute a subjective element and contained problems. The subjective part of the examination did not affect the scores reported to the committee or the subsequent analyses. Part I contained more questions than Part II in each of the three forms, and received the time allowance of 25 minutes, while Part II was given only 20 minutes. Analyses were made of the scores on Part I in relation to the scores on Part II for each of the nine groups. The results presented in Table II show clearly that the two parts measure different skills.

7. Construction of the 1936 pre-tests. Because the scores were low on Part II and it was not possible to draw satisfactory conclusions from the separation of Parts I and II, it was decided to build the new examinations in a single series of items. The 1935 tests showed gaps in range of difficulty and an attempt was made in constructing the 1936 tests to produce an examination of more carefully graduated questions. This necessitated the construction of questions of unknown validity and difficulty in the effort to fill the gaps. The experience of the committee with the 1935 examinations and the analyses of items which could be scored either in whole or in part showed that there were slight advantages, if any, to be gained by combining several units in a single item, and there were some obvious disadvantages.

The new forms contained fifty items for convenience in analysis and were easier and shorter than those of 1935. This produced a higher mean score, a psychological advantage, and a better distribution of the low scores. As the predictive value of a forty-five minute test proved to be small for the students of

highest quality, the lack of spread in the higher scores is not a serious disadvantage. The correlations between test scores and grades are about the same for the two sets of forms, 1935 and 1936.

8. Value of a short examination. A study was made of the predictive power of a short test. A group of 10 items was selected from Form A and the scores on these items were compared with scores on Form A and also with grades. The correlation between the first semester grades and scores for a group of 189 students on these 10 questions was .39 with a probable error of .04. The correlation of partial* scores on Form A for a sample of 200 students with the scores on the selected items is .84 with probable error of .01.

TABLE II

Analysis of Pre-Test Scores (1935)
Correlation between Scores on Part I and Part II

Form	Preparation	N	Partial scores*		Absolute scores	
			r	$P.E._r$	r	$P.E._r$
A	$P_1=1$ yr.	50	.52	.07	.45	.07
	$P_2=1\frac{1}{2}$ yrs.	100	.61	.04	.54	.05
	$P_3=2$ yrs.	50	.56	.06	.49	.07
	all	200	.63	.03	.55	.03
B	P_1	50	.37	.08	.33	.09
	P_2	50	.35	.08	.23	.09
	P_3	50	.66	.05	.60	.06
	all	150	.59	.04	.53	.04
C	P_1	50	.48	.07	.50	.07
	P_2	50	.53	.07	.55	.07
	P_3	50	.76	.04	.71	.05
	all	150	.70	.03	.70	.03

* As stated in paragraph 3, a number of items in the pre-tests of 1935 were divided in parts. For example, item 3 of Form A reads:

3. a) $3(-6)-4(-5)=$
   b) $2+0=$
   c) $0\cdot 3=$
   d) $\frac{0}{4}=$

Scorers were instructed to count this item only in case all parts were answered correctly. The purpose of dividing an item into parts was to avoid overweighting certain topics in the text and to secure information regarding a student's capacity to respond to part of a question which he could not answer completely.

For the samples studied in the tables above, the partial scores were computed and compared with the official test scores which are, for convenience, called absolute scores.

* *Cf.* paragraph 3.

TABLE III

Comparison of Scores on Form A (1935) and Scores on Part I of that Form with the Number of Correct Responses to the Ten Items 2, 4, 5, 9, 15, 17, 20, 23, 29, 32

Preparation	N	Scores		Partial scores on Part I	
		r	$P.E._r$	r	$P.E._r$
P_1	50	.79	.04	.88	.02
P_2	100	.83	.02	.89	.01
P_3	50	.76	.04	.77	.04
all	200	.84	.02	.89	.01

9. **Preparation and grades.** The following table presents a distribution of a sample of 874 students with respect to grades secured in their first semester in college and their preparation in college algebra.

TABLE IV

Percentages of 874 Students in each Preparation Group receiving a Given Grade

	Preparation group		
	P_1	P_2	P_3
Number in group	280	438	156
A	7.5%	11 %	10.9%
B	19.6	20.2	28.2
C	33.2	29	29.5
D and E	23	23.3	23.1
Failed	14.2	7.3	7.7
Dropped	2.5	3.7	0.6
Miscellaneous	0	2.3	0

These tables show a tendency for increased preparation to increase the number of high grades and decrease the number of failures.

We have no evidence that the amount of preparation affects the number of non-passing grades except that we note a tendency for a smaller percentage of failures among students of first-year mathematics who have had two years of high school algebra. For example, in one group of 166 students with one and one-half years of preparation in algebra there were 33 (20%) non-passing grades, while of 17 with two years of preparation there were no non-passing grades and 9 (50%) attained the grade B.

Vernon Price has computed tetrachoric correlations between satisfactory grades (A, B, or C) and unsatisfactory grades for a large number of items. A few items provide significantly large values of tetrachoric r. For some items r is negative.

10. A special study of Form A (1935). Miss Nura Turner analyzed 300 papers selected at random from the 1499 papers written on Form A in the fall of 1935. These papers were divided into 3 groups, representing the 3 degrees of preparation in algebra. She found that in general students with $1\frac{1}{2}$ years of algebra in high school omitted fewer items than people who had had only 1 year of preliminary training and responded correctly to more items than the less prepared group. The difference in rights, wrongs, and omissions corresponding to different degrees of preparation, are indicated by the following table.

TABLE V

Percentages of Right, Wrong, and Omitted Responses

Character of response	Preparation group		
	P_1	P_2	P_3
Right	40	51	60
Wrong	25	22	20
Omitted	34	26	20

11. Frequency of satisfactory and unsatisfactory grades. The following frequency table is based on a study of 275 papers from Miss Turner's sample of 300 and gives the distribution of scores of students receiving satisfactory (A, B, or C) grades or unsatisfactory (D, F, or Withdrawn).

TABLE VI

Frequency of Satisfactory and Unsatisfactory Grades

Score	9	12	14	18	21	24	27	30	33	36	39	42	45	48	51	Total
Satisfactory	1	3	10	9	8	22	20	23	21	21	18	9	7	3	3	178
Unsatisfactory	1	7	14	14	15	11	13	11	6		5					97

The bi-serial correlation is found to be $-.51$, indicating that unsatisfactory grades decrease substantially with increasing score. No failures occurred among students who completed the course and scored above 39.

12. A group of related items. In the construction of the 1935 pre-tests certain groups of items were introduced for purposes of comparison. One of these groups is given below. It consists of items 13, 14, 15, 16 of Form A. The relation between difficulty and complexity is evident.

	Difficulty	Validity
13. Solve for x: $3x - 5 = 8x + 10$.	70	8
14. Solve for W: $\frac{2}{3}W - W = 3$.	52	8
15. Solve for y: $\frac{2y}{3} - \frac{2y+1}{5} = \frac{1}{3}$.	42	9
16. Solve for T: $L = \frac{MT-G}{T}$.	35	11

13. A Comparison of college and high school students. In the spring of 1936, the 1936 State Scholarship Contest was given to 52 Iowa high school students who were completing their first year of high school algebra. These students ranged in age from 13 to 16, and were exceptional pupils since they were final competitors in the state academic contest.

The test given to these 52 pupils was composed of 51 items, the first 33 of which coincided with Part I of the pre-test, Form A, 1935. For a number of reasons the comparison between the work of the college and high school students is limited to items 1–24 which are counted as 27 items because item 3 of Form A has three parts. The expected superiority of the gifted contestants is shown in Table VII below.

TABLE VII

Percentage of Items Right, Wrong, and Omitted

Character of response	Preparation group			High school students
	P_1	P_2	P_3	
Right	52	66	75	85
Wrong	31	24	19	15
Omitted	16	9	5	

TABLE VIII

Frequency of Scores on 33 Items of Part I of Form A (1935) made by 52 High School Students

Score	20	21	22	23	24	25	26	27	28	29	30	31	32	33
Frequency	1	0	1	4	2	4	5	10	5	5	7	4	1	3

14. Constancy of the difficulty ratio. A comparison of the per cents right on items taken from 1935 tests with the per cents right on the *same* items in 1936 shows remarkably close agreement. For the entire list of items in the 1936 pre-tests which were used in 1935 the correlation is Form A, $r=.904$; Form B, $r=.99$. In evaluating this result we must consider the fact that a year elapsed between administration of the tests and the fact that there was a considerable degree of inhomogeneity of the population examined. It should also be noted that in 1935 the test was administered in separate parts, with a time allowance for each part. This agreement indicates that these items refer to common elements in the training of the groups examined.

15. Effect of the position of an item in a test. F. E. Satterthwaite kindly made an analysis of items from the 1935 tests repeated in 1936 with respect to the effect of position. The changes due to position fall within the range of probable errors.

16. A study of a group of first-year students. The response of a (P_2) preparation group of 117 students of first-year mathematics at the University of Iowa who were given the pre-test Form A (1936) was studied in detail with respect

to a number of factors. The distribution of scores obtained is normal in type. The sub-classes of this group obtaining the grades A, B, or C made scores with a wide range, distribution skewed toward the higher scores, and means increasing with grades.

An effort was made to find a response to the form which was more characteristic of the group of twelve receiving the grade A in the gross score. It was found that no reliance could be placed on the response to individual items, however difficult, on the response to groups of items toward the end of the test, or on several other factors. But, *all of these students scored well on the first forty items*, and there were only three other students of the 117 who scored as well on these items, and of these two made B, the third C. Thus the high grade students of this group were: (1) informed, (2) rapid, and (3) accurate workers. If a high degree of intelligence signifies ability to respond rapidly and accurately to *new* and *complex* situations, there is no evidence here that this group of twelve students possesses outstanding intelligence. Is it generally true that the three qualities listed above are sufficient for high grade work in college mathematics? This is an educational problem of some interest.

17. Tests at the college level. During the winter of 1935–36 a list of several hundred questions on college mathematics was submitted to the various coöperating institutions. Many of these questions were used as examination questions in final examinations in courses and the scores on these questions were reported to the committee. This practice was continued through the winter of 1936–37, and resulted in a substantial list of tested questions which may be used by instructors who wish to compare the work of their classes with that done in other institutions. Those items in the list which have been tried extensively give interesting information regarding the level of ability of students of college mathematics. The list of items with difficulty ratings was available in an appendix to this report.

The following pages contain a selection of items which were extensively used. With each item we give the number N of students examined, the number R of correct respones, and the percentage D of correct responses obtained.

TRIGONOMETRY

	N	R	$D=R/N$
23. Given $\cos x=-2/3$ and angle x lying between 180° and 270°, evaluate $\dfrac{\sin x+\cot x}{\sec x+\tan x}$ and simplify the result.	457	96	.21
30. Evaluate: $\dfrac{\sin(-120°)+\cos 330°+\tan 60°}{2\cot 135°\cdot\cos 180°}$.	575	246	.43
36. Express $\sin(-156°)$ in terms of a function of a positive acute angle.	543	290	.53
76. What trigonometric function of A is equal to $\dfrac{\sqrt{\sec^2 A-1}}{\sqrt{\tan^2 A+1}}$?	620	424	.68
120. If $\log A=m$, $\log B=n$, then $\log AB^2=$	525	233	.44

Algebra

23. Which of the following equalities are identities? Answer by number. (1) $x^2=\frac{1}{x+1}$, (2) $x^2-4=(x-2)(x+2)$, (3) $\frac{x}{x-1}=1+\frac{x}{x-1}$, (4) $x^2=4x+1$, (5) $\sqrt{a^2+b^2}=a+b$.	552	362	.65
46. If 2 is a root of $5x^3-kx^2+4=0$, what is the value of k?	571	425	.74
60. The sum of the roots of a quadratic equation is 1/2 and their product is $-15/2$. What are its roots?	613	376	.61
75. Write and simplify the first four terms of $(2x^3-y^2)^6$.	655	397	.61

Plane Analytic Geometry

	N	R	D
5. The vertices of a triangle are $A(0, 0)$, $B(a, 0)$, $C(b, c)$. Find the coördinates of the midpoints of the sides AC, BC.	103	93	.90
22. Name the curve: $x^2+3y^2-x-y=0$.	105	44	.42
38. Find the equation of the locus of a point which moves so that its distance from the origin is twice its distance from the line $x=4$.	176	58	.33
50. Find the equation of the line through the intersection of the lines $x+y+1=0$, $x-2y-3=0$, and parallel to the line $2x-y=0$.	110	57	.52
86. Find the equation of a parabola whose vertex is (2, 5) and directrix is the x-axis.	139	27	.19

Differential and Integral Calculus

1. If $y=e^{-t^2}$, find dy/dt.	324	274	.84
4. If $x^2-4xy+x+y+3=0$, find dy/dx.	275	182	.66
7. If $y=(\log x)^2$, find dy/dx.	274	219	.80
8. If $x=a\ (\theta+\sin\theta)$, $y=a\ (1-\cos\theta)$, find dy/dx.	289	192	.66
33. Find the values of x for which the tangents to the curves $y=x^2$ and $y=x^3$ are parallel.	264	156	.60

18. Comprehensive examinations in first-year college mathematics. Because of the practical impossibility of producing standardized examinations which could be used as final examinations throughout the coöperating institutions, the committee abandoned its plan to construct a comprehensive examination in first-year mathematics, and undertook the construction of examinations over portions of first-year college mathematics. A test covering a full year's work in college algebra was prepared in two forms on the basis of the committee's experience and approved in the summer of 1936. For a variety of reasons this test was not printed until the fall of 1937. Copies of the test were given in an appendix to the full report.

The revised pre-tests in two forms and one form of the comprehensive algebra test are now offered for general use by the Coöperative Test Service and listed in their catalog.

A two-hour examination in trigonometry in two forms has been prepared and typed copies were submitted in the full report. No significant trials of these examinations have been made.

CONCLUSIONS AND RECOMMENDATIONS

19. Test construction. The committee has considered seriously the question of the types of items best adapted for tests in mathematics. The multiple choice type of question has been used very extensively in tests prepared by the Coöperative Test Service and others. This type of item is not used in tests constructed by the committee because it is possible to construct tests in mathematics which are entirely objective without the use of multiple choice items, and the direct type of question used demands that the student produce the answer from the given data without other assistance. This makes each question a positive measure of achievement. The attempt to select a correct answer from a given selection of answers leads to guessing, to the process of working for the answer, and in some cases to the diversion of energy in an attempt to verify erroneous answers to numerical problems. The direct type of question gives information regarding the student's ability to check his work as well as to obtain answers.

The committee observes a tendency for authors of objective tests to be so concerned about the power of a test to discriminate markedly between able students, that tests contain too many questions. The use of a large number of easy questions puts a premium on speed and tends to overstrain students taking the examination. The committee recommends that examinations be moderate in length and that the desired discrimination between able students be effected by careful graduation of the questions with respect to difficulty. This graduation of questions can be obtained by the introduction of complications and by analyses of questions with respect to difficulty.

As far as practicable, it is desirable that each question refer to a single unit of instruction. It is also desirable to distribute topics considered in the examination well over the field of the examination. It is desirable that the number of topics considered be reasonably large in order to give a fair estimate of the student's accomplishment.

20. The objectives of test construction. It is the opinion of the committee that the following objectives are desirable and obtainable in test construction. Questions relative to a knowledge of the vocabulary and concepts should form part of a good examination in mathematics. There should be questions in an examination to test comprehension and reason; other factors are well known devices for testing technique and accuracy in computation.

It is desired to call attention to the use of classroom tests for the purpose of emphasizing important topics. The classroom test can be used for review or even to propose questions which are actually preparatory for later work. The use of tests as a teaching device is recommended.

A final examination should be broad enough to require preparation in the essential features of the subject of the examination.

21. Comprehensive examinations. The trend toward the use of comprehensive examinations covering the work of one or more years in college mathematics is approved by the committee. The committee undertook the preparation of standard tests in first and second-year mathematics and accumulated a mass of information regarding the difficulty of objective questions in algebra, trigonometry, and calculus. It was later found that funds for the printing of examinations in these subjects were not available and that it was also impracticable to test them in advance of publication. Since it was not possible to submit carefully tested examinations to the public, the plan to construct these tests was abandoned.

We did not obtain norms for comprehensive examinations because college courses in mathematics announced under the same title differ widely with respect to content and duration. Even in the case of the coöperating institutions no uniform final examinations could be given. At present final examinations in college mathematics are a local problem.

Many teachers of college mathematics are opposed to the introduction of standardized examinations. Since the teaching of mathematics is undergoing transformation under the influence of a number of factors including changes in college entrance requirements and the desire to introduce into the college world some of the features already extensively used in secondary school work, it is apparent that the publication of a set of standardized examinations under the auspices of the Association might interfere with the introduction of desirable changes in the curriculum.

A factor of considerable importance in this situation is the great difference which exists between the preparation of students in colleges in different parts of the country and corresponding differences in the intellectual level of first-year college courses. It is, of course, important to recognize that junior college mathematics has been dominated by the curriculum requirements of students of engineering.

22. The use of pre-tests. Pre-tests published by the University of Iowa have been quite extensively used by colleges throughout the country over a period of years and the sustained demand for these tests indicates that they are valued by teachers of mathematics in colleges. As a part of a program of the committee, pre-tests were prepared because it was found possible to secure joint action involving large numbers of students. Pre-tests published by the committee through the Coöperative Test Service are available for the purposes of measuring the relative capacities of entering students, diagnosis of weaknesses which are likely to seriously interfere with progress, and estimating probable success in college mathematics.

These tests can be used to section students on the basis of ability, but the separation into more than two (possibly three) grades of students may not be ideally successful. No statistics are available on this point.

23. A recommendation. The committee believes that the study of examinations is of great importance to the progress of the teaching of college mathematics and recommends that the Mathematical Association of America form a permanent Committee on Tests with an allowance for expenses in the budget of the Association.

24. Acknowledgment. Finally the members of the committee desire to express their appreciation of the patient coöperation and financial support of the Coöperative Test Service and its director, Dr. Ben D. Wood, and of the cordial support which has been given by the officers of the Association. Many of the studies reported here were made possible by the NYA. The work of Sarah Patterson, Vernon Price, F. E. Satterthwaite, and Nura Turner, graduate students of the University of Iowa who have assisted with the statistical analyses, is gratefully acknowledged. Much of the labor of preparation of the 1936 tests was done by Marie M. Johnson, who also compiled the list of questions on Junior College Mathematics, and the comprehensive examinations in trigonometry. The assistance given by Secretary W. D. Cairns has been of great value.

Editorial Note. Appendices of the report by the committee were presented as follows:

I. The report of the Everett Committee.
II. Conclusions reached by the Committee on Tests at its conference at Chicago, April 18, 19, 1935.
III. Comparison of items on College Board Examinations with items of Pre-Tests, by E. W. Chittenden.
IV. Iowa Placement Examinations in Mathematics, by E. W. Chittenden.
V. The 1935 Pre-Tests, Forms A, B, C with validities, difficulties, and keys.
VI. A circular letter on the Pre-Tests.
VII. The 1936 Pre-Tests, Forms A, B, with validities, difficulties, and keys.
VIII. A budget of tested questions on Junior College Mathematics.
IX. Comprehensive Examination in College Algebra, Forms A, B, with keys.
X. Comprehensive Examination in Trigonometry, Forms A, B, with keys.
XI. Report on the Mathematics Examinations of the College Board, by Ralph Beatley.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON THE EQUATIONS OF CONICAL SURFACES

T. F. MULCRONE, S.J., Spring Hill College

Many text-books in solid analytic geometry propose a comparatively lengthy method of finding the equation of a conical surface. The following artifice, apart from the fact that it often requires no more than mental calculation, draws attention to the fact that the nature of a conic surface depends primarily upon the nature of the base assumed as a directing curve, and that the equation of such a surface is homogeneous if the vertex of the cone is at the origin.

Consider the most important conical surface, that of the second order. The general equation of the directing curve is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

$$Lx + My + Nz = K. \tag{2}$$

Using the customary notation, let $P(x, y, z)$ be any point on the conical surface, and $Q(Q_x, Q_y, Q_z)$ the point where the generator through P intersects the directing curve. And since we are concerned only with cones having their vertices at the origin, $V(0, 0, 0)$, we may write the equations of the generator VQ in the symmetric form $x/Q_x = y/Q_y = z/Q_z$, and

$$Q_x = xQ_y/y, \tag{3}$$

$$Q_y = Q_y, \tag{4}$$

$$Q_z = zQ_y/y. \tag{5}$$

Now since Q is on the directing curve, by the substitution in (1) of Q_y for y, and the value of Q_x from (3) for x, we have, after collecting terms,

$$Q_y^2(Ax^2 + Bxy + Cy^2) + Q_y y(Dx + Ey) + Fy^2 = 0; \tag{6}$$

and likewise upon the substitution in (2) of (3), (4), and (5), we have, after collecting terms,

$$Q_y(Lx + My + Nz) = Ky. \tag{7}$$

Eliminating Q_y between (6) and (7) we have the general equation of a conical surface of the second order,

$$K^2(Ax^2 + Bxy + Cy^2) + K(Lx + My + Nz)(Dx + Ey) + F(Lx + My + Nz)^2 = 0,$$

a homogeneous equation of the second degree.

But we arrive at precisely the same equation when we *convert equation* (1)

to a homogeneous equation using the operator $(Lx+My+Nz)/K$, *or powers of it as multipliers of those terms in* (1) *which are not of the second degree.* Note that the operator is derived from (2), the equation of the plane of the directing curve, and that it is equal to unity for points on this plane.

Although we have confined the consideration to the conical surface of the second order, it is readily seen from the nature of the operations involved that the method is applicable to surfaces of any order.

A few examples will make the procedure clearer.

Find the equations of the following conical surfaces with vertices at the origin, where:

a) The trace in the plane $z=2$ is $y^2=8x$.

Solution. The operator is $z/2$. Hence $y^2=8x\cdot z/2$; that is, $y^2=4xz$.

b) The directing curve is $y=x^4$, $z=2$.

Solution. The operator is $z/2$. Hence $y\cdot(z/2)^3=x^4$; that is, $8x^4=yz^3$.

c) The directing curve is $y^2+z^2=y$, $x+y=1$.

Solution. The operator is $x+y$. Hence $y^2+z^2=y(x+y)$; that is, $z^2=xy$.

d) The directing curve is the circle $x^2+y^2+z^2=1$, $x+y+z=1$.

Solution. The operator is $x+y+z$. Hence $x^2+y^2+z^2=(x+y+z)^2$; that is, $xy+xz+yz=0$.

This method is also applicable when the vertex is not at the origin, provided the axes are translated so that the new origin is at the vertex before the homogeneous equation is formed.

THE RELATION BETWEEN THE TERMINAL RESERVE AND THE AMOUNT OF INSURANCE

G. M. Ewing, University of Missouri

Texts which treat the subject of life insurance explain that the reserve for a given policy is a fund accumulated during the early policy years to help bear the increased cost of insurance later on. Though held by the company, the reserve belongs to the policy holder and may be withdrawn, perhaps subject to a penalty, if he surrenders his policy. The text-books seem to give little explanation of how the reserve helps carry these increased costs except as the reader extracts this information from the prospective and (or) retrospective methods for computing it. Often the (false) impression is given that a \$1000 insurance policy provides term insurance for \$1000 during each policy year, and that the difference between the cost of such term insurance and the premium paid is supplied by the reserve.

Consider a \$1000 ordinary life policy taken at age 30. Using the American Experience Table of Mortality, the terminal reserve increases monotonely from 0 at age 30 to \$1000 at age 96. If rates are computed with the interest rate $i=.035$, the net annual premium is \$17.19. The terminal reserve at age 80 is found to be \$773.70, whereas the natural premium (*i.e.*, the cost of one-year term insurance) at age 80 is \$139.58 per thousand. It is clear, therefore, that if the insured lives to be very old his premiums of \$17.19 plus interest on the re-

serve will not be enough to provide term insurance for \$1000 each year without exhausting the reserve, contrary to the known fact that the reserve must increase. The explanation is that an \$$R$ insurance policy means one whose total death benefit is \$$R$ and not one which provides insurance for \$$R$ during each year.

Again, well informed people as well as students taking courses in the mathematics of finance may wonder why a \$1000 policy with a reserve of \$200 does not pay \$1200 upon death. Does not the \$200 reserve belong to the insured? Yes, but the \$1000 refers to the total death benefit and not to what is called in this article the "amount of insurance."

Consider any \$1 insurance policy taken at age x. Let V_n denote the terminal reserve for (or reserve at the end of) the nth policy year. Suppose for simplicity that death claims are payable at the end of the year in which death occurs. If the insured should die in the $(n+1)$th policy year, the accumulated reserve at the end of that year would be $(1+i)V_n$; and, in order for the total death benefit to be \$1, it is necessary and sufficient that the policy holder shall have been insured for $1-(1+i)V_n$ during that year. This difference between the total death benefit mentioned in the policy and the accumulated value of the reserve will be called the "amount of insurance." For an \$$R$ policy, multiply this result by R. It is convenient for our present purposes to think of every life contract as a series of one-year term contracts embodied in a single policy. The "amount of insurance" provided in a given year by an \$$R$ policy is never \$$R$ unless the reserve is zero. This will happen only in the first policy year.

In our example of the man aged 80 with an ordinary life policy taken at age 30, there is a reserve of \$773.70 which will accumulate to \$800.78 in one year. Hence the amount of insurance required is only \$199.22.

The cost of one-year term insurance for \$$T$ at age $(x+n)$ is known to be TC_{x+n}/D_{x+n}, where C_{x+n} and D_{x+n} are defined in any text on the mathematics of insurance. The cost of insurance, Z_{n+1}, for the $(n+1)$th policy year for an amount equal to the amount of insurance is therefore

$$(1) \qquad Z_{n+1} = [1-(1+i)V_n]C_{x+n}/D_{x+n}.$$

One verifies readily that

$$(2) \qquad (1+i)V_n l_{x+n+1} + (1+i)l_{x+n}(P_{n+1} - Z_{n+1}) = V_{n+1}l_{x+n+1},$$

l_k denoting the number of people, shown to be living at age k by the mortality table. Equations (1) and (2) are equivalent to the well known Fackler accumulation formula, which thus furnishes a check on our analysis of the problem.

Is it possible to devise a life contract for which the amount of insurance is constant, say \$1 for each year, and whose death benefit is $1+(1+i)V_n$? To do so replace (1) by

$$(3) \qquad Z_{n+1} = C_{x+n}/D_{x+n}.$$

From (2) and (3) with $n=95-x$ we find $P_{96-x}=1/(1+i)\equiv v$; and, if we assume

equal net annual premiums, $P_n = v$ for $n = 1, 2, \cdots, 95 - x$. We are led to $V_n = S_{\overline{n}|i}$, the amount of an annuity certain. A contract with these provisions meets the stated conditions; but it has lost the essential feature of life insurance in that the value of the contract at the end of a given year does not depend upon whether or not its holder is alive but only upon how many premium payments have been made.

INVESTIGATION OF LAGRANGE'S TANGENT METHOD ON DIOPHANTINE BINARY CUBICS

H. D. Grossman, New York City

Lagrange in his *Oeuvres*, vol. 4, pp. 396, 397, gives what he calls the tangent method (mentioned in Dickson's *History of the Theory of Numbers*, vol. 2, p. 595) for deriving from one rational solution of the general binary cubic another rational solution and so in general an infinite chain of them. His method in simpler notation is essentially this.

If (x, y) is a rational solution of $f(x, y) = k$ where f is a cubic form and k is a constant, then another is $(x - f_y g, y + f_x g)$ where

$$g = \frac{-3(f_x^2 f_{yy} - 2f_x f_y f_{xy} + f_y^2 f_{xx})}{f_x^3 f_{yyy} - 3f_x^2 f_y f_{xyy} + 3f_x f_{xy}^2 f_{xxy} - f_y^3 f_{xxx}}.$$

Let Δ be the differential operator

$$\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x};$$

then g may be written most elegantly and simply in the symbolic form $g = (-3\Delta^2 f)/(\Delta^3 f)$. This solution may be easily proved by expansion:

$$f\left(x - g\frac{\partial f}{\partial y},\ y + g\frac{\partial f}{\partial x}\right) = f(x, y) + g\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right)f$$
$$+ \frac{g^2}{2}\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right)^2 f + \frac{g^3}{6}\left(\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial}{\partial x}\right)^3 f.$$

The series ends here because f is only of third degree. Hence

$$k = k + 0 + \tfrac{1}{2}g^2\Delta^2 f + \tfrac{1}{6}g^3\Delta^3 f,$$

and

$$g = 0 \qquad \text{or} \qquad (-3\Delta^2 f)/(\Delta^3 f).$$

This method gives for the following equations the corresponding solutions if in each case (x,y) is a solution:

$$(1) \qquad x^2y + xy^2 = k, \qquad \left(-\frac{x(x+2y)^2}{(2x+y)(x-y)}, \frac{y(2x+y)^2}{(x+2y)(x-y)}\right);$$

$$(2) \qquad ax^3 + by^3 = k, \qquad \left(\frac{x(ax^3 + 2by^3)}{ax^3 - by^3}, \quad -\frac{y(2ax^3 + by^3)}{ax^3 - by^3}\right);$$

$$(3) \qquad ax^3 - by^2 = k, \qquad \left(\frac{9ax^4}{4by^2} - 2x, \ \frac{27a^2x^6}{8b^2y^3} - \frac{9ax^3}{2by} + y\right).$$

Merely a special case of (3) with $a=b=1$ is Euler's solution

$$\left(\frac{9x^4 - 8xy^2}{4y^2}, \frac{27x^6 - 36x^3y^2 + 8y^4}{8y^3}\right) \quad \text{of} \quad x^3 - y^2 = k$$

if (x, y) is a solution (Euler's *Algebra*, vol. 2, 1770, chap. 8, art. 121, cited in Dickson's *History of the Theory of Numbers*, vol. 2, p. 535).

Lagrange's method fails when $\Delta^3 f = 0$; *e.g.*, when $x=y$ in (1), $ax^3 = by^3$ in (2), or $y=0$ in (3). It sometimes fails to give new solutions; *e.g.*, when $x=0$ in (1), (2), or (3). It sometimes gives only a finite chain; *e.g.*, the chain (2, 3), (0, -1) for $x^3+1=y^2$. And it may not give the general solution even when it gives an infinite chain; *e.g.*, it omits the solution $(\frac{17}{5}, \frac{99}{5})$ in giving the chain (1, 3), $(\frac{1}{2}, \frac{1}{2})$, $(\frac{37}{8}, \frac{503}{16})$, $\cdots$ for $10x^3 - y^2 = 1$. The solutions (4, 9) of $x^3+x^2+1=y^2$ and $(\frac{1}{4}, \frac{9}{8})$, (72, 611) of $x^3+x+1=y^2$ can be only initial members of chains.

Note by the Editor. The significance of the term "tangent method" is obvious when the geometric meaning of the method is considered. If $P(x, y)$ is a point of the curve C whose equation is $f(x, y)=k$, then for any value of t the point $(x - f_y t, \ y + f_x t)$ lies on the line tangent to C at P; and g is the value of t which gives the residual intersection of the tangent line with C. Each of the special types of behavior considered above has a simple geometric interpretation.

There are some obvious generalizations of the tangent method. If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are points of C with rational coördinates, the third intersection of the line P_1P_2 with C also has rational coördinates. Writing the line P_1P_2 in the parametric form

$$x = x_1 + t\Delta x, \qquad y = y_1 + t\Delta y,$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, we find that the third intersection is given by $t = -2L/(Q+2L)$, where $L = f_x\Delta x + f_y \Delta y$, $Q = f_{xx}\Delta x^2 + 2f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2$, the derivatives being evaluated at P_1. This "secant method" does not give rise to a chain of solutions, but taken in conjunction with the tangent method it will often give solutions otherwise unobtainable. Another generalization of the tangent method would be the determination of the sixth point of intersection with C of the conic having five-point contact with C at P. This would usually lead to a chain of solutions, but the calculations involved would be so complicated that the method would be of little value. R. J. W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department at the Mathematical Association of America, 531 West 116th St., New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Rules of Partitions. By Hansraj Gupta. Madras, Indian Mathematical Society, 1939. 81 pages. Rs.5.

Mathematics in Action, Book Three. By W. W. Hart and Lora D. Jahn. Boston, D. C. Heath and Co., 1940. 6+442 pages. $1.28.

Advanced Algebra. By S. Barnard and J. M. Child. London, Macmillan and Company, 1939. 10+280 pages. $4.00.

Textes Mathématiques Babyloniens. Edited and translated by F. Thureau-Dangin. Leiden, E. J. Brill, 1938. 8+245 pages. 25 guilders.

Encyklopädie der Mathematischen Wissenschaften .... Second edition. Band I_1, Heft 5. Leipzig and Berlin, B. G. Teubner, 1939. 54+54+28 pages. RM 7.20 for foreign countries; RM 9.80. Allgemeine Modul-, Ring-, und Idealtheorie, by W. Krull; Theorie der Polynomideale und Eliminationstheorie, by W. Krull; Die Theorie der Verbände, by H. Hermes and G. Kothe.

Encyklopädie der Mathematischen Wissenschaften .... Second edition. Band I_1; Heft 4, Teil 1. Leipzig and Berlin, B. G. Teubner, 1939. 51 pages. RM 2.85 for foreigners; RM 3.80. Allgemeine Gruppentheorie, by W. Magnus.

Differentialgeometrien in den Kugelräumen. Band II: Laguerresche Differentialkugelgeometrie. By T. Takasu. Tokyo, Maruzen Company, 1939. 20+444 pages. $3.50.

Theory of Probability. By Harold Jeffreys. (The International Series of Monographs on Physics.) Oxford, Clarendon Press, 1939. 7+380 pages. $7.00.

Mathematical Methods in Engineering. An Introduction to the Mathematical Treatment of Engineering Problems. By Theodore v. Kármán and Maurice A. Biot. New York and London, McGraw-Hill Book Company, 1940. 12+505 pages. $4.00.

Elements of Statistical Reasoning. By A. E. Treloar. New York, John Wiley and Sons; London, Chapman and Hall, 1939. 11+261 pages. $3.25.

REVIEWS

A Supplement to Magic Squares of $(2n+1)^2$ *Cells*. By M. J. van Driel. London, Rider and Company, 1939. 31 pages.

This is a supplement to the author's book, *Magic Squares of* $(2n+1)^2$ *Cells*, which was reviewed in this MONTHLY, vol. 45, 1938, p. 467.

The supplement is confined entirely to a discussion of magic squares of order 5. The author's purpose is to show how the number of magic squares, of this order, found in the book, may be considerably increased. This is done in two ways: first, by certain permutations of rows and columns; and, second, by find-

ing more general transformations of the fundamental semi-magic squares of the book which will make the principal diagonals magic.

The expression in the second line of page 10 contains a confusing misprint. It should read $(n-3)(n-4)n!^2/8$. The statement that the total number of pandiagonal magic squares of prime order n is given by this expression is incorrect. It gives the correct number for the cases $n=3$ and $n=5$ only. For n prime and greater than 5 the total number of pandiagonal magic squares exceeds the number given by this formula.

G. E. RAYNOR

Mathematics in Action, Book Two. By W. W. Hart and Lora D. Jahn. New York, Heath and Company, 1939. 9+374 pages. $0.96.

The second volume of this series, as the preceding one, aims to present students with an "understanding and appreciation of the rôle of mathematics in many phases of modern life"; and to this end, fundamental skills serve as a means of solving problems arising in real life.

The book meets the requirements of a second year's work of a practical, socialized course in mathematics for the junior high school. The twelve units of work cover a review of fundamentals, geometry in industry, formulas, mensuration, percentages and its uses, insurance, banks and banking, taxes, investments, equations, indirect measurement, and positive and negative numbers. Each unit is provided with diagnostic tests, cross reference to re-instruction, cumulative review material, as well as unit and survey tests. In short, the majority of techniques of modern text-book writing are employed.

The units devoted to the subjects of insurance, banking, taxes and investments are especially noteworthy. Emphasis is placed upon content rather than skills in computation, and timely material includes group and hospital insurance, income taxes, and unemployment and Federal old-age insurance. The final units of the book are "exploratory previews" of topics to be studied in subsequent grades and are offered as adequate first instruction for an introduction to algebra.

It seems rather unfortunate that in presenting a rule for the order of operations, the authors should decide upon the rule calling for multiplications first, divisions next, and finally additions and subtractions, instead of the customarily accepted rule of performing multiplications and divisions in order from left to right, followed by additions and subtractions. Since the book is not provided with answers, it is impossible to discover how consistently the rule has been applied throughout the remaining pages.

The reviewer feels the typography of this book suffers from the same drawback as its predecessor—poor page make-up. Presumably to condense the total number of pages, material has been crowded into every available space, giving the impression of endless pages of printed matter with nothing to relieve the monotony. A wise deletion of some of this material would have improved the effect immeasurably.

R. A. HARRISON

Mathematics of Statistics. By John F. Kenney. New York, D. van Nostrand Company, Inc., 1939. Part I, 10+248 pages. $2.50. Part II, 9+202 pages. $2.25. Parts I and II bound together, $4.00.

These books are intended to serve as text-books for a college course in mathematical statistics. Part I is elementary, and deals with the type of material which may be understood by students with no mathematical preparation beyond the usual freshman algebra course. Part II is for students of the calculus, and is concerned with such topics as the Pearson curves, continuous bivariate distributions, multiple and partial correlation, and the theory of small samples. Part I is by no means the first text-book to appear in its field; but Part II at the moment seems to stand quite alone as an up-to-date statistics text-book which can be used as a *mathematical* introduction to the more modern theories.

A survey of the contents reveals that in spite of certain important omissions, the author's choice of topics is, on the whole, a good one. Chapters I–V of Part I contain a description of observed frequency distributions, their graphical representation, and their moments. There is a brief but adequate discussion of Sheppard's corrections. Chapter VI is concerned with a descriptive treatment of the normal distribution. The notation of the integral calculus is used here, but a detailed explanation is given as to what the symbols mean. Chapter VII contains a treatment of curve fitting. Several methods are given for fitting a straight line to empirical data. The fitting of exponential curves, parabolas, and the Gompertz curve are also discussed. The last chapter of Part I is on simple correlation. Although this chapter contains one or two unnecessary complications (Theorem IV is an immediate consequence of Theorem IX, p. 44, and Theorem V is an immediate consequence of the work on p. 96), nevertheless it is on the whole a good introduction to the observational phase of the subject. Blakeman's test is presented, together with Fisher's criticism of it. The formula for the rank correlation coefficient ρ is derived, and Pearson's relation $r=2\rho \sin(\pi/6)$, which may puzzle some teachers as well as students, is stated without a reference.* Part I concludes with a table of ordinates and areas of the normal curve, a five-place table of logarithms, and an index.

Chapter I of Part II contains an elementary discussion of the probabilistic groundwork of statistics. The theorems of total and compound probability, Bernoulli's Theorem, and the DeMoivre-Laplace Theorem are stated without proof (the last two, however, are proved in later chapters). The Poisson exponential is described. Chapter II is entirely mathematical, and introduces the reader to the Gamma and Beta functions. Chapter III is concerned with the general concept of a continuous distribution in one dimension, and in particular with the Pearson and Gram-Charlier systems of curves. Chapter IV is devoted to continuous joint distributions. A number of important basic definitions are given and theorems proved without the assumption of normality; the normal surface is then treated as a special case. Chapter V presents the observational

* Karl Pearson, Mathematical Contributions to the Theory of Evolution, XVI. Drapers' Company Research Memoirs, Biometric series IV.

theory of multiple and partial correlation. Chapter VI is concerned with the theory of large samples, and in particular, the distribution of the mean. The derivations are given by applying the fundamental theorems of expected value to a set of N identically distributed variables representing undetermined observations. Tchebycheff's inequality is proved and applied to the mean of a sample. The concepts of a null hypothesis and of a significance test are introduced and are illustrated by appropriate examples.

Chapters VII and VIII are of especial interest in that they lead the reader into some of the more recently developed fields of statistics. A fairly complete set of references is given at the end of each of these chapters, and of Chapter VI. Chapter VII commences with a discussion of unbiased estimates and of degrees of freedom. Then a derivation of the distribution of "Student's" t is presented; the author chooses (probably wisely) to transform the volume element by appeal to N-dimensional geometry. This leads to Fisher's z-distribution, which is applied to the analysis of variance. The chapter is concluded with various modern tests of the significance of the correlation coefficient. Chapter VIII consists of two quite distinct parts. In the first part, the χ^2 distribution is derived as an approximation to the multinomial distribution and is applied to testing simple and composite hypotheses concerning frequency distributions. In the second part, the concept of statistical inference is studied. After deriving Bayes' Theorem and indicating its limitations, the author reproduces an exposition recently given by Rietz in this MONTHLY, and gives applications to finding fiducial limits for the mean and variance of a sample for the difference of two means. The appendix contains tables of ordinates and areas of the normal curve, Snedecor's table of the 5 per cent and 1 per cent points of F, and a table of the χ^2 scale. There is an adequate index.

In the main, adverse criticism is on minor points. In Part I, as might be expected, the most vulnerable sections are those which attempt to present concepts from the calculus to readers who are supposed to be ignorant of this subject. Thus both Chapters VII and VIII of Part I, which to a certain extent must necessarily be based on such concepts, contain several inaccurate statements and inadequately defined terms.

In Part II, in addition to minor inaccuracies, the reviewer noticed several more important slips. The first of these is that the statement of Bernoulli's Theorem on page 9 is practically meaningless. The second is that the restriction to a random sample of a Bernoulli distribution, stated on page 27 in connection with Pearson's derivation of the standard error of a class frequency, is quite unnecessary and likely to be rather confusing to the beginner. Also, the demonstration under II on page 146 seems to be invalid as it stands. The error can be rectified by writing $w=s_1-s_2$ instead of $w=s_1^2-s_2^2$ and altering the numerators of (47) and (47a) correspondingly. There is a misprint in the sixth line from the bottom on that page: N_1-N_2 should be replaced by N_1+N_2. (Incidentally, there is a fairly large number of misprints in both Parts I and II.) Finally, the curves on page 158 were taken from the wrong page of Fisher's *Statistical Methods*. They

are the curves for the distribution of the *intraclass* correlation coefficient, on page 205 of Fisher's book (fifth edition), rather than the curves for the *interclass* correlation coefficient (as alleged), which will be found on page 187 of Fisher's book.

The nature of these criticisms, together with certain other pieces of internal evidence (such as the fact that one or two of the exercises are completely worked out in the text), suggests that the final draft of the manuscript for Part II may have been prepared somewhat hurriedly. However, the errata which the reviewer noted are all of the type which may easily be corrected in a revised edition, and they are by no means characteristic of the work as a whole. On the contrary, as compared with the average elementary text-book on applied mathematics, both Parts of this one are really quite carefully written from the mathematical point of view. Definitions are usually stated clearly and emphatically, and their consequences are developed logically, often in formally stated and proved theorems. Proofs are eventually given for almost all of the important formulas. At the same time, much attention is paid to the practical aspects of the subject. The author never for a moment forgets that he is writing a text-book on a branch of applied mathematics in which the mathematics is very easily misapplied. Detailed common-sense motivations are provided for a number of the developments, and there are a number of passages containing useful advice concerning the application of the theory to experimental work. A good set of exercises is provided with each chapter both in Part I and in Part II.

The reviewer is willing to venture the opinion that at the present moment, Part I of Kenney is the best available text-book on elementary statistics using the mathematical type of presentation. The usefulness of Part II will be admitted without question by most teachers of statistics, as well as by students with some mathematical background who desire to look into the proofs of the newer developments in statistics. Both parts taken together certainly constitute one of the most important contributions to the field of text-books on mathematical statistics that has appeared in some time.

J. H. CURTISS

Portraits of Famous Philosophers Who Were Also Mathematicians. With biographical accounts by Cassius J. Keyser. New York, Scripta Mathematica, 1939. 12 folders. $3.00.

This set of portraits and accompanying essays will be a delight to all mathematicians philosophically inclined and all philosophers mathematically inclined. The list of "heroes" included may be rather startling to the mathematician—Pythagoras, Plato, Aristotle, Epicurus, Roger Bacon, Descartes, Pascal, Spinoza, Leibniz, Berkeley, Kant, Charles Peirce—but Professor Keyser has good reasons for including them all.

Each portrait, quarto size, is inserted in a four-page double sheet containing the biographical account. The portraits themselves would be a valuable addition to any library. The whole set is enclosed in a heavy folder fronted with Raphael's

Plato and Aristotle at the School of Athens. It is throughout an artistic work.

The biographies are written in the same style which makes all Professor Keyser's works worth reading. It is to be hoped that the reception of this latest effort will encourage him to make this set one of a series of *Portraits*.

A reviewer can not possibly express the satisfaction obtained in studying these portraits and essays. The portraits can not be shown here, but it is possible to reproduce a few sentences from the essays. They may help to convey the spirit of the whole.

"It is no exaggeration to say that in the history of philosophy after 347 B.C. there is to be found neither insight nor utterance quite so divine as the vision and voice of Plato."

Speaking of Kant, "On principle he addressed his lectures mainly to the students of middle ability, for dunces, said he, cannot be helped and geniuses can help themselves."

"Was Spinoza a great mathematician? If the term is to designate only such as have made important contributions to so-called 'pure' mathematics, the answer is No. But if it be applied also to those who have masterfully applied the mathematical method to no matter what kind of concrete subject-matter, the answer is Yes."

This makes one wonder if perhaps the second type of mathematician may not be just as great as, or sometimes greater than, the first.

HARRIET F. MONTAGUE

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.

SUGGESTIONS FOR PRESENTING A TOPIC BEFORE THE MATHEMATICS CLUB

Better presentation of topics at club meetings has become the aim of a number of clubs during the past few years. Since practice in public speaking is not always a part of the background of those majoring in mathematics, it has seemed practical for the mathematics club to supply some training in this field in connection with oral presentation of mathematical topics. Professor C. W. Munshower, adviser, has sent in the following list of suggestions which are handed to the members of the *Mathematics Club* of *Colgate University* before the preparation of the first paper.

1. Dress neatly.
2. Know the topic so well that a small, concise set of notes is sufficient.
3. Start by telling what you propose to do—stating the problem or reading the title of the paper to be presented usually will not suffice; elaborate so that the listeners will appreciate what you are trying to do. The history of the problem, simple examples to illustrate the meaning of new concepts and terms included in the problem statement, and the relation of the problem to facts familiar to the listeners are some means of motivating the audience.
4. Speak to the audience and not to the blackboard; speak distinctly.
5. Place a record of your work on the blackboard; in general, proceed from left to right; write neatly, on horizontal lines, with good alignment. Paragraph the work. Number the important relations so that you can refer to them by their numbers.

6. Use the eraser only when an error has been made; the blackboard should contain a complete record of your talk when you finish. To make clear some minor point, step to a blackboard at one side and erase your work when finished.
7. Words, as well as symbols, should appear on the blackboard; use symbols which are standard.
8. Be very careful to use the correct word in writing or in speaking, particularly when it is a mathematical term; know the definitions of the terms you are using.
9. Do not leave out a single "step" the omission of which might confuse the listeners.
10. Do not obstruct the blackboard work; after a short interval of writing upon the board move far enough away so that all can see; repeat this movement continually.
11. Do not remain silent for longer than a few seconds at any time.
12. Simple examples are useful to illustrate what you are doing at any point in the discussion.
13. Conclude by summarizing briefly; make reference to the blackboard record.
14. When you have concluded, inform the chairman that you will try to answer questions and that you will appreciate hearing comments related to the topic discussed.
15. Do not erase the work from the blackboard until the chairman indicates that the discussion period following the presentation has ended.
16. When acting as chairman: (1) arrive early and be certain that the necessary supplies are present and that the meeting room is ready for use, (2) start the meeting on time, (3) introduce each speaker briefly, (4) direct the discussion at the end of each paper, being prepared to take the lead yourself if necessary, (5) know how long each paper will take so that you can limit the discussions to proper lengths, (6) close the meeting promptly.

READINGS IN MATHEMATICS FOR SUMMER LEISURE

Many clubs have found that members enjoy spending some time during the summer months reading in mathematical fields for which time is not always available during the rush of the college year's activities. Those interested in preparing such a list for use this summer are referred to the list of Books for Clubs which have appeared in past numbers of this department.*

In addition, we wish to call attention to the following titles taken from the list which Professor R. C. Archibald of Brown University has presented to his students in the spring for several years. We quote: "There are many publications more or less mathematical in character which enable the general reader to enlarge and to enrich his intellectual outlook, while exploring regions of no little interest. May the following brief list arouse curiosity and excite the desire for adventure and voyages of discovery, especially on the part of those students who have had a year of mathematics in college. We suggest such cruising as one form of summer recreation especially profitable."

Aspects of Science, by J. W. N. Sullivan, New York, Knopf, 1926, first series, "Mathematics and music," pp. 185–206; second series, "Mathematics as an art," pp. 80–96.

Omar Khayyam as a Mathematician, by W. E. Story, 1918, 13 pages; may be purchased from G. E. Stechert and Co., New York.

Three Students, by Haldane Macfall, New York, Knopf, 1926, 359 pages. Delightfully written romance of the life of the mathematician, astronomer, and poet of the *Rubaiyat*.

The Torchbearers: *Watchers of the Sky*, by Alfred Noyes, New York, Stokes, 1922. The first of the trilogy by a gifted poet; see especially the sections on Galileo, Newton, William Herschel (astronomer and musician), pp. 131–243.

The Fourth Dimension Simply Explained, a Collection of Essays Selected from those Submitted in the Scientific American Prize Competition, edited by H. P. Manning, New York, Munn, 1910, 251 pages.

Through Space and Time, by James H. Jeans, New York, Macmillan, 1934, 238 pages. Based on lectures by the distinguished scientist.

Numbers and Numerals, A Story Book for Young and Old, by D. E. Smith and J. Ginsburg, New York, Columbia Univ., 1937, 52 pages. (Contributions of Mathematics to Civilization Series, no. 1.)

* This MONTHLY, vol. 44, pp. 656–7; vol. 45, pp. 44–5, 183, 245–6, 317–8, 385, 688–9.

The Living Past, A Sketch of Western Progress, by F. S. Marvin, New York, Oxford Univ. Press, fourth edition, 1920, 312 pages. A happy sketch of the history of civilization with special reference to the contributions made by mathematicians.

On Growth and Form, by D'Arcy W. Thompson, Cambridge Univ. Press, 1917, 809 pages. This beautifully written work by the veteran professor of natural history at the University of St. Andrews is filled with discussions of mathematical interest such as those concerning leaf arrangements, forms in the shell of the "chambered nautilus" (read the poem of O. W. Holmes), and the bee's cell. This volume is out of print but may be procured in libraries or second-hand book shops.

"Mathematics and Astronomy," in *The Legacy of Greece*, by T. L. Heath, edited by R. W. Livingstone, Oxford, 1922, pp. 97–136. The whole volume, by a group of specialists, is fascinating.

Isaac Newton 1642–1727, by J. W. N. Sullivan, New York, Macmillan, 1938, 295 pages. "I have long been impressed by the fact that Isaac Newton, besides being the greatest of scientific geniuses, was also one of the most singular and fascinating characters of which we have any record." From the "prefatory note" of the late gifted author who was one of the constant editorial contributors to the *London Times Literary Supplement*.

Outline of the History of Mathematics, by R. C. Archibald, fourth edition, Oberlin, O., Mathematical Association of America, 1939, 66 pages. From the time of the Babylonians (2500 B.C.) to the early nineteenth century.

CLUB REPORTS, 1938–39

Reports from some clubs contain interesting topics used for papers and discussions that might well be added to our Club Topics. The following list is representative of some of the subjects discussed at meetings of the respective organizations. This does not attempt to note the complete club program in every instance, since many duplicates would arise.

Pi Mu Epsilon, Oklahoma A. and M. College

The parallel postulate from plane geometry
A different approach to spherical trigonometry
Prime numbers,—is there a last one?
The history of zero
Some topics and problems in permutations and combinations

Mathematics-Physics Club, Haverford College

Some interesting theorems of Newton
Orders of infinity
Development of logarithms
Lightning calculation exposed
The problem of Plateau

Junior Mathematical Club, University of Chicago

Green's function
Interpolation by polynomials
The theory of lattices
The mechanics of cell deformation in biology
Finite projective geometries
Matrix equations
Transfinite numbers
Families of planes in five-dimensional space
Analytic functions of hypercomplex variables

Delta Rho, Southern Illinois Normal University at Carbondale

Life and work of Lobatchewsky
Philosophy of mathematics
Theory of aesthetic measure as applied to polygons
Algebraic fallacies
Group theory

Mathematics Club, Rutgers University

The Chordel spiral
Continued fractions
Trisection of any angle by the use of marked straight edge
Transfinite numbers
How to become an actuary

Mathematics Club, Adelphi College

The use of mathematics in art
The use of mathematics in psychology
The influence of mathematics in training for citizenship
The golden section
The history of arithmetic, algebra, geometry, trigonometry

Mathematics Club, Cooper Union Institute of Technology

Paraboloids of revolution
The method of differences
Mathematical and physical research
Prime number theory

Mathematics Club, University of Cincinnati

The use of matrices in the study of plane analytic geometry
Finite calculus
Mean values and convex functions
When is a number irrational?
Certain Diophantine problems
Flatland
The Laplace transformation

Kappa Mu Epsilon, State Teachers College, Pittsburg, Kansas

The duodecimal system
Standards of weights and measures
Repeating decimals
Computing devices
Indian knot records
Japanese mathematics
Ancient and modern operations with fractions

Pythagorean Club, Milwaukee State Teachers College

Arithmetic and emotional disturbances
Stewart's theorem
Ceva's theorem
Mathematics and fortune telling
Mathematics and World's Fair architecture

Harvard Mathematical Club

Number theory
Integration theory
What is topology?
The fundamental theorem of the integral calculus
The hypergeometric function
The rigidity of polyhedra
Map projections
Stirling's formula

Mathematics Club, Boston University

Evolution of logarithms
Dimensional equations*
Women in mathematics
Some figurate numbers
Rational triangles

Mathematics Club, Oberlin College

A certain quartic surface and its reflecting properties†
Constructions with compasses
The new 200-inch telescope
Circle inversions
Complex numbers

Mathematics Club, Hunter College of the City of New York

Applications of calculus to interest
Savings bank life insurance
Some aspects of continuity
Construction of a mortality table
Indeterminate equations
Games and puzzles

Mathematics Club, University of Buffalo

Absolute values
Relation between descriptive and analytic geometry
Consistency of axiomatic systems
The monad and the swastika
Graphical solution of the cubic
Parametric equations
Curve of pursuit

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* See Dimensional Analysis, by Bridgman.
† See this MONTHLY, vol. 33, pp. 132–36, article by A. R. Williams.

RECREATIONS

In the April number of the MONTHLY several games and recreations were given in this Department. Another is as follows:

5. *How old is Pi Delta Theta?* An interesting graphical solution to the problem "How old is Pi Delta Theta" which appeared in this department* has been submitted by Professor L. S. Shively of Ball State Teachers College.

Problem: "The combined ages of the University of Denver and Pi Delta Theta are 86 years. The University of Denver is fifteen-sixteenths as old as Pi Delta Theta will be when the University of Denver is nine-sixteenths as old as Pi Delta Theta will be when Pi Delta Theta is twice as old as the University will be when the University of Denver is twice as old as Pi Delta Theta. When was Pi Delta Theta founded, and how old is the University of Denver?"

Solution: Four different times are referred to in the problem and they are indicated by marginal numbers (1), *etc.* The abbreviations D and P are used for the University of Denver and Pi Delta Theta, respectively. The problem may be restated as follows:

The sum of the ages of D and P is 86 years.

(1) D is 15/16 as old as P will be
(2) when D is 9/16 as old as P will be
(3) when P is twice as old as D will be
(4) when D is twice as old as P.

Required the age of D and the date of founding of P.

In the following diagram (Fig. 5), (4) represents the ages of D and P in terms of an arbitrary length a taken as P's age at that time. In passing from (4) to (3), *etc.*, we observe that the difference in ages at any time is constantly represented by the length a. The remainder of the graphic work is obvious.

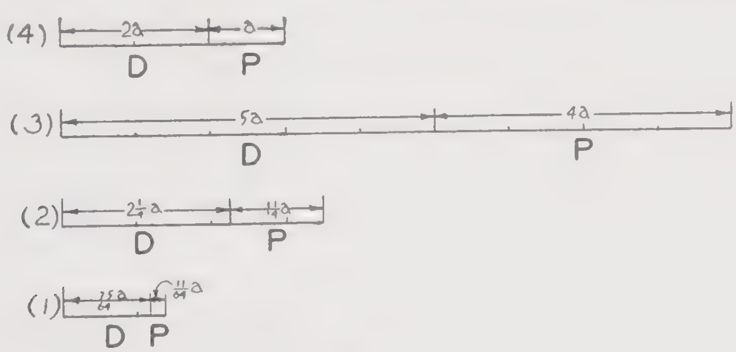

FIG. 5

Therefore $75a/64+11a/64=86$, and $a=64$.

Thus D's age at present is 75 years and P's is 11 years. Pi Delta Theta was founded in 1928.

* This MONTHLY, vol. 46, Aug.–Sept. 1939, p. 451.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 421. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres having a point in common, construct a secant through this common point, meeting the spheres again in points P, Q, R, S, so that we shall have, both in magnitude and in sign,

$$PQ:PR:PS = u:v:w,$$

where u, v, w are given.

E 422. *Proposed by J. E. Trevor, Cornell University.*

Let s_n $(n<10)$ be the sum of the first n terms of the series

$$1 + 12 + 123 + 1234 + \cdots,$$

and let S_n $(n<10)$ be the sum of the first n terms of

$$1 + 21 + 321 + 4321 + \cdots.$$

Find closed expressions for s_n and S_n, and deduce the linear relation

$$10S_n - (9n-2)s_n = n(n+1)(n+2)/2.$$

E 423. *Proposed by C. W. Trigg, Los Angeles City College.*

Squares are constructed on the sides of a right triangle ABC. Denote the centroid of the square on BC and exterior to ABC by A', and the centroid of the square on BC and "interior" to ABC by A''. Use corresponding notation for the centroids of the other four squares. Show that:

(i) the centroids of ABC, $A'B'C'$, $A''B''C''$ coincide;

(ii) $A'B'C'$ and $A''B''C''$ are never equilateral;

(iii) two vertices of $A'B'C'$ (or $A''B''C''$) fall on an altitude of $A''B''C''$ (or $A'B'C'$), and the third vertex falls on the side to which that altitude is drawn;

(iv) one side of $A'B'C'$ (or $A''B''C''$) and the altitude to that side are equal; the foot of this altitude divides the side into segments proportional to the legs of the right triangle;

(v) the sum of the areas of $A'B'C'$ and $A''B''C''$ equals one-half the area of the square on the hypotenuse;

(vi) the difference of the areas of $A'B'C'$ and $A''B''C''$ equals twice the area of ABC.

E 424. *Proposed by V. Thébault, Le Mans, France.*

If the number $123 \cdots (n-2)n$, in the scale of $n+1$, is multiplied by a two-digit number, the sum of the two digits being n, show that the digits of the product are all alike, save that the final digit may be zero.

SOLUTIONS

E 381 [1939, 297]. *Proposed by W. B. Clarke, San Jose, California.*

Show how to construct a square with one corner on each of four generally placed straight lines in a plane. How many solutions are there in general? What constitute special cases? What happens if the lines are placed askew in space?

Solution by Edward C. Phillips, Woodstock College.

This problem (in the plane) was solved trigonometrically by L. N. M. Carnot (*Géométrie de Position*, Paris, 1803, pp. 374–377), but he failed to notice that four given lines generally determine not only three but six squares. For a simpler and more complete solution we use a preliminary problem and a preliminary theorem. The problem is: to draw a square with two opposite vertices on given lines and a third vertex at a given point. The theorem is: when three vertices of a variable square move along fixed lines, the locus of the fourth vertex is a fourth line.

To solve the problem, let P be the given point, b and d the given lines. Let PL be the perpendicular from P to d. On the line through P perpendicular to PL lay off a segment PN equal to PL. Draw NM perpendicular to d, meeting b in Q. Join PQ, and let the perpendicular line through P meet d in S. Complete the rectangle $PQRS$. This will be a square, since triangles PQN, PSL are congruent. Since the segment PN can be laid off in either of two opposite directions, the problem has two solutions; the order of the vertices P, Q, R, S is clockwise for one, counterclockwise for the other. To cover the case when b and d are perpendicular, we have to allow Q, R, S to be at infinity, unless P is equidistant from b and d.

To prove the theorem, let P be the point (x, y), b and d the lines $y=mx$ and $y=0$. Then L is $(x, 0)$, N is $(x\pm y, y)$, Q is $(x\pm y, m\{x\pm y\})$, S is $(\{1\pm m\}x+\{m\mp 1\}y, 0)$, and R is $(\{1\pm m\}x+my, mx-\{1\mp m\}y)$. If P moves along a line, so that x and y are linear functions of a parameter, the coördinates of R are linear functions of the same parameter, and the locus of R is another line. The case when b and d are parallel requires separate treatment, but presents no difficulty. This theorem was proved trigonometrically by Thomas Clausen (Bull. de l'Acad. des Sciences de St. Petersbourg, vol. 7, 1864, cols. 177–181) and by C. M. Hebbert (Annals of Math., ser. 2, vol. 16, 1915, pp. 38–42, 67–71). The latter observes that if one vertex of the variable square moves along its line with constant velocity, the other three vertices also move with constant velocity, each along its own line.

We come now to the main problem: to construct a square with its vertices on four given lines a, b, c, d. First draw any two squares with three of their vertices lying in (say) clockwise order on b, c, d, respectively. (This is most easily done by taking one square with a side along b and a vertex where b meets c, and the other with a side along d and a vertex where d meets c.) Next join the fourth vertices of these two squares by a line e, thus getting the locus of the fourth vertices of the infinite set of such squares. The point P, where e meets a, gives the position of a vertex of that particular square of the system which has its vertices on all four lines. (So far, this is Clausen's solution.) Finally, we use the "preliminary problem," constructing the clockwise square $PQRS$ with Q on b, and S on d. Then R will lie on c, as required.

Since the lines b, c, d could have been so designated in any one of the six possible orders of arrangement, *six* squares can, in general, be drawn with their vertices on four given lines of a plane. To save space, we refrain from describing the special cases where there are fewer than six. Of course, if the four lines happen to be situated as b, c, d, e, above, the number of squares will be infinite. The above method is easily generalized from squares to rectangles of given shape; there are then, in general, twelve solutions.

For further literature on this and related problems, see Smith and Bryant, *Euclid*, London, 1901, p. 407; Arnold Emch, Amer. J. of Math., vol. 35, 1913, pp. 407–412; R. C. Archibald, this MONTHLY, vol. 28, 1921, p. 185; and E. C. Phillips, Bull. Amer. Ass. of Jesuit Scientists, December, 1939.

Also solved by L. M. Kelly, using the method advocated by Smith and Bryant (*op. cit.*).

Editorial Note. The corresponding problem for lines placed askew in space has, in general, no solution. For, if three vertices of a variable square describe given lines, the locus of the fourth vertex will be a curve (or straight line?) which will not, in general, meet the fourth line.

E 387 [1939, 513]. *Proposed by Thorold Gosset, Cambridge, England.*

A rabbit runs straight at a constant speed. A dog (not directly behind him) runs on an ever-changing course directly towards him, at a constant (greater) speed, until he catches him. Show that the distance traversed by the dog is the same as it would have been if he had run straight half-way to the rabbit's initial position and then straight to the point of capture.

Solution by Jack Lotsof, University of Buffalo.

Let P_1 and P_2 be the initial positions of the rabbit and the dog, O the point of capture, and M the midpoint of P_1P_2. Let $P_1P_2=\rho$, $OP_1=D$, angle $OP_1P_2=\phi$, and let the ratio of the rabbit's speed to the dog's be ϵ. Then we know that

$$D = \rho\epsilon(1 - \epsilon \cos \phi)/(1 - \epsilon^2).$$

(See A. J. Lotka, *Families of curves of pursuit, and their isochrones*, this MONTHLY, vol. 35, 1928, p. 422.) By the cosine law,

$$MO^2 = \tfrac{1}{4}\rho^2 + D(D - \rho\cos\phi) = \tfrac{1}{4}\rho^2 + \rho^2\epsilon(1 - \epsilon\cos\phi)(\epsilon - \cos\phi)/(1 - \epsilon^2)^2$$
$$= \tfrac{1}{4}\rho^2(1 + \epsilon^2 - 2\epsilon\cos\phi)^2/(1 - \epsilon^2)^2.$$

Hence

$$MO = \tfrac{1}{2}\rho(1 + \epsilon^2 - 2\epsilon\cos\phi)/(1 - \epsilon^2) = (D/\epsilon) - \tfrac{1}{2}\rho.$$

Since D/ϵ is the distance travelled by the dog, this proves the desired result.

Also solved by R. E. Gaines, L. S. Johnston (referring to E. M. Berry's solution to Problem 3573 [1933, 436]), H. D. Larsen, F. C. Smith (referring to Osgood's *Advanced Calculus*, p. 332), E. P. Starke, W. I. Thompson, F. Underwood (referring to S. L. Loney's *Dynamics of a Particle and of Rigid Bodies*, pp. 42, 43), H. H. Downing and the proposer.

E 388 [1939, 512]. *Proposed by V. Thébault, Le Mans, France.*

On the lateral surface of any right prism, find the length of the shortest route from end to end of one lateral edge, winding n times round the prism on the way.

Solution by H. D. Larsen, University of New Mexico.

Let p be the perimeter of the base of the right prism, and h the length of a lateral edge. Consider the prism cut along a lateral edge and laid flat to form a rectangle. Place n such rectangles edge to edge. Clearly, the required shortest distance is the diagonal of the resulting rectangle, *i.e.*, $(h^2+n^2p^2)^{1/2}$.

Also solved by W. E. Buker, E. P. Starke, C. W. Trigg, and the proposer.

E 389 [1939, 512]. *Proposed by C. W. Trigg, Los Angeles City College.*

Show that there is but one five-digit integer whose last three digits are alike and whose square contains no duplicate digits.

Solution by W. E. Buker, Pittsburgh, Pa.

We see immediately that the last digit is 1, 2, 4, 5, or 7. If the square contains all ten digits, it is divisible by 9. So the number itself must be divisible by 3, and the first two digits form a multiple of 3 between 32 and 100. This reduces the number of possibilities to 115. Using a four-place table of squares, we examine the first three and the last four digits of the square. Duplicate digits are found in all but nine cases: 63777, 69444, 78555, 81111, 81222, 84222, 84555, 87111, and 96222. Squaring all of these reveals the unique solution

$$(81222)^2 = 6597013284.$$

There remains the possibility that the square contains nine digits, so that the number itself begins with 10, 11, $\cdots$, or 31. Proceeding without support from "casting out nines," we try the 110 possible numbers as above, but get no more solutions.

Also solved by E. P. Starke, Alan Wayne, and the proposer.

E 390 [1939, 512]. *Proposed by H. T. R. Aude, Colgate University.*

Two freshmen work on a problem. At one point in their progress they compare their calculations and find that they have the same number N which is greater than 1. They agree to four significant figures, which are all they use. Thereupon one takes the cube root, while the other freshman divides by 3. Again they compare and find that their results agree—reading from the left—in the first, second, and fourth figures, but differ by 1 in the third figure. Find the number N.

Solution by E. P. Starke, Rutgers University.

If $x/3=x^{1/3}$ with $x>1$, we have $x=27^{1/2}=5.196$ to four significant figures. But we desire $|N/3-N^{1/3}|$ to equal .01 when the difference is correct to the third decimal place. Put $N=27^{1/2}+\delta$. Then

$$N^{1/3} = 3^{1/2} + \delta/9 + (\text{terms involving higher powers of } \delta),$$

$$N/3 = 3^{1/2} + \delta/3.$$

Thus $|N/3-N^{1/3}|=2|\delta|/9$ is approximately .01, whence δ is approximately $\pm.045$. These values give the two solutions

$$N = 5.196 + .045 = 5.241, \qquad N/3 = 1.747, \qquad N^{1/3} = 1.737;$$

$$N = 5.196 - .045 = 5.151, \qquad N/3 = 1.717, \qquad N^{1/3} = 1.727.$$

Since these calculations were approximate, other values of N, differing slightly from the above, may satisfy the conditions. By actual calculation we determine that N must be one of the above values, or differ from one of them by $\pm.001$.

Also solved by Ida M. Becker, Daniel Finkel, H. D. Larsen, Herbert Sauer, C. W. Trigg, E. E. Whitford, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3958. *Proposed by Michael Goldberg, Washington, D. C.*

What is the probability P_n of making a sequence of at least r throws of a number in n throws of a die when the probability of throwing that number in a single trial is p?

3959. *Proposed by N. A. Court, University of Oklahoma.*

The four pairs of reciprocal transversals a, a'; b, b'; c, c'; d, d' are situated, respectively, in the faces BCD, CDA, DAB, ABC of the tetrahedron $ABCD$. (1) If the lines a, b, c, d are coplanar, so are also the lines a', b', c', d' (the proposer's *Modern Pure Solid Geometry*, p. 121). (2) If the lines a, b, c, d form a hyperbolic group, so also do the remaining four lines.

3960. *Proposed by R. E. Gaines, University of Richmond.*

If a series of triangles be constructed so that the sides of each are equal to the medians of the following one, then (1) the area of each triangle is three-fourths of the area of the one following; (2) the alternate triangles are similar; (3) excluding the case of equilateral triangles, no two consecutive triangles of the series are similar.

3961. *Proposed by V. Thébault, Le Mans, France.*

Each face angle of a given trihedral angle $O-XYZ$ is $\pi/3$, and on the respective edges the points A, B, C are located. Show that the Monge point of the tetrahedron $OABC$ describes a sphere as A, B, C vary on the edges so that $OA^2+OB^2+OC^2$ remains constant.

3962. *Proposed by V. Thébault, Le Mans, France.*

A circle (C) with a given radius rolls on a fixed circle (C'). Find the form of the locus of the points of intersection of (C) with the polar of a fixed point P with respect to (C). Consider the cases where (C') reduces to a point, or a straight line.

SOLUTIONS

3869 [1938, 253]. *Proposed by Otto Dunkel, Washington University.*

Two perpendicular planes are tangent to a paraboloid and their intersection has a given direction. Determine the form of the surface generated by the intersection as the pair of planes varies. This supplements problem 3773 [1936, 111] solved 1938, page 124.

Solution by the Proposer.

A paraboloid S is a quadric surface which is tangent to the plane at infinity π_∞ at a point T_∞. There are two types: for one type S cuts π_∞ in only one real point T_∞, while for the other the intersection consists of two real straight lines, generators of S, g_∞ and g_∞', intersecting in T_∞. The intersections d of pairs of perpendicular tangent planes to S cut π_∞ in the point P, and we shall suppose that P is not at T_∞ or on any one of the two generators in the second type. The polar plane π of P with respect to S cuts S in a conic C, and it cuts π_∞ in a straight line through T_∞ which is necessarily tangent to C at T_∞. Thus C is a non-degenerate parabola, and it is also the locus of the points of contact of planes through P tangent to S. Since each d is the intersection of a pair of perpendicular tangent planes to S, each d cuts π in a point which is the intersection of tangents to C cut from π by the corresponding pair of planes. These intersec-

tions have a locus K in π, and we shall show that K is a straight line. From this it will follow that the locus of d is a plane.

The envelope of all tangent planes through P is a parabolic cylinder with C as a directrix curve. Let π' be a plane chosen perpendicular to the parallel lines d; then π' cuts from this cylinder a right section C' which is the orthogonal projection of C upon π'. Hence C' is also a parabola. The pair of tangents to C which intersect on K project into a pair of tangents to C' intersecting in a point of the projection K' of K. Since this corresponding pair of tangents to C' are perpendicular, the curve K' must be the directrix of the parabola C'; and it now follows that K must also be a straight line. In the excluded cases above the problem loses its meaning.

An analytic solution using the method of Springer in his solution of 3773 [1938, 124] is as follows. Write the equation of S in the form $ax^2+by^2-4z=0$. Then

$$l_1n_1x + m_1n_1y + n_1^2z = -\left(\frac{l_1^2}{a} + \frac{m_1^2}{b}\right)$$

is a tangent plane to S whose normal has the direction cosines l_1, m_1, n_1; and a perpendicular tangent plane has the same form with the subscripts 2. Let α, β, γ be the constant direction cosines of their intersection d, and choose the sets of direction cosines and their order so that their determinant is unity. The addition of each member of the equations of the two perpendicular planes gives as the locus of d

$$\alpha\gamma x + \beta\gamma y + (\gamma^2 - 1)z = \frac{1-\alpha^2}{a} + \frac{1-\beta^2}{b}.$$

3872 [1938, 323]. *Proposed by Ellis R. Ott, University of Buffalo.*

One man has m coins and another has n. They match coins until one player has won all the coins. Find the average number of tosses required to end the game.

Solution by George W. Petrie, State School of Mines, Rapid City, S. D.

Let M_x be the average number of tosses required to end the game if one player has x coins and the other has $m+n-x$. Thus $M_0=0$ and $M_{m+n}=0$. Also by symmetry $M_x=M_{m+n-x}$. Since after the next match the man who had x coins will either have $x-1$ or $x+1$ coins with equal probability, the following recurrence relation is deduced:

$$M_x = (M_{x-1} + 1)/2 + (M_{x+1} + 1)/2,$$

or

$$-M_{x-1} + 2M_x - M_{x+1} = 2, \quad \text{for} \quad 0 < x < m + n.$$

By summing both sides for values of x from 1 to $m+n-1$ inclusive and using

$M_0=M_{m+n}=0,$

$$M_1+M_{m+n-1}=2(m+n-1),$$

or since $M_1=M_{m+n-1}$ by the symmetry relation

$$M_1=M_{m+n-1}=m+n-1,$$

M_2 is found by the original recurrence relation for $x=1$,

$$M_2=2M_1-M_0-2=2(m+n-1)-0-2=2(m+n-2).$$

This suggests $M_x=x\ (m+n-x)$ which is proved by induction, *i.e.*, assume the relationship holds for $x=k$ and $x=k-1$,

$$M_k=k(m+n-k) \qquad \text{and} \qquad M_{k-1}=(k-1)(m+n-k+1).$$

Then

$$\begin{aligned} M_{k+1} &= 2M_k-M_{k-1}-2=2k(m+n-k)-(k-1)(m+n-k+1)-2 \\ &= (k+1)(m+n-k-1), \end{aligned}$$

or for $x=m$,

$$M_m=m(m+n-m)=mn,$$

the required average number of tosses required to end the game.

Solved also by J. P. Ballantine and the proposer.

Editorial Note. The remaining solutions obtained the same result by deriving a similar recurrence relation, but differed in the treatment of this relation. Ballantine considered also the case of three players where the result depends upon the rules of the game. The proposer stated that the problem was suggested by a similar problem in Whitworth's *Choice and Chance*, Fifth edition, p. 288, Ex. 836, where $(m, n)=(5, 6)$ and the given result is 30.

The recurrence relation may be written $\Delta^2M_x=-2$, and so its solution must be $M_x=-x^2+Ax+B$. The initial conditions then give $M_x=x\ (m+n-x)$.

3873 [1938, 323]. *Proposed by G. T. Coate, Student, Tulane University.*

Evaluate the nth order determinant whose elements are given by

$$e_{11}=e_{1j}=e_{i1}=1; \quad \text{and} \quad e_{ij}=e_{i-1,j}+e_{i,j-1}, \qquad 1<i,j\leqq n.$$

Solution by V. W. Graham, High School, Dublin, Ireland.

Denote the given determinant by D_n, the jth column by c_j, and the ith row by r_i. (1) Replace c_j by c_j-c_{j-1}, $(j=2, 3, \cdots, n)$. (2) In the resulting determinant replace r_i by r_i-r_{i-1}, $(i=2, 3, \cdots, n)$. This results in replacing e_{ij} by e_{ij}', where $e_{11}'=1$, $e_{1j}'=e_{i1}'=0$, and $e_{ij}'=e_{i-1,j-1}$, as consequences of the definition of e_{ij}. Hence $D_n=D_{n-1}=\cdots=D_1=1$.

Solved also by Norman Anning, Wallace Givens, Emma Lehmer, B. C. Moore, M. A. Scheier, F. Underwood, and the proposer.

Editorial Note. The proposer, Anning, and Moore used reductions similar to the above; Underwood used only (2) above. The remaining solutions used the fact that $e_{ij}={}_{i+j-2}C_{j-1}$. Givens wrote $e_{ij}=\sum_{r=1}^{n}{}_{i-1}C_{r-1}\,{}_{j-1}C_{r-1}$, where ${}_pC_q=0$, if $q>p$. Hence $D_n=\left|{}_{i-1}C_{r-1}\right|^2=1$, since the determinant on the right has unity for its value. Mrs. Lehmer replaced c_n by $\sum_{j=1}^{n}(-1)^{n-j}\,{}_{n-1}C_{j-1}c_j$, and this makes all elements of the last column zero except $e'_{nn}=1$. Hence $D_n=D_{n-1}=\cdots=D_1=1$. Rev. M. A. Scheier replaced r_i by $\sum_{k=0}^{i-1}(-1)^k\,{}_{i-1}C_k\,r_{i-k}$, $(i=2,\ 3,\cdots,n)$, and this reduces D_n to a determinant with a principal diagonal of unit elements and zeros below this diagonal. Hence $D_n=1$.

The proposer stated that this problem, for $n=4$, is given in Dickson's *First Course in the Theory of Equations*, p. 114, Ex. 4. Anning referred to p. 82 in Fischer, *Determinanten*, Sammlung Göschen 402.

The reduction may be put in a different form, which we state briefly. From the definition we have $\Delta e_{ij}=e_{i+1,j-1}$, and hence $\Delta^{i-1}e_{1j}=e_{i,j-i+1}$. The member on the right of this equality is zero if $j<i$, and unity if $j=i$. A determinant with elements e_{ij}, with any definition, is unchanged in value if each such element is replaced by $\Delta^{i-1}e_{1j}$, or, what is the same, if r_i is replaced by $\Delta^{i-1}r_1$, $(i=2,\ 3,\cdots,n)$. In the present case the ith row becomes $0,\ 0,\cdots,\ 0,\ 1,\ e_{i2},\ e_{i3},\cdots,\ e_{i,n-i+1}$; and the determinant has unity for its value. Or we may leave the other rows unchanged and replace r_n by $\Delta^{n-1}r_1$; and this shows that $D_n=D_{n-1}$. The explicit form of the transformation is given by $\Delta^{i-1}r_1=(U-1)^{i-1}r_1=\sum_{k=0}^{i-1}(-1)^k\,{}_{i-1}C_kU^{i-1-k}r_1=\sum_{k=0}^{i-1}(-1)^k\,{}_{i-1}C_kr_{i-k}$. This shows the relation of these reductions to the above solutions, excluding the interesting method of Givens.

3875 [1938, 323]. *Proposed by John Tom Hurt, Texas A. and M. College.*

Let $h_1,\ h_2,\cdots,\ h_k,\cdots$ be a sequence of positive numbers which tend to infinity and which are such that the sum of their reciprocals is a divergent series. Let $H_n^{-1}=(1+h_1)(1+h_2)\cdots(1+h_{n+1})$, and let the elementary symmetric functions of $h_1,\ h_2,\cdots,\ h_n$ be $B_{n,k}$, that is, $B_{n,k}$ is the sum of all products taken k at a time formed without repetitions from the first n of the h's. Prove that $\sum_{n=p}^{\infty}H_nB_{n,n-p}=1$, for $p=0,\ 1,\ 2,\ 3,\cdots$.

Solutions by the Proposer.

(I) The identity was originally obtained as follows. In the paper by Hurt and Ford entitled *Polynomial expansions in the Borel region*, Proc. Edinburgh Math. Soc., Ser. 2, vol. 5, Pt. II, pp. 82–89, it is shown, under the conditions of the problem, that an analytic function $f(z)$ may be expanded in a series of polynomials

$$f(z)=P_0(z)+P_1(z)+P_2(z)+\cdots.$$

If both sides of this equation are differentiated p times, using the explicit form of $P_n(z)$, and z is set equal to zero, we have the identity of the problem.

Once the identity was obtained it became of interest to prove it by more direct means, to regard the problem as one in algebra.

(II) Let

$$f_m(p) = \sum_{n=p}^{p+m-1} H_n B_{n,n-p}, \qquad (m = 1, 2, 3, \cdots), \tag{1}$$

so that the problem is to prove that

$$\lim_{m\to\infty} f_m(p) = 1 \qquad \text{for each} \qquad p = 0, 1, 2, \cdots.$$

Since by definition

$$0 < f_m(p) < f_{m+1}(p),$$

the existence of the limit

$$\lim_{m\to\infty} f_m(p) \leqq 1$$

follows upon showing that $f_m(p) < 1$ for all m and p. This is done by establishing that

$$f_m(p) = H_{p+m-1} \sum_{k=0}^{m-1} B_{p+m,k}. \tag{2}$$

The definition of $B_{n,k}$ gives the relation

$$H_{n-1}^{-1} = \prod_{k=1}^{n} (1 + h_k) = \sum_{k=0}^{n} B_{n,k} \tag{3}$$

and the fundamental recurrence formula

$$B_{n,k} = B_{n-1,k} + h_n B_{n-1,k-1}. \tag{4}$$

For consistency define

$$\begin{aligned} B_{n,k} &= 1 \quad \text{for} \quad k = 0, \\ &= 0 \quad \text{for} \quad k > n \quad \text{or} \quad k < 0. \end{aligned} \tag{5}$$

From (1) we have that

$$f_1(p) = H_p B_{p,0} = H_p$$

so that (2) holds for the value $m=1$. Assume that (2) is true for the value m; then from (1),

$$f_{m+1}(p) = H_{p+m-1} \sum_{k=0}^{m-1} B_{p+m,k} + H_{p+m} B_{p+m,m}$$

which is, by the definition of H_{p+m},

$$f_{m+1}(p) = H_{p+m}\left\{(1 + h_{p+m+1})\sum_{k=0}^{m-1} B_{p+m,k} + B_{p+m,m}\right\}$$
$$= H_{p+m}\left\{\sum_{k=0}^{m} B_{p+m,k} + \sum_{k=0}^{m-1} h_{p+m+1}B_{p+m,k}\right\}.$$

Using (5) we have

$$f_{m+1}(p) = H_{p+m}\sum_{k=0}^{m}(B_{p+m,k} + h_{p+m+1}B_{p+m,k-1}),$$

and from (4)

$$f_{m+1}(p) = H_{p+m}\sum_{k=0}^{m} B_{p+m+1,k}$$

which is precisely (2) with m replaced by $m+1$. Thus (2) has been established by induction.

Using (3) we may write (2) in the form

$$(6) \qquad f_m(p) = \frac{\sum_{k=0}^{m-1} B_{p+m,k}}{\sum_{k=0}^{p+m} B_{p+m,k}} = 1 - \frac{\sum_{k=m}^{p+m} B_{p+m,k}}{\sum_{k=0}^{p+m} B_{p+m,k}} = 1 - \frac{\sum_{i=0}^{p} B_{p+m,m+i}}{\prod_{k=1}^{p+m}(1+h_k)}.$$

This shows that $f_m(p) < 1$ for all m and p, and thus that

$$\lim_{m\to\infty} f_m(p) \leqq 1$$

exists for each p.

With the relation (3) we have that

$$\frac{B_{p+m,m+i}}{\prod_{k=1}^{p+m}(1+h_k)} = \prod_{k=1}^{p+m}(1+h_k^{-1})^{-1}\,\frac{B_{p+m,m+i}}{B_{p+m,p+m}}.$$

The fraction on the right is readily seen to be an elementary symmetric function of $h_1^{-1}, h_2^{-2}, \cdots, h_{p+m}^{-1}$ where the products consist of $p-i$ of the h^{-1}'s. The terms of this symmetric function are included among those of

$$\left(\sum_{k=1}^{p+m} h_k^{-1}\right)^{p-i}, \qquad (i = 0, 1, 2, \cdots, p).$$

Hence the fraction

$$\frac{B_{p+m,m+i}}{B_{p+m,p+m}} \leq \left(\sum_{k=1}^{p+m} h_k^{-1}\right)^{p-i}, \qquad (p-i = 0, 1, 2, \cdots, p).$$

Since by hypothesis $h_k \to \infty$, $h_k^{-1} < 1$ except for a finite number of the first of the h's. If $h_k^{-1} < 1$, then

$$1 + h_k^{-1} > \exp\left(\tfrac{1}{2} h_k^{-1}\right).$$

Therefore, for values of m sufficiently large,

$$\prod_{k=1}^{p+m} (1 + h_k^{-1}) > K \exp\left(\frac{1}{2} \sum_{k=1}^{p+m} h_k^{-1}\right).$$

The constant K, $0 < K \leqq 1$, takes care of the finite number of the h's greater than or equal to one. It follows that

$$\frac{B_{p+m,m+i}}{\prod_{k=1}^{p+m} (1 + h_k)} < \frac{\left(\sum_{k=1}^{p+m} h_k^{-1}\right)^{p-i}}{K \exp\left(\frac{1}{2} \sum_{k=1}^{p+m} h_k^{-1}\right)} \to 0 \quad \text{as} \quad \sum_{k=1}^{p+m} h_k^{-1} \to \infty.$$

In the expression (6) for $f_m(p)$ there are only a finite number, $p+1$, of such terms, therefore $\lim_{m\to\infty} f_m(p) = 1$, and the identity is proved.

3877 [1938, 389]. *Proposed by Louis Weisner, Hunter College, New York.*

Show that it is possible to construct a set S consisting of an infinite number of positive, and an infinite number of negative integers, which has the following properties: (1) If the integers of S are arranged in ascending order of magnitude, the difference between consecutive positive integers, and the difference between consecutive negative integers of S, tend to infinity. (2) If $f(x)$ is *any* non-constant polynomial with integral coefficients, S contains an infinite number of the integers represented by $f(x)$ for integral values of x. The problem becomes trivial if the first requirement is omitted; we could then take S as the set of all integers.

Solution by the Proposer.

The set of all non-constant polynomials with integral coefficients and positive leading coefficient is enumerable. Let

$$f_1(x),\ f_2(x),\ f_3(x),\ \cdots$$

be one enumeration. Choose x_1 so that $f_1(x_1) > 0$. Now construct the set S consisting of the integers

$$f_1(x_1),\ f_2(x_2);\ f_1(x_3),\ f_2(x_4),\ f_3(x_5);\ f_1(x_6),\ f_2(x_7),\ f_3(x_8),\ f_4(x_9);$$
$$f_1(x_{10}),\ f_2(x_{11}),\ f_3(x_{12}),\ f_4(x_{13}),\ f_5(x_{14});\ \cdots$$

and their negatives, the integers $x_1, x_2, \cdots$, being chosen so that if $f_i(x_n)$ and $f_j(x_{n+1})$ are consecutive terms, then

$$f_j(x_{n+1}) > n + f_i(x_n).$$

This is always possible as the leading coefficient of $f_j(x)$ is positive and x_{n+1} may be chosen so that $f_j(x_{n+1})$ is arbitrarily large. The set S thus constructed clearly has the required properties

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A symposium on the foundations of topology was held at the University of Notre Dame on April 10–11, 1940. The principal speakers were R. L. Moore, Karl Menger, E. W. Chittenden, J. W. Tukey, A. N. Milgram, and Solomon Lefschetz.

The National Council of Teachers of Mathematics will hold its 1940 summer meeting on July 1, 2, and 3 in Milwaukee, Wisconsin. The headquarters will be the Hotel Pfister. The theme of the meetings is: *How Mathematics Serves Our Community*. Questions and suggestions should be addressed to A. E. Katra, 525 West 120th Street, New York, N. Y.

The Division of Mathematics at Harvard University has awarded the William Lowell Putnam Memorial Scholarship for 1940 to A. M. Gleason of Yale University. This scholarship is awarded to one of the five ranking highest in the Putnam Competition for the year.

Assistant Professor D. H. Lehmer of Lehigh University has been appointed an associate professor at the University of California, Berkeley.

At Columbia University Assistant Professors B. O. Koopman and P. A. Smith have been promoted to associate professorships, and Associate Professor L. P. Siceloff has been promoted to a full professorship.

Associate Professor Helen M. Walker of Teachers College, Columbia University, has been promoted to a full professorship.

Professor Oscar Zariski of Johns Hopkins University is on leave February–October, 1940, on a Guggenheim Fellowship.

Dr. G. H. Dowker and J. L. Kelley have been appointed to instructorships at Johns Hopkins University beginning September 1940.

C. A. Lindemann, professor emeritus at Bucknell University, died April 28, 1940. He joined the staff at Bucknell University in 1902, retired in 1938. He had been a member of the Mathematical Association since 1924.

Professor emeritus F. N. Willson of Princeton University died November 15, 1939. He was a charter member of the Mathematical Association.

Professor George Sarton of Harvard University writes: "I am preparing a study on the French mathematician, Joseph Louis Lagrange (1736–1813), and

would welcome any information concerning MSS of him (letters from him or to him) in public or private libraries. I would gladly pay for photostatic copies of such MSS and the owner's courtesy would be fully acknowledged."

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The following results of the third annual William Lowell Putnam Mathematical Competition held March 2, 1940, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the readers only by number up to the time of this announcement.

The first prize, five hundred dollars, is awarded to the department of mathematics of the University of Toronto, Toronto, Ontario. The members of the team were W. J. R. Crosby, J. C. Maynard, G. H. K. Strathy; to each of these is awarded a prize of fifty dollars.

The second prize, three hundred dollars, is awarded to the department of mathematics at Yale University, New Haven, Connecticut. The members of the team were J. E. Brewster, A. M. Gleason, G. R. MacLane; to each of these is awarded a prize of thirty dollars.

For the third prize, two hundred dollars, there is a tie between the department of mathematics of Columbia University and the department of mathematics of Cooper Union Institute of Technology, both in New York, and the prize is awarded half to each department. The members of the Columbia University team were Laurence Annenberg, Julius Ashkin, Paul Marcus; the members of the Cooper Union team were Murray Klamkin, Benjamin Lax, Samuel Manson; to each of these is awarded a prize of ten dollars.

The five persons ranking highest in the examination, named in alphabetical order, were: W. J. R. Crosby, University of Toronto; A. M. Gleason, Yale University; E. L. Kaplan, Carnegie Institute of Technology; J. C. Maynard, University of Toronto; R. M. Snow, George Washington University. Each of these will receive a prize of fifty dollars. The order of the names in this list has no relation to their rank in the examination.

The following teams won honorable mention: Department of Mathematics, University of California, Berkeley, the members of the team being Julia H. Bowman, W. M. Kincaid, C. W. Lippman; Department of Mathematics, University of California at Los Angeles, Los Angeles, the members of the team being Richard Arens, Robert James, Harold Shniad; Department of Mathematics, Carnegie Institute of Technology, Pittsburgh, the members of the team being S. N. Foner, E. L. Kaplan, W. E. Stuermann.

Five individuals are given honorable mention, the names listed in alphabetical order: G. R. MacLane, Yale University; Samuel Manson, Cooper Union; Paul Marcus, Columbia University; G. H. K. Strathy, University of Toronto, J. E. Wilkins, Jr., University of Chicago.

The order of the names in both lists for honorable mention has no relation to their rank in the examination.

The following is a list of all colleges and universities which entered teams in the Competition. (This list is arranged alphabetically, and the order of the names here has no relation to the rank of the teams in the examination.) Adelphi College, University of

British Columbia, Bryn Mawr College, University of California, University of California at Los Angeles, Carnegie Institute of Technology, Colorado School of Mines, Cooper Union Institute of Technology, Columbia University, Denison University, Florida Southern College, George Washington University, Goshen College, Harvard University, Heidelberg College, College of Idaho, Iowa State College, John Brown University, Kansas State College, Lafayette College, Massachusetts Institute of Technology, College of the City of New York, New York University, Ohio State University, University of Oklahoma, University of Pennsylvania, Queen's University (Ontario), Rutgers University, St. Bonaventure College, College of St. Francis, St. Olaf College, College of St. Thomas, University of Saskatchewan, Swarthmore College, University of Tampa, University of Texas, University of Toronto, Union College (Schenectady, New York), Virginia Military Institute, University of Washington, Wayne University, Webb Institute of Naval Architecture, Western Reserve University, Whittier College, Yale University.

In addition to these forty-five teams, there were seventy-three individual contestants from these and twenty-three other institutions, making a total of two hundred eight individuals representing sixty-eight institutions.

W. D. Cairns, *Secretary-Treasurer*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va., April 20; Grove City, Pa., November 2.
Illinois, Bloomington, May 10–11.
Indiana, Richmond.
Iowa, Mt. Vernon, April 19–20.
Kansas, Wichita, March 30.
Kentucky, Lexington, April 27.
Louisiana-Mississippi Oxford, Miss., March 8–9.
Maryland-District of Columbia-Virginia, Richmond, Va., May 11.
Michigan, Ann Arbor, April 26–27.
Minnesota
Missouri, Warrensburg, April 19.
Nebraska, Omaha, May 9–11.
Northern California, Berkeley, January 27.
Ohio, Columbus, April 5.
Oklahoma
Philadelphia, November 23 or 30.
Rocky Mountain, Fort Collins, Colo., April 19.
Southeastern, Athens, Ga., March 29–30.
Southern California, Compton, March 2.
Southwestern, Tucson, Ariz., April 22–23.
Texas, Dallas, March 29–30.
Wisconsin, Milwaukee.

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics.

THE APRIL MEETING OF THE OHIO SECTION

The twenty-fifth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Friday, April 5, 1940, with an afternoon session, dinner, and evening session. Professor Wayne Dancer, chairman of the Section, presided at these sessions, assisted by Professor Lincoln La Paz, chairman of the Program Committee. Professor and Mrs. Emil Artin were guests of the Section.

Sixty-three persons registered attendance, including the following forty-two members of the Association: W. E. Anderson, Emil Artin, Max Astrachan, F. R. Bamforth, I. A. Barnett, H. M. Beatty, H. A. Bender, Henry Blumberg, J. B. Brandeberry, O. E. Brown, R. S. Burington, I. W. Burr, V. B. Caris, Rufus Crane, Wayne Dancer, C. R. Eason, T. M. Focke, Margaret E. Jones, E. M. Justin, H. W. Kuhn, A. C. Ladner, Lincoln La Paz, R. H. MacCullough, H. M. MacNeille, Florentina Mathias, C. C. Morris, Max Morris, J. R. Musselman, W. A. Patterson, Ruth M. Peters, Jesse Pierce, H. S. Pollard, S. E. Rasor, R. F. Rinehart, N. S. Risley, S. A. Rowland, S. A. Singer, C. F. Thomas, C. C. Torrance, J. H. Weaver, R. B. Wildermuth, C. R. Wylie, Jr.

The following officers were elected for the coming year: Chairman, J. R. Musselman, Western Reserve University; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; member of Executive Committee, C. R. Wylie, Jr., Ohio State University; member of Program Committee, H. S. Pollard, Miami University. A committee was appointed to investigate the possibility of some sort of coöperation with the Ohio Academy of Science. It is expected that the next meeting will be held at the Ohio State University, probably on Thursday, April 3, 1941.

The following eight papers were read:

1. "Some loci connected with a triangle" by Professor J. R. Musselman, Western Reserve University.

2. "A remark on the area of surfaces" by Dr. Paul Reichelderfer, Ohio State University, introduced by Professor Radó.

3. "Some results of a mathematical test" by Professor H. A. Bender, University of Akron.

4. "An inverse problem in the calculus of variations" by Professor W. A. Patterson, Fenn College.

5. "Products of consecutive integers equal to kth powers" by William Scott, Ohio State University, introduced by Professor La Paz.

6. "Mathematics entrance requirements in Ohio colleges today" by Professor O. L. Dustheimer, Baldwin-Wallace College.

7. "Fundamental concepts in undergraduate mathematics" by Professor Wayne Dancer, University of Toledo.

8. "Introduction of coördinates in affine geometry" by Professor Emil Artin, Indiana University, by invitation of the program committee.

In the absence of Professor Dustheimer, paper number six was presented in mimeographed summarized form by Professor Bender.

Abstracts of these papers follow:

1. This paper appears in this issue of the MONTHLY.

2. Let $L(S)$ denote the Lebesgue area of a continuous surface S: $z=f(x, y)$ defined over the unit square Q. If S_n: $z=f_n(x, y)$ is a sequence of continuous surfaces defined over Q and converging to S, for which $L(S_n)$ and $L(S)$ are finite, then McShane (*On a certain inequality of Steiner*, Annals of Mathematics, vol. 33, 1932, pp. 125–138) has shown that a necessary condition in order that $L(S_n)$ converge to $L(S)$ is that the quantity $F_n \equiv [(f_{nx}-f_x)^2+(f_{ny}-f_y)^2]^{1/2}$ converge to zero in measure. Dr. Reichelderfer showed that, if $L(S_n)$ converges to $L(S)$, then F_n exhibits a much stronger type of convergence, namely, convergence to zero with respect to every positive exponent less than 1; that is, $\iint_Q F_n^\lambda dxdy$ converges to zero for every exponent λ between 0 and 1. Since McShane (*loc. cit.*) has shown that $\iint_Q F_n dxdy$ does not generally converge to zero when $L(S_n)$ converges to $L(S)$, this result is the best obtainable in this direction.

3. A mathematical test given to 257 high school students and to 93 college freshmen showed that all classes or sections rate about the same on any given question, and the high school students just finishing algebra are only about five per cent better than the college freshmen. The same error is not repeated by different students but the mistakes are for the most part different. Professor Bender stated that the greatest number of different answers was given to $2(x-3)/(x+2)-3$, which was answered correctly by only 34% of the high school students and 23% of the college students, with 67 different answers counted from 179 students.

4. In this paper Professor Patterson considered the problem of finding the most general first order double integral variation problem of which the extremal surfaces are the integral surfaces $z=z(x, y)$ of the equation $R(p, q)r+2S(p, q)s+T(p, q)t=0$.

5. It has been shown that the equation

$$x(x+1)\cdots(x+n-1)=y^k, \qquad (x>0,\ n>1,\ y>0,\ k>1), \tag{1}$$

has no solutions in integers for the following cases: for all k if $n=2$, 3, or k; for all n if $k=2$; for $k=3$ or 5 if $4\leqq n\leqq 13$; and, finally, when any one of the factors in the left member of (1) is prime, and therefore in particular when $x=1$. It has also been shown that for any fixed k, if solutions of (1) exist at all they are finite in number. Mr. Scott showed that for all k, (1) has no solutions if $4\leqq n\leqq 13$, and that (1) has no solutions for all $k\geqq n-1$ if $14\leqq n\leqq 19$. The proof is based principally on the following:

LEMMA. *If $4\leqq n\leqq 16$, then in any set of n consecutive integers, $x, x+1, \cdots, x+n-1$, there is at least one integer which is relatively prime to all of the others of the set.*

6. Professor Dustheimer presented a fourteen-page report on college entrance requirements for mathematics, in which he summarized the replies to a questionnaire sent to the heads of mathematics departments of all the Ohio colleges and several of the colleges of Pennsylvania, Kentucky, Indiana, and

Michigan. This report revealed the following important facts: (1) Most of the colleges still require at least one year of elementary algebra and one year of plane geometry. (2) For the most part, if a student is deficient at entrance, the responsibility for making up the deficiency is placed on the student. (3) Several institutions have special courses for such students. In one or two cases, college credit is being given for such work, but in most cases only high school credit is permitted. (4) A complete analysis of replies would lead one to feel that certain pressure has been brought to bear upon the mathematics departments of some institutions in regard to this matter.

7. Emphasizing the function of mathematics in replacing hazy notions by precise ideas, Professor Dancer outlined the way in which the fundamental notions of unity, aggregate, order, and correspondence form the basis for the study of mathematics. He traced the need for the concept of infinity through the courses in secondary schools and colleges, and showed how the idea of infinite set is related to the concepts of order and correspondence. Other notions of primary importance in college mathematics, such as function, limit, number, continuity, *etc.*, depend upon the fundamental concepts. The author pleaded for greater clarity in presenting the underlying ideas of mathematics.

8. The group of dilatations of an affine geometry has an invariant sub-group, the group of all translations. If there are translations in different directions, then this sub-group is commutative. It can be shown, without the use of Desargues's theorem, that the homomorphisms of this group form a field. Professor Artin showed this as a very easy way of introducing coördinates and establishing the equations of the straight lines.

RUFUS CRANE, *Secretary*

THE ANNUAL MEETING OF THE MISSOURI SECTION

The annual meeting of the Missouri Section of the Mathematical Association of America was held at Central Missouri State Teachers College at Warrensburg on Friday, April 19, 1940, in conjunction with the annual meeting of the Missouri Academy of Science. Professor L. M. Blumenthal, Section chairman, presided during the presentation of the first four papers. Following a recess, the Secretary, Professor G. M. Ewing, presided.

The total attendance was about twenty-five including the following eleven members of the Association: V. W. Adkisson, L. M. Blumenthal, J. H. Butchart, Theodosia T. Callaway, J. E. Case, B. F. Finkel, G. M. Ewing, W. L. Graves, Lena Cole Hadley, U. G. Mitchell, F. W. Urban.

The officers elected for the coming year were: Chairman, J. E. Case, St. Louis University; Secretary, J. H. Butchart, William Woods College.

The following papers were presented:

1. "Tabulation of positive reduced binary quadratic forms" by Raymond Allen, St. Louis University, introduced by the Reverend J. E. Case.

2. "The altitude quadric of a tetrahedron" by Professor J. H. Butchart, William Woods College.

3. "Curves of the fourth harmonic of a C_n with respect to conics" by the Reverend J. E. Case, St. Louis University.

4. "On approximate cubature" by Professor G. M. Ewing, University of Missouri.

5. "Testing infinite series" by Professor W. C. Doyle, Rockhurst College, introduced by the Secretary.

6. "On the theory of matrices in a non-commutative field" by L. J. Heider, St. Louis University, introduced by the Reverend J. E. Case.

7. "On a new method of tabulation of reduced indefinite binary quadratic forms" by A. Lorenz and J. Andrews, St. Louis University, introduced by the Reverend J. E. Case.

8. "Some remarks concerning a new class of spaces" by Professor L. M. Blumenthal, University of Missouri.

Abstracts of the papers follow:

1. Mr. Allen has constructed a table of positive reduced binary quadratic forms for determinants -1 to -1000 following the method of H. N. Wright (*Quadratic Forms of a Negative Determinant*, University of California Press, 1914). Mr. Allen explained the method of constructing the table and showed a reproduction of it on film.

2. Professor Butchart's paper appears in this issue of the MONTHLY.

3. Professor Case pointed out that curves of the fourth harmonic of a given curve C_n of order n with respect to conics are curves C_{2n} of order $2n$, the transformation being birational. Properties of the C_{2n} and corresponding properties of the C_n were discussed.

4. Professor Ewing exhibited a simple generalization of the prismoid formula applicable to n-fold integrals and discussed the form which the integrand must have in order for the formula to be exact.

5. Professor Doyle discussed a method for testing infinite series for convergence. The method is applicable to the generalized Lambert series with $(a_nb_nz^{\lambda n})/(1-a_nz^{\mu n})$ as general term. Many familiar series are special cases.

6. Mr. Heider considered the theory of matrices in a quasi (*i.e.*, non-commutative) field. He explained that his development follows that of E. H. Moore but pointed out that the use of partitioned matrices simplifies the notation and the proofs.

7. Mr. Lorenz presented the joint paper. He discussed the method used for constructing the table of reduced indefinite binary quadratic forms, which involves the making of a basic table and the expansion of the entries in this table into chains. The method is believed to be much simpler than the classic one of Cayley.

8. Professor Blumenthal designated as "ptolemaic" a metric space with the property that, for each quadruple of points p_1, p_2, p_3, p_4 of the space, the determi

nant $C(p_1, p_2, p_3, p_4) = |(p_ip_j)^2|$, $(i, j = 1, 2, 3, 4)$, formed for the six distances p_ip_j determined by the four points is negative or zero. A very brief proof shows that the euclidean four-point condition $D(p_1, p_2, p_3, p_4) \geqq 0$ implies the ptolemaic inequality $C(p_1, p_2, p_3, p_4) \leqq 0$, this implication being valid even when hypotheses weaker than metricity are made on the space. An investigation of the betweenness relation in ptolemaic spaces was sketched.

G. M. EWING, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-ninth regular meeting of the Iowa Section of the Mathematical Association of America was held at Cornell College, Mt. Vernon, Iowa, on Friday and Saturday, April 19–20, 1940, in conjunction with the fifty-fourth regular meeting of the Iowa Academy of Science. Professor Henry Van Engen, chairman of the Section, presided.

The attendance was about forty-five, including the following twenty-six members of the Association: J. W. Beach, J. O. Chellevold, E. W. Chittenden, L. M. Coffin, N. B. Conkwright, Marian E. Daniells, Franc C. Earhart, H. E. Ellingson, Cornelius Gouwens, Gertrude A. Herr, Dora E. Kearney, L. A. Knowler, O. C. Kreider, Ruth O. Lane, R. B. McClenon, F. M. McGaw, J. V. McKelvey, Mrs. J. V. McKelvey, E. E. Moots, E. N. Oberg, J. F. Reilly, Fred Robertson, W. J. Rusk, Henry Van Engen, L. E. Ward, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the Section elected for 1940–41 are as follows: Chairman, Fred Robertson, Iowa State College; Vice-Chairman, H. E. Ellingson, Luther College; Secretary-Treasurer, Cornelius Gouwens, Iowa State College.

A motion was made and passed that the new chairman appoint a committee of five to make a thorough study of the status of and the need for survey courses in college mathematics. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by the host, Cornell College and the Department of Mathematics, was adopted at the business meeting. The invited address was given by Professor Roscoe Woods of the University of Iowa. The following twelve papers were read:

1. "Some functions resembling Bessel functions of order $n+1/2$" by Professor L. E. Ward, State University of Iowa.
2. "Exponential operators" by Fred Robertson, Iowa State College.
3. "The solution of certain integral equations" by Louis Garfin, State University of Iowa, introduced by Professor Oberg.
4. "A topological formula of Kuratowski" by J. W. Kitchens, State University of Iowa, introduced by Professor Chittenden.
5. "A sphere-point transformation" by Professor J. V. McKelvey, Iowa State College.
6. "Transformations of the complex plane which regularize certain differen-

tial equations" by Professor R. S. Jacobsen, Luther College, introduced by the Secretary.

7. "An application of stratified sampling" by Vernon Price, State University of Iowa, introduced by Professor Rietz.

8. "Enumeration of isomeric hydrocarbons" by Professors E. S. Allen and Harvey Diehl, Iowa State College, introduced by the Secretary.

9. "A note on a Bolzano function" by Professor Marian E. Daniells, Iowa State College.

10. "Joint life annuities on a mortality table given by Makeham's first modification of the law of Gompertz supplemented by double geometric laws" by J. S. McCollum, State University of Iowa, introduced by Professor Rietz.

11. "Report of the Joint Commission on the Place of Mathematics in the Secondary Schools" by Dr. Ruth O. Lane, University High School.

12. "Some applications of a certain elementary theorem in modern geometry" by Professor Roscoe Woods, State University of Iowa.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Ward discussed a solution of the differential equation,

$$x^2y'' + xy' + (x^2 + \mu x - n^2)y = 0,$$

where μ is a real constant and n is any one of the numbers $0, \pm 1/2, \pm 1, \pm 3/2, \cdots$ in the form of a contour integral along a closed path encircling the points 1 and -1 in the positive sense. Several deformations of the path were made and an asymptotic formula for the solution, valid when the real part of x is positive and large, was given.

2. Mr. Robertson developed the generalized Leibnitz theorem for operators $\psi(x, D)$, $\phi(x, D)$ from the theorem of the same name in ordinary calculus. The exponential form of the operators in which D means d/dx was used. The Leibnitz formula was used to prove the transfer theorem of Robert Murphy (*Theory of analytical operations*, Phil. Trans., London, 1837, pp. 179–210.), namely, $F(D)e^{\phi(x)} = e^{\phi(x)} F[D+\phi'(x)]$. The Leibnitz theorem was applied to a class of exponential operators which yields trigonometric operators. An example of the meaning of the operators sin $(x)D$, and cos $(x)D$ was given. The application of these operators to the solution of a class of infinite order differential equations was discussed, and two examples illustrating the method were given.

3. Mr. Garfin indicated how the solution of some special types of integral equations can be made to rest on the solution of a differential equation or a difference equation, depending upon the nature of the kernel in the particular problem considered. The methods for these cases were presented and their application illustrated.

4. Mr. Kitchens discussed the formula $eieiA = eiA$ established by Kuratowski for point sets, where the closure operation eA satisfies $e(A+B) = eA + eB$, $A \leqq eA$, $e^2A = eA$, $e0 = 0$ and the interior operation is defined in terms of complements by $iA = cecA$, and showed it can be extended to non-complemented lat-

tices where e and i are postulated and satisfy the conditions $e(A+B)=eA+eB$, $i(AB)=iAiB$, $iA\leqq A\leqq eA$, $e^2A=eA$, $i^2A=iA$.

5. In the transformation discussed in this paper, Professor McKelvey made the sphere defined by $x^2+y^2+z^2-2hx-2ky-2lz+m=0$ in space of three dimensions S_3 correspond to the point (h, k, l, m) in S_4. He examined the properties of the linear system of spheres $S+aS'=0$, the correspondence between circles in S_3 and lines in S_4, found the condition that two spheres intersect at a fixed angle and associated this condition with certain pole and polar properties with respect to the hyperquadric defined by $x^2+y^2+z^2-u=0$.

6. Several transformations which regularize the differential equations of motion in the restricted problem of three bodies were determined by Professor Jacobsen. The transformations were made to fulfill the same conditions as those which G. D. Birkhoff used in determining a similar transformation in an article in *Rendiconti del Circolo Mat. di Palermo*, 1915, vol. 39.

7. The standard error of a given proportion P, as calculated from a random sample, is given by the formula $\sigma_P=\sqrt{PQ/N}$. When calculated from a stratified sample, this formula is $\sigma_P=\sqrt{PQ/N-\sigma_P^2/N}$, where σ_P^2 is the variance of the proportions in the strata taken about P. To illustrate the use of these formulas, Mr. Price presented information concerning test papers written by groups of college freshman students classified according to preparation in algebra. Stratified samples were drawn from each group and the percentages of correct responses to each item calculated. The paired percentages were then composed by referring the ratio $(P_1-P_2)/\sqrt{\sigma_{P_1}^2+\sigma_{P_2}^2}$ to a normal probability table.

8. In the early thirties H. M. Henze and C. R. Blair gave the best series of studies, up to that time, of the number of isomers and of stereoisomers among certain types of hydrocarbons. In the case of alcohols $(C_nH_{2n+1}OH)$ and methanes (C_nH_{2n+2}) Professors Allen and Diehl followed recursion methods similar to those of Henze and Blair; the results are more comprehensive, in that the number of isomers with a given number of forms—differing as to right- and left-handed orientation—is obtained. A similar study of glycols $(C_nH_{2n}(OH)_2)$ yields a recursion method for their enumeration.

9. More than thirty years before Weierstrass constructed his well known continuous, non-differentiable function, Bernhard Bolzano devised a function that is continuous for all values of the independent variable but does not possess a derivative for any value of that variable. Professor Daniells explained the construction of this function and gave a demonstration of its properties.

10. Many attempts have been made to graduate the American Men table by means of Makeham's Law of Mortality but with unsatisfactory results because of the flatness of the curve of mortality in the younger ages. Mr. McCollum suggested a method of graduation by means of a double geometric law, supplementary to the Makeham law, thereby permitting the substitution of equal ages for unequal ages in the computation of joint life annuities.

11. The Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics is published

as the Fifteenth Yearbook of the National Council of Teachers of Mathematics. This report aims to bring up-to-date the epoch-making 1923 report, *The Reorganization of Mathematics in Secondary Education*. It is prepared for all those interested in the mathematics curriculum in grades seven to fourteen inclusive as well as for the classroom teacher. The reader is oriented by an introduction on "The Rôle of Mathematics in Civilization" and then for several chapters the attempt is made to look at education in a broad way and the place of mathematics in it. The curriculum plans which follow are neither revolutionary nor arbitrary, but rather present suggestions in regard to patterns which are in use in some of our schools at the present time. The aim is to leave opportunity for adaptation and experimentation. Several chapters of especial interest to teachers deal with evaluation, the problem of retardation and acceleration, analysis of mathematical needs, the transfer of training, vocabulary and equipment.

12. Professor Woods dealt with the properties of points which are isogonally conjugate with respect to a given triangle ABC. He showed that the foci of every conic inscribed in the triangle ABC are a pair of isogonally conjugate points. Special attention was called to the inscribed conic which has the circumcenter and the orthocenter as foci, since its eccentricity is $\sqrt{1-8\cos A\cos B\cos C}$. The properties of the so-called isogonal transformation were then discussed. Under this transformation, a general line is transformed into a conic circumscribing the triangle ABC; the conic being a hyperbola, a parabola, or an ellipse according as the line cuts, touches, or does not intersect the circumcircle. Diameters of the circumcircle transform into rectangular hyperbolas, each of which passes through the orthocenter. The transforms of various conics were then considered with particular reference to the Apollonian circles of the triangle ABC.

CORNELIUS GOUWENS, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The twentieth regular meeting of the Southern California Section of the Mathematical Association of America was held at Compton Junior College, Compton, California, on Saturday, March 2, 1940. Professor G. R. Livingston, chairman of the Section, presided.

The attendance was eighty-five, including the following forty members of the Association: L. J. Adams, O. W. Albert, C. K. Alexander, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, L. T. Black, Jessie R. Campbell, Myrtie Collier, G. S. Cook, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, H. H. Gaver, Harriet E. Glazier, W. H. Glenn, J. R. Gorman, W. L. Hart, G. H. Hunt, C. G. Jaeger, G. R. Kaelin, G. R. Livingston, I. P. Maizlish, A. B. Mewborn, A. D. Michal, B. C. Moore, P. M. Niersbach, W. T. Puckett, Jr., Lena E. Reynolds, J. M. Robb, G. E. F. Sherwood, D. V. Steed, W. I. Thompson, C. W. Trigg, S. E. Urner, Morgan Ward, W. M. Whyburn, B. R. Wicker, Euphemia R. Worthington.

The following officers were elected for the coming year: Chairman, O. W.

Albert, University of Redlands; Vice-Chairman, L. J. Adams, Santa Monica Junior College; Program Committee, D. V. Steed, Chairman, H. J. Hamilton, and the Secretary. The next meeting was tentatively scheduled to be held on March 8, 1941, at the University of Redlands.

The following eight papers were read:

1. "Ideal numbers defined over the field of rational numbers" by L. F. Walton, San Diego State College, introduced by Professor Livingston.

2. "Junior college mathematics" by L. J. Adams, Santa Monica Junior College.

3. "Some aeronautical applications of Heaviside's operators" by Dr. W. R. Sears, California Institute of Technology, introduced by Professor Bateman.

4. "The problem of determining the rank of certain correlation matrices" by Dr. P. G. Hoel, University of California at Los Angeles, introduced by Professor Daus.

5. "A view-point for the curriculum in secondary mathematics" by Professor W. L. Hart, University of Minnesota.

6. "Projections in Minkowski spaces" by Dr. H. F. Bohnenblust, Princeton University.

7. "Generalization of a formula involving an inverse differential operator" by Professor D. V. Steed, University of Southern California.

8. "The theorem of Jordan-Hölder" by Professor Morgan Ward, California Institute of Technology.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Walton presented a few concrete examples of the methods used by J. von Neumann in his development of the ideal numbers of Heinz Prüfer.

2. Mr. Adams called attention to the fact that comparatively little has been published concerning junior college mathematics in the educational and mathematical journals. He described a typical mathematics program common to many junior colleges, and made brief remarks dealing with each item of the program.

3. Dr. Sears described the operational treatment of the differential equation of motion of airplanes with various initial conditions and applied (control) forces, and the operational treatment of certain integral equations in the theory of airfoils in non-uniform motion.

4. The connection between the problem of determining significant factors in test analysis and the rank of certain correlation matrices was pointed out by Dr. Hoel, as well as the assumptions underlying such problems and the rôle of statistical methods in handling them.

5. In his discussion of secondary mathematics, Professor Hart placed his main emphasis on a suggestion that the "college preparatory" label for substantial secondary mathematics should be dropped and this field should be placed strongly on its own feet without college props, to aim at independent objectives in addition to its essentially automatic objectives as an introduction to college mathematics. In developing this main point, he recommended (1) a

placement division of secondary students into two (or more) groups on the basis of mathematical intelligence rather than unreliable and confusing collegiate intentions, and (2) special curricula of suitable types for the groups. The substantial $3\frac{1}{2}$- or 4-year program recommended for the upper group was consistent with all classical aims of secondary mathematics and, besides, included a one-year course in applied mathematics to present the high school students with authentic attained objectives, particularly of the socialized variety.

6. Professor Bohnenblust discussed in his address the relation between the notion of base in a Banach space and the existence of certain projections with bounded norms. He gave examples of Minkowski spaces in which no projections on more than one-dimensional sub-spaces have norm one.

7. The well known formula for finding the particular integral of the linear differential equation with constant coefficients, $f(D^2)y = \sin ax$, by applying the inverse differential operator fails in the case where $f(-a^2)=0$. Professor Steed developed a new formula which is valid when $-a^2$ is an m-fold root of $f(D^2)=0$ and which includes the old formula as a special case.

8. Professor Ward discussed the analog of the theorem of Jordan-Hölder with respect to lattices by the introduction of a non-transitive binary relation and an appropriate set of four postulates, so that the length of the chain of inclusions connecting two lattices was of precise length.

P. H. DAUS, *Secretary*

THE SPRING MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The fourteenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at West Virginia University, Morgantown, West Virginia, on Saturday, April 20, 1940. Professor J. S. Taylor, chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was forty-six, including the following twenty-two members of the Association: C. S. Atchison, O. F. H. Bert, Elizabeth F. Brown, P. N. Carpenter, H. A. Davis, L. L. Dines, H. L. Dorwart, J. A. Eiesland, H. C. Hicks, M. L. Manning, David Moskovitz, J. H. Neelley, E. G. Olds, F. W. Owens, Helen B. Owens, C. N. Reynolds, J. B. Rosenbach, J. S. Taylor, Bird M. Turner, C. H. Vehse, M. L. Vest, W. J. Wagner.

At the close of the afternoon session, the ladies of the Mathematics Department of the University served tea for the guests. A vote of thanks was extended to the staff of the University for their generous hospitality.

The Section accepted the invitation of Grove City College to hold its fall meeting there. The date of this meeting was set for November 2, 1940. The invitation from Grove City College includes an invitation to the wives of visiting mathematicians, for whom a social program is being planned.

After the opening address by President C. E. Lawall of West Virginia University, the following eight papers were read:

1. "The Westinghouse educational program in Sharon, Pennsylvania" by

M. L. Manning, Secretary, Educational Committee, and Research Engineer, Westinghouse Electric and Manufacturing Company at Sharon.

2. "A birational $T_{9,6}$ associated with the secants of the twisted cubic" by Professor H. A. Davis and A. B. Cunningham, West Virginia University.

3. "Non-involutorial space transformations associated with a $Q_{1,2}$ congruence" by A. B. Cunningham, West Virginia University, introduced by Professor Davis.

4. "Mathematical Reviews" by Professor H. L. Dorwart, Washington and Jefferson College.

5. "A projected compendium of mathematical applications" by Professor E. G. Olds, Carnegie Institute of Technology.

6. "Families of conics with a trigonometric parameter" by Professor H. L. Dorwart, Washington and Jefferson College.

7. "Any root of any number" by Elizabeth F. Brown, Arsenal Junior High School, Pittsburgh, Pa.

8. "A birational T_{10} associated with the secants of the twisted cubic" by M. L. Vest, West Virginia University.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Mr. Manning outlined a successful graduate educational program which has been in existence for the past three years in the Transformer Division of the Westinghouse Electric and Manufacturing Company in coöperation with the University of Pittsburgh. Three questions were discussed in the paper: (1) What qualities in engineers does industry demand? (2) What courses should be offered to develop these qualities? (3) What is the tangible evidence that these qualities are developed? Temperamental traits, mental characteristics and education were the qualities of engineers discussed. Suggestions for the organization of courses to develop these qualities were outlined. A knowledge of applied partial and differential equations, vector analysis, complex variable theory, operational calculus, the design and the application of transformers, applied symmetrical components, human relations in engineering, technical foreign languages, and a sound written and oral course in English were emphasized. The attainment of a rich and useful vocabulary by wide reading in scientific and humanistic articles and books was stressed. The cultivation of effective oral English by much practice in public speaking was urged. The subject-matter and special teaching devices used for these courses were discussed in detail. And the concrete results such as the increasing percentage in enrollment of engineers from year to year, the development of insight and self-reliance, the improvement in business correspondence, the stimulation of research and the advantages of the coöperation of industry and school were evaluated.

2. The authors called attention to the work of Caldarera who has discussed the most general birational transformations I and T_{15} which belong to the complex of secants of a twisted cubic. Davis and Black have discussed a T_{13} which differs from Caldarera's T_{15} in several respects. The present authors presented

a $T_{9,6}$ which is quite unlike either of the aforementioned transformations. Consider the twisted cubic r, the points of which are projectively related to the quadrics of a pencil $|F|:r$ and the planes of a pencil $|F'|$. Through a generic point P passes one quadric F of $|F|$. The line t through P and the point P_r of r associated with F, cuts the associated F' of $|F'|$ in a point P', image of P in the T thus defined.

3. Mr. Cunningham presented some space transformations associated with the congruence of lines which meet a conic r and a secant s of it. Given two projective pencils of surfaces

$$|F_{n+m+1}|:r^n s^m; \qquad |F'_{n'+m'+1}|:r^{n'} s^{m'},$$

where $n \leqq m+1$, $n' \leqq m'+1$. A point P determines an F of $|F|$, and the unique line t through P, r, s meets the associated F' of $|F'|$ in one residual point P', image (T) of P. The transformations are of three types; Case I: $n=m+1$, $n'=m'+1$; Case II: $n<m+1$, $n'<m'+1$; Case III: $n=m+1$, $n'<m'+1$.

4. Professor Dorwart outlined the activities and progress of the new periodical *Mathematical Reviews*, and exhibited copies of those numbers which have already been published.

5. Professor Olds discussed the Seventeenth Yearbook of the National Council of Teachers of Mathematics which will be entitled, *Compendium of Mathematical Applications*. It is contemplated that the principal topics of arithmetic, elementary algebra, geometry and trigonometry will be listed alphabetically, together with type problems illustrating their application in various fields, such as home economics, agriculture, biology, and architecture. A cross index by fields will enable the reader to find the illustrations drawn from any particular field. In the selection of problems the probable breadth of information and interest of students will be taken into account and references will be supplied to provide necessary background. All teachers who find any problems which exemplify the practical value of mathematics are asked to send them to the speaker, who is acting as chairman of the committee in charge of preparing the Yearbook.

6. Professor Dorwart raised the question of the nature of one-parameter families of conics whose semi-axes are trigonometric ratios, and discussed some of the more interesting of these families. This paper will be published in *The Mathematics Teacher*.

7. Miss Brown presented a method by means of which any root (or power), integral, fractional, or decimal, of any rational number may be found to any desired degree of accuracy, using only the processes of elementary arithmetic and the laws of exponents, not the binomial theorem or Horner's method. The same method may be applied to the determination of integral roots of polynomials. The process is self-correcting in that accidental errors are automatically eliminated. By an extension of the process, exponential equations of the form $a^x=b$ may be solved directly for x, leading to a method for calculating logarithms as actual powers of 10, or of any other rational base.

8. Mr. Vest considered the twisted cubic r, the points of which are projectively related to the quadrics of two pencils $|F|:r$ and $|F'|:r$. Through a generic point P passes one quadric F of $|F|$. The line t through P and the point P_r of r associated with F meets the associated F' of $|F'|$ in P_r and a residual point P', image of P in the T thus defined. DAVID MOSKOVITZ, *Secretary*

THE FEBRUARY MEETING OF THE OKLAHOMA SECTION

The regular meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the annual convention of the Oklahoma Education Association at Oklahoma City on Friday morning, February 16, 1940. Professor J. O. Hassler, chairman of the Section, presided.

Seventy-five representatives of high schools and colleges attended the meeting, including the following seventeen members of the Association: E. F. Allen, Joseph Barnett, Jr., J. C. Brixey, N. A. Court, A. H. Diamond, E. P. R. Duval, W. V. N. Garretson, H. L. Hall, J. O. Hassler, E. E. Heimann, J. E. LaFon, Dora McFarland, W. C. Randels, W. T. Short, C. E. Springer, E. B. Wedel, B. S. Whitney.

At the business session the following officers were elected: Chairman, A. H. Diamond, Oklahoma A. and M. College; Secretary, J. C. Brixey, University of Oklahoma.

The program consisted of the following four papers:

1. "On the presentation of certain fundamental concepts in elementary calculus" by Professor A. H. Diamond, Oklahoma A. and M. College.

2. "A one-parameter Lie group" by Professor E. F. Allen, Oklahoma A. and M. College.

3. "The relation of three conics with the vertices of a triangle for foci" by Professor W. T. Short, Oklahoma Baptist University.

4. "Incidence matrices in combinatorial topology" by Professor C. E. Springer, University of Oklahoma.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Diamond discussed definitions and proofs relating to limits, to derivatives, and to certain physical concepts.

2. In this MONTHLY, vol. 33, 1926, p. 31, Professor Allen obtained a formula for an involution of the nth order. It was now shown that the transformation given by the formula upon assuming that the variables are complex numbers is a representation of a one-parameter Lie group. The differential equations invariant under the group, and the path curves were studied.

3. Professor Short showed that if three ellipses have two vertices of a triangle for foci and pass through the third vertex, then the line of intersection of two of the ellipses is the line joining the symmetric of the orthocenter with respect to the circumcenter to the excenter corresponding to the common vertex of the two ellipses.

4. In his expository paper Professor Springer explained how to calculate the homology groups of a finite simplicial complex by means of incidence matrices.

J. C. BRIXEY, *Secretary*

"A PARADOX, A PARADOX, A MOST INGENIOUS PARADOX"*

L. M. BLUMENTHAL, University of Missouri

1. Introduction. Frederic, the simple, artless hero of Gilbert and Sullivan's operetta, *Pirates of Penzance*, shared with most individuals an interest in paradoxes. Indentured when a lad to a *pirate* instead of to a *pilot* by a nurse somewhat hard of hearing, Frederic has attained his majority when the play opens and believes himself free to abandon the vile life to which he was so inadvertently delivered. But a technicality dashes his hopes. For the luckless boy was born on the twenty-ninth of February and the indentures bound him to the pirates until his twenty-first birthday. Confronted with this news, it is the paradox involved rather than the wrecking of his plans that engages his attention, and he exclaims:

"How quaint the ways of paradox!
At common sense she gaily mocks!
Though counting in the usual way,
Years twenty-one I've been alive,
Yet, reckoning by my natal day,
I am a little boy of five!"

Are there mathematical theorems that also mock at common sense? Are there paradoxes in mathematics, and, if so, can some of the most remarkable of them be appreciated by those who are without an extensive mathematical education? These questions admit, I believe, an affirmative answer, but first let us be quite clear about the word "paradox." This word is derived from $\pi\alpha\rho\alpha$ "contrary to" and $\delta\acute{o}\xi\alpha$ "opinion," and I shall use it in almost exactly this sense. More precisely, an assertion will be called a paradox provided it is *seemingly* contradictory or *opposed to common sense.* Thus, a paradox may be (1) a true statement that seems to be false, or (2) a false statement that seems to be true. This makes, to be sure, the labelling of an assertion as paradoxical a purely subjective matter—a kind of test of an individual's sophistication. This is as it should be. To the omniscient there are no paradoxes!

An *effective* paradox of type (1) is a true statement that "most" people would unhesitatingly pronounce to be false, if required to give an immediate opinion

* Presented to the Mathematics Section of the Missouri Academy of Science, by invitation, at Springfield, Missouri, April 29, 1939.

concerning it, since it appears to violate previously held conceptions. These remarks are illustrated in the examples given in this paper. Some, perhaps, will be paradoxical only to the mathematically immature, while, on the other hand, some are considered paradoxes by the most mature mathematicians. Even a clear understanding of the proofs of these latter paradoxes does not entirely deprive them of their astonishing character.

2. Paradoxes for the naïve. All of the paradoxes mentioned in this paper derive their paradoxical nature from the fact that they seem to contradict common sense notions concerning the *congruence of figures*. Now anyone who has had high school geometry knows what is meant by saying that two figures (in the plane or in space) are congruent; namely, that the figures may be made to coincide, point for point, by superposition. Thus, two figures are congruent in the sense of elementary geometry when one can be *moved rigidly* so as to coincide with the other. Whenever two figures (*i.e.*, two sets of points) are called congruent in this paper that is exactly what is meant, for all the point sets to be considered will be part of the line, plane, or three space. Since each rigid motion may be accomplished by a rotation which leaves a given point (the origin) fixed and a translation (or either alone), it follows that two point sets in euclidean space are congruent if and only if one set may be "sent into" the other by rotating it about a fixed point and then translating it, or by either of these two operations alone.

Evidently each figure is congruent with itself, but *certain figures are congruent with parts of themselves* as well.* This seems rather surprising if the word "figure" makes one think of those figures (*e.g.*, line, triangle, parallelogram, pyramid, sphere) treated in elementary geometry, for no one of these figures is congruent to a part of itself. But consider the figure consisting of an arbitrarily selected point A of a horizontal line L, together with all points of L to the right of the point A. Call this figure R_1 and let B be a point of R_1 distinct from A. Then the figure R_2 consisting of B and all points of R_1 to the right of B is clearly a part of R_1 and $R_1 \approx R_2$, since R_1 can be made to coincide with R_2 by translating R_1 along the line L so that A coincides with B.†

As soon as the notion of a figure is extended to include sets like R_1 it no longer seems remarkable that a figure may be congruent with a part of itself. Other examples (*e.g.*, the analogs of R_1 and R_2 in the plane) will doubtless occur to the reader. It may be observed that the sets R_1, R_2 do not have *lengths* (linear measures), and the analogs of these sets in the plane do not have *areas* (plane measures). We shall see that this lack of *measure* is a peculiarity of some of the sets that enter into the paradoxes to be given later.‡

* Throughout this paper, "part" is used in the sense of "proper part."

† The notation $R_1 \approx R_2$ is used to denote the congruence of these two sets. If cartesian coördinates are introduced in L, then R_1 is the closed ray $x \geqq a$ and R_2 is the closed ray $x \geqq b$, where a, b are the coördinates of A, B, respectively.

‡ In this paper, *measure* always refers to (finite) *Lebesgue measure*. A brief discussion of the notion is given in Section 4.

It is natural now to ask whether there is any *bounded* sub-set of the line congruent to a part of itself. It may be shown that *such sets do not exist.* On the other hand, bounded sub-sets of the plane with this paradoxical property are easily described. Let θ denote an angle whose radian measure is incommensurable with π (*e.g.*, θ may be one radian), and mark the points of the plane with polar coördinates $(1, n\theta)$, $(n=0, 1, \cdots)$. The set A of points so obtained is situated on the circumference of the unit circle with center at the origin, and hence A is surely bounded. The sub-set B of A obtained by deleting the point $(1, 0)$ from set A is congruent with A, for a (counter-clockwise) rotation of A about the origin through the angle θ evidently makes this set coincide with its sub-set B.

Though, as remarked above, no bounded sub-set of a line is congruent to a part of itself, there is a paradoxical property of a line segment (a bounded portion of a line consisting of all points of the line between and including two points) that is of much interest. Obviously no segment can be sub-divided into infinitely many non-overlapping congruent segments, but *every segment S* (and it is immaterial whether two, one, or no end-points of S are included) *can be decomposed into a denumerable infinity of sub-sets $S_1, S_2, \cdots, S_n, \cdots$ such that no two of the sets $S_i, S_j, (i\neq j)$, have points in common, and $S_i \approx S_j, (i, j=1, 2, \cdots)$.** This decomposition, which derives its paradoxical character from the fact that the sets S_i are *denumerably infinite* (such a decomposition of S into *uncountably* many sets could evidently be obtained by letting the sets consist of single points of S, while a decomposition of S into a *finite* number of such sub-sets offers no difficulty) is effected with the use of the axiom of choice, to be discussed in section 6. It may be noted here that no one of the sets S_i has a measure, though each S_i is, of course, bounded. We shall verify this later.

3. The Sierpiński-Mazurkiewicz paradox. In the examples offered in the preceding section, we saw how certain sets A may be divided into two parts, $A=A_1+A_2$, such that† $A_1\neq 0$, $A_2\neq 0$, $A_1\cdot A_2=0$, and $A\approx A_1$. What of the remaining part A_2? Beyond assuring ourselves that it was not empty, so that A_1 is a *part* and not the whole of A, we disregarded it. Let us now take it into consideration and ask whether it too might be congruent to A! That is, does there exist a set A which may be decomposed into two sets A_1, A_2 which have no points in common, are congruent to each other and to the set A? This question, proposed by the Polish mathematician W. Sierpiński, was answered affirmatively by his fellow countryman S. Mazurkiewicz, who defined such a set A and the sub-sets A_1, A_2 in the following way:‡

The set A consists of a point O of a plane together with all points of the plane derivable from O by the operations of (1) rotation (counter-clockwise) about O through one radian and (2) translation through one unit in a fixed direction.

* J. v. Neumann, Die Zerlegung eines Intervalles in abzählbar viele kongruente Teilmengen, Fundamenta Mathematica, vol. 11, 1928, pp. 230–238. Translations carry S_i into S_j.

† That is, A_1, A_2 are non-empty sets without common points.

‡ S. Mazurkiewicz and W. Sierpiński, Sur un ensemble superposable avec chacune de ses deux parties, Comptes Rendus de l'Académie des Sciences, Paris, vol. 158, 1914, pp. 618–619.

(These operations are applied to O and to each point obtained from O by their use.) Take O, for example, as the origin of a cartesian coördinate system in the plane, and perform the translation in the direction of the positive x-axis. A point P of A belongs to A_1 provided the final operation in the sequence of operations yielding the point P is the operation (1) of rotation; in the contrary case, P belongs to the set A_2.

Evidently, $A=A_1+A_2$ and $A_1\neq 0$, $A_2\neq 0$. But $A_1\cdot A_2=0$, for it is easy to show that it is not possible for a point P of A to be obtained by two sequences of operations, one of which has operation (1) for its final operation, while the other terminates with operation (2).* If, now, the set A be rotated about O through one radian, then the rotated set is seen to coincide with the sub-set A_1, for if P is any point of A, the rotation transforms it into a point of A_1, and each point of A_1 is obviously a point of the rotated set. Hence $A\approx A_1$. Similarly, if A is translated one unit to the right, then the translated set coincides with A_2. Hence we have $A_1\approx A_1+A_2\approx A_2$. It is observed that the definitions of the sets A, A_1, A_2 do not involve the axiom of choice.

4. The Hausdorff paradox. We have mentioned several times the measure of a set. To the reader who is unacquainted with this notion it suffices to remark that the concept generalizes the familiar notions of length, area, volume (applied to figures in elementary geometry) to complicated sets for which these intuitive notions of content are hardly applicable. Denoting the measure of S by $m(S)$, we list some important properties of this function: (1) $m(S)\geqq 0$ for each set S that has a measure; (2) $m(S_1)=m(S_2)$ if $S_1\approx S_2$; (3) the measure of a sum (finite or denumerably infinite) of sets equals the sum of their measures provided the sets have, pairwise, no elements in common. (It is supposed in property (3) that the measures of all sets concerned exist.)

Though the notion of measure has a large domain of applicability, there are bounded euclidean sub-sets—even sub-sets of the line—to which it does not apply. It was remarked in Section 2 that the mutually exclusive, pairwise congruent sets $S_1, S_2, \cdots, S_n, \cdots$ into which the segment S was decomposed are not measurable. This is verified at once by noting that an assumption to the contrary is easily shown, by the properties listed above, to lead to a contradiction.

Now the set A defined in Section 3 is denumerable and hence has zero measure. It follows that $m(A_1)=m(A_2)=m(A)=0$. With the aid of the axiom of choice, S. Ruziewicz has shown how a non-denumerable plane set may be "constructed" which enjoys the paradoxical property of set A. Though this set is non-denumerable, a brief consideration shows that it cannot have positive measure; indeed, plane sets of positive measure admitting such a decomposition do not exist. For suppose A is such a set. Then $m(A)>0$, and there are sub-sets A_1, A_2 of A such that $A=A_1+A_2$, $A_1\cdot A_2=0$, and $A_1\approx A\approx A_2$. Then from the

* An assumption that A_1 and A_2 have a point in common implies that e^i satisfies an algebraic equation with integral coefficients. This is impossible since e^i is transcendental.

properties of the measure function it follows that $m(A_1)=m(A_2)=m(A)$, and $m(A)=m(A_1)+m(A_2)$. But these equalities imply $m(A)=0$.

Thus a plane set of positive measure cannot be the sum of *two* mutually exclusive congruent sub-sets with each sub-set congruent to the sum of the two sub-sets. This suggests the following question: *Can a bounded plane set of positive measure be decomposed into three pairwise mutually exclusive and congruent sub-sets, each of which is congruent to the sum of the other two?* The argument used above may be applied to answer this question in the negative provided one is assured that the three sets whose sum forms the given set have measures. Each of these sets is, of course, bounded but, as we have seen, this does not imply that they are measurable.

The answer to our question is furnished at once, however, by results due to S. Banach, who proved the following highly important theorem:*

THEOREM OF BANACH. *To each bounded sub-set A of a euclidean space of one or two dimensions a real non-negative number $f(A)$ may be attached such that* (1) *if $A \approx B$, then $f(A)=f(B)$*; (2) *if $A \cdot B=0$, then $f(A+B)=f(A)+f(B)$*; (3) *if A is measurable then $f(A)=m(A)$.*

Suppose, now, that S is a bounded plane set of positive measure, and A, B, C are sub-sets of S such that $S=A+B+C$, $A \cdot B=B \cdot C=C \cdot A=0$, $A \approx B \approx C$, and $A \approx B+C$. Then, according to the Banach theorem, $m(S)=f(S)=f(A)+f(B)+f(C)$, $f(A)=f(B)=f(C)$, $f(A)=f(B+C)=f(B)+f(C)$. It follows readily that $2f(A)=m(S)=3f(A)$; *i.e.*, $m(S)=0$, contrary to the assumption that S has positive measure.

If we abandon the plane and consider the question of the existence of such sets in three space, quite a different situation is encountered. The Banach theorem, by virtue of which we easily showed that such plane sets do not exist, is not valid if the dimension of the euclidean space exceeds two—indeed, the theorem fails even for sub-sets of the surface of a sphere. Of course, the fact that the Banach theorem cannot be applied to contradict the existence of such sets does not imply that these sets actually exist. The fact that there really are sets in three space, bounded and of positive measure, admitting this paradoxical decomposition was shown by F. Hausdorff and constitutes the following celebrated paradox:†

HAUSDORFF PARADOX. *The surface S of a sphere may be decomposed into three pairwise mutually exclusive and congruent sets, each of which is congruent with the sum of the other two.*

The surface S has positive measure $4\pi r^2$, where r is the radius of the sphere,

* S. Banach, Sur le problème de la mesure, Fundamenta Mathematica, vol. 4, 1923, pp. 7–33.

† F. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914, pp. 469–472. In the example as given by Hausdorff the sets A, B, C fill out *S except for a denumerable set D*. It has, however, been shown by Banach and Tarski (see following footnote) that this denumerable set D, whose presence detracts a little from the extraordinary character of the example, may be dispensed with.

but no one of the three constituent sets has a measure. The remarkable decomposition of the spherical surface S given by Hausdorff (with the aid of the axiom of choice) is astonishing to the sophisticated mathematician as well as to the naïve amateur.

5. The Banach-Tarski paradox.

"We've quips and quibbles heard in flocks,
But none to match *this* paradox!"

The pinnacle of paradoxes concerning the notion of congruence is achieved by results due to Banach and Tarski.* According to these mathematicians, two figures (sets of points) A and B are *equivalent by finite decomposition* provided sets $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ exist with the following properties:

(1) $A = A_1 + A_2 + \cdots + A_n$; $B = B_1 + B_2 + \cdots + B_n$,

(2) $A_k \cdot A_j = B_k \cdot B_j = 0$, $1 \leqq k < j \leqq n$,

(3) $A_k \approx B_k$, $1 \leqq k \leqq n$.

The notion of equivalence by denumerable decomposition is defined in an analogous manner.

Let T_1 be a triangle with area 1000 square inches, and T_2 a second triangle with area 1 square inch. Are these two triangles equivalent by finite decomposition? Can each be cut into the same (finite) number of "pieces" such that corresponding "pieces" may be made to coincide after appropriate rigid motions? Even though the term "piece" is a little misleading, the reader will be correct if he allows his intuition to answer the question in the negative. Indeed, it is easy to show with the aid of the Banach theorem that if A and B are two bounded sub-sets of the line or the plane which are measurable, and A, B are equivalent by finite decomposition, then the measure of A equals the measure of B. On the other hand, if two polygons A and B have the same area they are easily seen to be equivalent by finite decomposition. Hence two polygons are equivalent by finite decomposition if and only if they have the same area.

On the other hand, the triangles T_1 and T_2 are equivalent by denumerable decomposition. In fact, *any two arbitrary sub-sets (bounded or not) of any euclidean space are equivalent by denumerable decomposition provided they contain interior points.*

Let us consider the analog in three space of the question answered above concerning the triangles T_1, T_2. Is a tetrahedron T_1 of 1000 cubic inches equivalent by finite decomposition to a tetrahedron T_2 of 1 cubic inch? Here the answer suggested by intuition is false, for *the two tetrahedra are actually equivalent by finite decomposition*! This is a consequence of the famous

BANACH-TARSKI PARADOX. *In any euclidean space of dimension $n > 2$, any two arbitrary bounded sets are equivalent by finite decomposition provided they contain interior points.*

An analogous statement is valid for sub-sets of a sphere (surface).

* S. Banach and A. Tarski, Sur la décomposition des ensembles de points en parties respectivement congruentes, Fundamenta Mathematica, vol. 6, 1924, pp. 244–277.

Consider, for example, an orange O and the earth E. If the paradoxical decomposition insured by the above theorem is given by $O=O_1+O_2+\cdots+O_n$, $E=E_1+E_2+\cdots+E_n$, then, since $O_i\approx E_i$ for every index $i\leqq n$, it follows that the parts $O_1, O_2, \cdots, O_n$ may be made, by *rigid motions*, to fill out the whole earth. The other way of stating this is no less astonishing; that is, the n parts $E_1, E_2, \cdots, E_n$ into which the entire earth has been divided may be displaced *rigidly* so as to fit into the orange—each two parts remaining mutually exclusive. This fact justifies, perhaps, an oft-repeated assertion regarding the smallness of the world!

The reader may draw additional entertaining consequences from this surprising theorem, whose demonstration makes use of the axiom of choice. It can be shown, for example, that the solid sphere of radius 1 may be decomposed into nine pairwise mutually exclusive sets such that, after carrying out certain rotations, the first *five* of these sets as well as the last *four* of them fill out a solid unit sphere.

6. The axiom of choice. The more important of the paradoxes discussed in the preceding sections involve, as remarked in each appropriate case, the axiom of choice. What is this axiom, and why should its use in proving a theorem call for a special declaration of the fact? The axiom of choice, announced by E. Zermelo in 1904, is the following assertion:

If M is any collection of pairwise mutually exclusive, non-empty sets P, there exists at least one set N that contains one and only one element from each of the sets P of the collection M.

Since the set N can evidently be formed if one may choose an element from each of the sets P of M, the axiom asserts that such a choice may be made. This aspect of the assertion accounts for the name, "axiom of choice."

Now this assertion that seems so very simple (indeed, so "self-evident" that the uninitiated would hardly trouble to formulate it at all) has been the subject of lively—and at times, acrimonious—debate among mathematicians of the first rank. Some mathematicians affirm that they do not understand what this axiom says. Their difficulty is caused principally by the word "exist." The statement asserts the existence of a certain set, but what is meant by the *existence* of a mathematical entity?* Nothing is said in the assertion about ways of finding the set N or even that it is possible to find it. The statement is merely that the set N exists!

Those mathematicians (*idealists*) who attach a meaning to the Zermelo statement make no attempt to define "existence." To say that the set N exists means simply that it exists. The notion of existence is not reducible to simpler concepts; it is not to be explained by any clearer terms. On the other hand, those who find the axiom of choice meaningless (the *realists*) say, in effect, that one must deny (at least for mathematical purposes) the "existence" of a set of objects if one knows, in advance, that there is no possible way of ascertaining

* We are ignoring difficulties due to the notion of *set*.

its members. Thus, from this view-point, a class of entities is a fit subject for mathematical investigation if and only if a *rule of construction* is given for the class. To give a method of construction is, for these individuals, the sole means of demonstrating "existence" in a mathematical sense. Propositions dealing with sets, objects, *etc.* whose claims to "existence" are not validated by the stern, uncompromising criterion of construction are metaphysical rather than mathematical. It is worth remarking that the realists do not deny the axiom of choice; they say that the assertion is, to them, incomprehensible, meaningless.

It should be clear now why it is desirable to properly tag those theorems whose proofs involve the axiom of choice. For the realists such theorems have not been established at all, while even for the idealists the question arises whether such theorems can be proved without the use of the axiom and hence be made acceptable to their feet-on-the-ground colleagues of the other school.*

Finally, let us consider briefly, by means of some examples, the difference between the axiom of choice (a pure existence axiom) and its effective realization. If infinitely many pairs of shoes were piled in a heap, no axiom of choice is needed to establish the existence of a set of shoes containing one and only one shoe from each pair. We need not assume the existence of such a set because the set may be effectively constructed by a simple *rule of procedure* according to which all *right* shoes are selected from the heap. But suppose infinitely many pairs of stockings (alike as to size, color, *etc.*) were piled in a heap. Does there exist a set containing one and only one stocking from each pair? According to the axiom of choice, such a set does exist, and in this case the axiom is indispensable for there evidently is no possibility of giving a rule by which the set may be constructed.

A more mathematical example may be in order. Consider the set of all infinite sequences of real numbers and arrange this set into classes by putting in the same class all sequences that differ only in the order of their terms. We have, then, a collection of classes which are non-empty and pairwise mutually exclusive. Without the aid of the Zermelo axiom there is no way of demonstrating the existence of a set N containing one and only one sequence from each of these classes. Such a set exists only by fiat.

* Questions raised concerning theorems whose proofs involve, directly or indirectly, the Zermelo assertion are somewhat reminiscent (if one may indulge in over-simplication) of the demands of ancient geometers according to which geometrical constructions were "allowable" only if they were accomplished with no instruments other than the ruler and compass.

SOME LOCI CONNECTED WITH A TRIANGLE*

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1. Introduction. If we take any point P in the plane of the triangle $A_1A_2A_3$ and determine the points B_1, B_2, and B_3 such that P is the midpoint of A_iB_i, $(i=1, 2, 3)$, the circles $B_2B_3A_1$, $B_3B_1A_2$, and $B_1B_2A_3$ will intersect on the circumcircle of $A_1A_2A_3$. In the study of the triangle it is convenient to use the circumcircle as the base circle and to let the coördinates of the vertices A_i be turns t_i, *i.e.*, complex numbers whose moduli are unity. We consider the t_i as roots of the equation $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$, *i.e.*,

$$\sigma_1 = t_1 + t_2 + t_3, \qquad \sigma_2 = t_1t_2 + t_2t_3 + t_3t_1, \qquad \sigma_3 = t_1t_2t_3.$$

The conjugates of these elementary symmetric functions of the t_i are

$$\bar{\sigma}_1 = \sigma_2/\sigma_3, \qquad \bar{\sigma}_2 = \sigma_1/\sigma_3, \qquad \bar{\sigma}_3 = 1/\sigma_3.$$

If we let p be the coördinate of the point P, then the coördinate of B_2 is $2p-t_2$, and the equation of the circle $B_2B_3A_1$, in parametric form is

$$(2p - \sigma_1 + 2t_2t_3\bar{p} - t_2t_3/t_1)x$$
$$= 2(2p^2 - \sigma_1 p + \sigma_3\bar{p}) - [4p^2 - 2(\sigma_1 + t_1)\, p + t_1\sigma_1 + t_2t_3]T.$$

This circle cuts the circumcircle of $A_1A_2A_3$ at A_1 and at a point D whose coördinate is

$$d = \frac{2p^2 - \sigma_1 p + \sigma_3\bar{p}}{2\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p}. \tag{1.1}$$

Since the coördinate of D is symmetrical in t_i, the circles $B_3B_1A_2$ and $B_1B_2A_3$ are likewise on D. Further the circles $A_2A_3B_1$, $A_3A_1B_2$, and $A_1A_2B_3$ meet on the circle $B_1B_2B_3$ [1] at a point D' such that P is the midpoint of DD'.

If we ask for the locus of the point P so that the point D should be the point of Feuerbach for the tangential triangle of $A_1A_2A_3$ we must set

$$\frac{2p^2 - \sigma_1 p + \sigma_3\bar{p}}{2\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p} = \frac{\sigma_2}{\sigma_1}.$$

Whence we obtain

$$2\sigma_1 p^2 - (\sigma_1^2 + \sigma_2)p = 2\sigma_2\sigma_3\bar{p}^2 - (\sigma_2^2 + \sigma_1\sigma_3)\bar{p}.$$

If we call the triangle formed by the midpoints of $A_1A_2A_3$ the medial triangle of $A_1A_2A_3$, the transformation $x=\sigma_1-2y$ sends the triangle $A_1A_2A_3$ into the medial triangle and sends the hyperbola of Jerabek [2] of $A_1A_2A_3$, namely

$$\sigma_1 x^2 + (\sigma_2 - \sigma_1^2)x = \sigma_2\sigma_3\bar{x}^2 + (\sigma_1\sigma_3 - \sigma_2^2)\bar{x},$$

* Read before the Ohio Section of the Mathematical Association of America, April 5, 1940

into

$$2\sigma_1 y^2 - (\sigma_2 + \sigma_1^2) y = 2\sigma_2\sigma_3 \bar{y}^2 - (\sigma_1\sigma_3 + \sigma_2^2)\bar{y};$$

whence we have the theorem: *If P be any point on the hyperbola of Jerabek of the medial triangle of $A_1A_2A_3$ and B_1, B_2, and B_3 be the symmetric points of A_1, A_2, and A_3 respectively as to P, the circles $B_2B_3A_1$, $B_3B_1A_2$, and $B_1B_2A_3$ all meet on the circle $A_1A_2A_3$ at the point of Feuerbach for the tangential triangle of $A_1A_2A_3$. Also the circles $A_2A_3B_1$, $A_3A_1B_2$, and $A_1A_2B_3$ meet at a point D' and as P traverses the hyperbola of Jerabek of the medial triangle of $A_1A_2A_3$, the point D' has for its locus the hyperbola of Jerabek of $A_1A_2A_3$.*

If we ask for the locus of the point P so that the point D will be the Steiner point [**2**, page 451] of the triangle $A_1A_2A_3$ we set

$$\frac{2p^2 - \sigma_1 p + \sigma_3\bar{p}}{2\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p} = \frac{\sigma_3(\sigma_1^2 - 3\sigma_2)}{\sigma_2^2 - 3\sigma_1\sigma_3},$$

obtaining

$$2(\sigma_2^2 - 3\sigma_1\sigma_3)p^2 + (2\sigma_1^2\sigma_3 - \sigma_1\sigma_2^2 + 3\sigma_2\sigma_3)p$$
$$= 2\sigma_3^2(\sigma_1^2 - 3\sigma_2)\bar{p}^2 + \sigma_3(2\sigma_2^2 - \sigma_1^2\sigma_2 + 3\sigma_1\sigma_3)\bar{p}.$$

One can easily verify that this is the hyperbola of Kiepert [**2**, page 442] of the medial triangle of $A_1A_2A_3$, whence we have the theorem: *If P be any point on the hyperbola of Kiepert of the medial triangle of $A_1A_2A_3$ and B_i be the symmetrics of A_i respectively as to P, the circles $B_2B_3A_1$, $B_3B_1A_2$, and $B_1B_2A_3$ all meet on the circle $A_1A_2A_3$ at the Steiner point of the triangle $A_1A_2A_3$. Also the circles $A_2A_3B_1$, $A_3A_1B_2$, and $A_1A_2B_3$ meet at a point D' and as the point P traverses the hyperbola of Kiepert of the medial triangle of $A_1A_2A_3$, the point D' has for its locus the hyperbola of Kiepert of $A_1A_2A_3$.*

To associate a definite equilateral hyperbola of the medial triangle of $A_1A_2A_3$ with each point D of the circumcircle of $A_1A_2A_3$ we note that the isogonal conjugate of the diameter OT, which passes through a point T on the circumcircle, is the equilateral hyperbola [**3**]

$$T^2x^2 + (\sigma_3 - \sigma_1 T^2)x + \sigma_2 T^2 = \sigma_3\bar{x}^2 + \sigma_3(T^2 - \sigma_2)\bar{x} + \sigma_1\sigma_3.$$

The transformation $x = \sigma_1 - 2y$ sends this into

$$2y^2 - \left(\sigma_1 + \frac{\sigma_3}{T^2}\right)y = 2\frac{\sigma_3^2}{T^2}\bar{y}^2 - \sigma_3\left(1 + \frac{\sigma_2}{T^2}\right)\bar{y}. \tag{1.2}$$

Now if we set

$$\frac{2p^2 - \sigma_1 p + \sigma_3\bar{p}}{2\sigma_3\bar{p}^2 - \sigma_2\bar{p} + p} = d,$$

we obtain

$$(1.3) \qquad 2p^2 - (\sigma_1 + d)p = 2d\sigma_3\bar{p}^2 - \sigma_3\left(1 + \frac{\sigma_2 d}{\sigma_3}\right)\bar{p}.$$

The equations (1.2) and (1.3) will be identical if $\sigma_3/T^2 = d$. Hence *if P be any point on the isogonal conjugate hyperbola H of the diameter $dx - \sigma_3\bar{x} = 0$ for the medial triangle of $A_1A_2A_3$ and B_i be the symmetrics of A_i respectively as to P, the circles $B_2B_3A_1$, $B_3B_1A_2$, and $B_1B_2A_3$ all meet on the circle $A_1A_2A_3$ at the point D. Also the circles $A_2A_3B_1$, $A_3A_1B_2$, and $A_1A_2B_3$ meet at a point D' and as P traverses the hyperbola H for the medial triangle of $A_1A_2A_3$, the point D' has for its locus the hyperbola H of the triangle $A_1A_2A_3$.*

2. The line of images. The author has shown [3, page 424] that if P be any point in the plane of the triangle $A_1A_2A_3$, with images C_1, C_2, C_3 respectively in the sides A_2A_3, A_3A_1, A_1A_2, then the circles $C_2C_3A_1$, $C_3C_1A_2$, and $C_1C_2A_3$ will meet on the circle $A_1A_2A_3$ at the point N whose coördinate is

$$n = \frac{\sigma_2 - \sigma_3\bar{p}}{\sigma_1 - p}.$$

This point has for its line of images the join PH, where H is the orthocenter of triangle $A_1A_2A_3$; and further, the line PH is the locus of points P whose images in the sides of the triangle $A_1A_2A_3$ are points C_i such that the circles $C_2C_3A_1$, $C_3C_1A_2$, and $C_1C_2A_3$ will meet at N.

Let us denote the further intersections of the lines A_iP with the circumcircle as D_i. The equation of the circle through D_1 and P with center on A_2A_3 is in parametric form

$$(\sigma_3\bar{p} - t_1p + t_1\sigma_1 - \sigma_2)x = (\sigma_2 - \sigma_3\bar{p})(t_1 - p) + t_1(p^2 - \sigma_1p + \sigma_2 - \sigma_3\bar{p})T.$$

This circle cuts the circumcircle of $A_1A_2A_3$ at D_1 and at a point whose coördinate is $(\sigma_2 - \sigma_3\bar{p})/(\sigma_1 - p)$. But this is the point N; and likewise, the circle with center on A_1A_3 and through D_2 and P, and the circle with center on A_1A_2 and through D_3 and P must be on N. If we ask for the locus of points P such that this point N be a given point T we set

$$\frac{\sigma_2 - \sigma_3\bar{p}}{\sigma_1 - p} = T,$$

or

$$Tp - \sigma_3\bar{p} = T\sigma_1 - \sigma_2.$$

But this is the line of images of the point T [3, p. 422]. We have found a further property of the line of images of a point T of the circumcircle, namely: *For every point R on the line of images of T, if we determine the images C_i of R in the sides of the triangle $A_1A_2A_3$, and if we determine the further intersections D_i of A_iR with the circumcircle of $A_1A_2A_3$, then the three circles $C_iC_jA_k$ and the three circles through RD_i with centers on A_jA_k, all meet at the same point T of the circumcircle of $A_1A_2A_3$.*

If we ask for the locus of the point P such that the two points D and N on the circumcircle of $A_1A_2A_3$ will coincide we are led to the curve

$$(2.1)\qquad 2p^3 - 3\sigma_1 p^2 + (\sigma_1^2 + \sigma_2)p = 2\sigma_3^2\bar{p}^3 - 3\sigma_2\sigma_3\bar{p}^2 + (\sigma_2^2 + \sigma_1\sigma_3)\bar{p}.$$

Let us denote the centers of the circumcircle and the nine-point circle of $A_1A_2A_3$ by O and F, respectively. The cubic (2.1) intersects the circumcircle of $A_1A_2A_3$ at the vertices and at the three points diametrically opposite the ones whose Simson lines intersect at the midpoint of OF. The intersections of (2.1) and the circumcircle are given by the sextic

$$(2.2)\qquad (T^3 - \sigma_1T^2 + \sigma_2T - \sigma_3)(2T^3 - \sigma_1T^2 - \sigma_2T + 2\sigma_3) = 0.$$

The equation of the Simson line of a point T on the circumcircle of $A_1A_2A_3$ is

$$(2.3)\qquad 2(Tx - \sigma_3\bar{x}) = T^2 + \sigma_1T - \sigma_2 - \sigma_3/T.$$

The first factor of (2.2) equated to zero gives the coördinates of the vertices of $A_1A_2A_3$. The points on the circumcircle diametrically opposite to the three given by the second factor of (2.2) are given by the cubic

$$(2.4)\qquad 2T^3 + \sigma_1T^2 - \sigma_2T - 2\sigma_3 = 0.$$

If in (2.3) we set $x=\sigma_1/4$ and $\bar{x}=\sigma_2/4\sigma_3$ we obtain (2.4) which proves our statement.

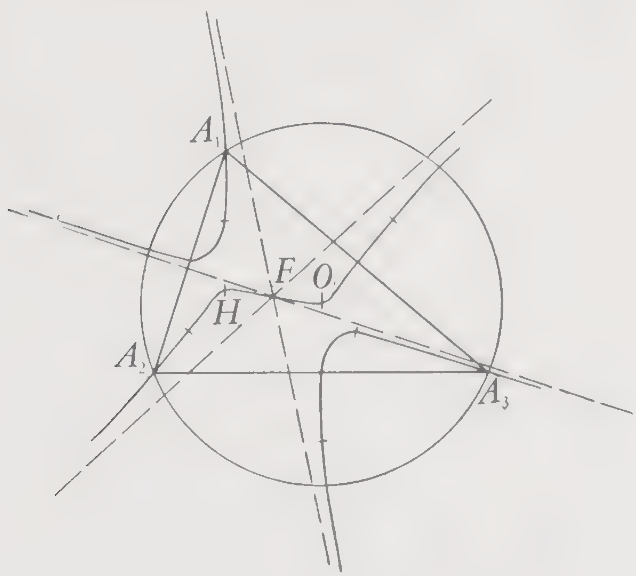

FIG. 1

The cubic (2.1) passes through the orthocenter H, the circumcenter O, and the nine-point center F of the triangle $A_1A_2A_3$; in fact, it is symmetric about the point F. (See Figure 1.) It passes through the midpoints of the sides of $A_1A_2A_3$ and through the midpoints of the segments A_iH, whence its intersections with the nine-point circle of $A_1A_2A_3$ are completely known. It is tangent at H to the hyperbola of Jerabek of $A_1A_2A_3$. Its three asymptotes are given by

$$2(x - \omega_i \sigma_3^{2/3} \bar{x}) = \sigma_1 - \omega_i \sigma_2 \sigma_3^{-1/3},$$

where ω_i are the three cube roots of unity. Hence *the three asymptotes of the cubic* (2.1) *meet at the nine-point center F of $A_1A_2A_3$ and make angles of 60° with each other.* This cubic has the property that for any point P of it, if the points B_i are the symmetrics of A_i in P and the points C_i are the images of P in the sides A_jA_k, the six circles $B_iB_jA_k$ and $C_iC_jA_k$ intersect at the same point on the circumcircle of $A_1A_2A_3$.

3. A problem of Thébault [4]. Let $A_1A_2A_3$ be any triangle with orthocenter H, and P any point in the plane. Let D_1, D_2, and D_3 be the symmetrics of H in A_iP, respectively. The three circles A_iD_iP meet on the circumference of $A_1A_2A_3$ at the point L.

To prove this we note that there is one equilateral hyperbola on $A_1A_2A_3$ and P which likewise passes through H. We recall a theorem of mine: *If we reflect any point X of an equilateral hyperbola in a chord A_1A_2, obtaining the point D_3, the circle $A_1A_2D_3$ intersects the hyperbola at the diametrically opposite point of X* [1, page 373]. Hence the symmetrics of H in the chords A_iP give rise to three points D_i, and the three circles A_iPD_i will meet on the hyperbola and also on the circumcircle of $A_1A_2A_3$ since the diametrically opposite point of H on the hyperbola is also on the circle $A_1A_2A_3$.

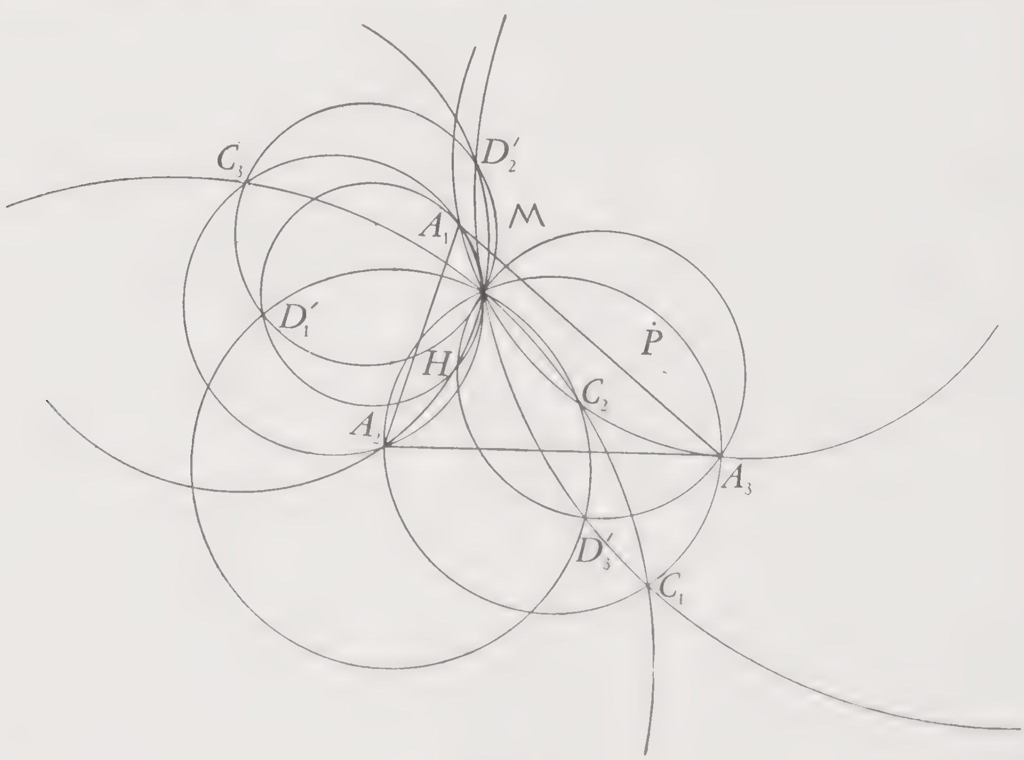

FIG. 2

We see further that if D_i' be the symmetrics of P in the lines A_iH, the three circles A_iHD_i' must also meet at a point M on the hyperbola, but not on the

circle $A_1A_2A_3$. The coördinates of the points L and M may be written as

$$l = \frac{p - n}{1 - \bar{p}n}, \qquad m = \sigma_1 - p + l, \qquad \text{where} \qquad n = \frac{\sigma_2 - \sigma_3\bar{p}}{\sigma_1 - p}.$$

The line of images of the point N of section 2 is the join HP and L is the second intersection of NP with the circumcircle of $A_1A_2A_3$. Also M is a known point [1, page 373]. It was shown that if the reflexions of P in the sides A_jA_k be C_i, then the three circles $A_iA_jC_k$ will intersect at the point M which is also on the circle $C_1C_2C_3$. Since the double ratio of the points $D_1' D_2' C_3M$, $D_2' D_3' C_1M$, and $D_3' D_1' C_2M$ are each real, the three circles $D_i' D_j' C_k$ are on the point M. Hence we have the theorem: *If P be any point in the plane of the triangle $A_1A_2A_3$ with orthocenter H and C_i be the symmetrics of P in the sides A_jA_k, and D_i' be the symmetrics of P in the lines A_iH, the nine circles $A_iA_jC_k$, A_iHD_i', and $D_i' D_j' C_k$ intersect at the same point which is also on the circle $C_1C_2C_3$.* (See Figure 2.)

If we ask for the locus of points P such that the two points N and L on the circumcircle of $A_1A_2A_3$ will coincide we are led to

$$\begin{aligned} p^3 - 2\sigma_1 p^2 + (\sigma_1^2 + 2\sigma_2)p - 2\sigma_3 p\bar{p} - 2\sigma_1\sigma_2 \qquad & \\ + \sigma_3^2\bar{p}^3 - 2\sigma_2\sigma_3\bar{p}^2 + (\sigma_2^2 + 2\sigma_1\sigma_3)\bar{p} = 0. & \end{aligned} \tag{3.1}$$

This cubic is tangent to the circumcircle at the vertices A_i, and cuts the side A_jA_k further at the point which is the symmetric in the midpoint of A_jA_k of the intersection of A_jA_k with the tangent to the circle $A_1A_2A_3$ at the diametrically opposite point of A_i. This cubic has the property that for any point P of it, if the symmetrics of P in the sides A_jA_k be C_i and the symmetrics of H in A_iP be D_i, the six circles $C_jC_kA_i$ and A_iD_iP meet at the same point of the circumcircle $A_1A_2A_3$.

If we ask for the locus of points P such that the points N and L will be diametrically opposite on the circle $A_1A_2A_3$ we obtain

$$p^3 - 2\sigma_1 p^2 + \sigma_1^2 p = \sigma_3^2\bar{p}^3 - 2\sigma_2\sigma_3\bar{p}^2 + \sigma_2^2\bar{p}. \tag{3.2}$$

This cubic intersects the circumcircle at the vertices A_i, and at the three points diametrically opposite to the three whose Simson lines meet at O, the circumcenter of $A_1A_2A_3$. The tangents to the cubic at the vertices are the symmedian lines of $A_1A_2A_3$; hence the tangents to the cubic at the vertices meet at the symmedian point K of $A_1A_2A_3$. The cubic passes through O, and has a double point at H. It cuts the side A_jA_k further at the point where the line through H parallel to OA_i cuts A_jA_k. The asymptotes meet at angles of 60° at the midpoint of GH, where G is the centroid and H the orthocenter of $A_1A_2A_3$. This cubic (3.2) has the property that for any point P of it, if the symmetrics of P in the sides A_jA_k be C_i, and the symmetrics of H in A_iP be D_i, the intersection of the three circles $C_jC_kA_i$ is diametrically opposite on the circle $A_1A_2A_3$ the intersection of the three circles A_iD_iP.

Finally if we ask for the locus of the point P such that L will lie diametrically

opposite D on the circle $A_1A_2A_3$ we obtain

$$(3.3)\quad \begin{aligned} &3p^3 - 4\sigma_1 p^2 + (\sigma_1^2 + \sigma_2)p + \sigma_2 p^2 \bar{p} \\ &\qquad = 3\sigma_3^2\bar{p}^3 - 4\sigma_2\sigma_3\bar{p}^2 + (\sigma_2^2 + \sigma_1\sigma_3)\bar{p} + \sigma_1\sigma_3 p\bar{p}^2. \end{aligned}$$

This cubic passes through the vertices A_i, the incenter and excenters, through H, O, G, K, the midpoints of the sides, and the midpoints of the altitudes—in other words, it is the 17 point cubic [**2**, page 460]. Hence the 17 point cubic is the locus of points P such that if D_i be the images of H in A_iP and B_i be points such that P is the midpoint of A_iB_i, the three circles $B_jB_kA_i$ meet on the circle $A_1A_2A_3$ at the point diametrically opposite the intersection of the three circles A_iD_iP.

4. A pencil of cubics. The 17 point cubic is a member of a pencil of cubics arising from requiring that a point P and its isogonal conjugate point P' should be collinear with a definite point Q, called the pivot point. If we ask that Q run over the Euler line of triangle $A_1A_2A_3$ the following pencil of cubics arises:

$$(4.1)\quad \begin{aligned} &(1+\lambda)y^3 + \sigma_2 y^2\bar{y} - \sigma_1\sigma_3 y\bar{y}^2 - (1+\lambda)\sigma_3^2\bar{y}^3 - \sigma_1(2+\lambda)y^2 \\ &\qquad + \sigma_2\sigma_3(2+\lambda)\bar{y}^2 + \{\sigma_2(\lambda-1) + \sigma_1^2\}y - \{\sigma_1\sigma_3(\lambda-1) + \sigma_2^2\}\bar{y} = 0. \end{aligned}$$

This pencil cuts the circumcircle $A_1A_2A_3$ at the vertices and in the three points given by

$$(4.2)\quad (1+\lambda)T^3 - \sigma_1 T^2 - \sigma_2 T + (1+\lambda)\sigma_3 = 0.$$

By an argument similar to one used in section 2 we can show that the points (4.2) are diametrically opposite the three points on the circumcircle whose Simson lines meet at $x = \lambda\sigma_1/2(1+\lambda)$. All the cubics in the pencil pass through the vertices A_i, H, O, the incenter, and excenters. The tangents at the four last points meet at the pivot point Q, those at the vertices meet at Q', the isogonal conjugate point of Q. The cubics are tangent at Q to the line QQ' [**5**]. Since the tangents at the incenter and excenters meet at Q which runs over the Euler line, the tangents at the vertices will meet at a point on the hyperbola of Jerabek.

For $\lambda = \infty$, we have McCay's cubic [**6**]; $\lambda = 2$ gives the 17 point cubic, $\lambda = -1$ the 21 point cubic [**7**], and $\lambda = -2$ the cubic of Darboux [**8**]. If $\lambda = 0$, we have the cubic

$$(4.3)\quad \begin{aligned} &y^3 + \sigma_2 y^2\bar{y} - 2\sigma_1 y^2 + (\sigma_1^2 - \sigma_2)y \\ &\qquad = \sigma_3^2\bar{y}^3 + \sigma_1\sigma_3 y\bar{y}^2 - 2\sigma_2\sigma_3\bar{y}^2 + (\sigma_2^2 - \sigma_1\sigma_3)\bar{y}. \end{aligned}$$

This cubic is tangent to the Euler line at the orthocenter of $A_1A_2A_3$, the tangents at the vertices meet at the circumcenter, while those at the incenter and excenters meet at the orthocenter. It passes through the feet of the altitudes of $A_1A_2A_3$, hence the pedal triangle of the pivot point H lies on the curve. It is the only cubic of the pencil (4.1) such that the pedal triangle of its pivot point

as to $A_1A_2A_3$ is on the curve. The further intersections of the cubic with the circumcircle are diametrically opposite those whose Simson lines as to $A_1A_2A_3$ meet at O, the circumcenter of $A_1A_2A_3$.

The loci of sections 2, 3, and 4 are, inversively speaking, special cases of bi-cubics; in projective geometry they are cubic curves. It is interesting to note that the pencil of bi-cubics

$$(2+\lambda)p^3 - (3+\lambda)\sigma_1 p^2 + (\sigma_1^2 + \overline{1+\lambda}\,\sigma_2)p = (2+\lambda)\sigma_3^2\bar{p}^3 - (3+\lambda)\sigma_2\sigma_3\bar{p}^2 + (\sigma_2^2 + \overline{1+\lambda}\,\sigma_1\sigma_3)\bar{p}$$

contains for $\lambda=0$, ∞, -1, -2, respectively the locus (2.1), the McCay cubic, locus (3.2), and the hyperbola of Jerabek. The members of this pencil pass through the vertices of $A_1A_2A_3$, the circumcenter, and have the same tangent line at the orthocenter. The three asymptotes of each member are parallel to the asymptotes of all the others and intersect at angles of 60° at the point on the Euler line whose coördinate is $[(3+\lambda)/3(2+\lambda)]\sigma_1$. If we call the further intersection of the pencil with the Euler line $P=\sigma_1/(2+\lambda)$, then the pencil of cubics cuts each side A_jA_k at the point where the parallel to OA_i through P cuts A_jA_k.

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ON THE USE OF CONFORMAL MAPPING IN SHAPING WING PROFILES*

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1. Introduction. The use of conformal mapping in fluid mechanics dates back to the early work of Gauss, Riemann, Weierstrass, C. Neumann, H. A. Schwarz, and Hilbert. The earliest comprehensive application of conformal mapping to aerodynamics may be attributed to Lord Rayleigh (1877) [**1**]. In the early part of the present century Kutta (1902) [**2**] and Joukowski (1906) [**3**] published a series of fundamental papers on airfoil theory which marked the beginning of a new era in fluid mechanics. Within a very short time many improvements of the theories of Kutta and Joukowski were advanced, the recent works of von Kármán, Tietjens, Prandtl, Mises, Trefftz, Höhndorf, Glauert and many others being of importance [**4**, **5**]. As the limitations of these theoretical methods became more fully appreciated, interest in the mathematical shaping of airfoils waned. However, the recent improvement in the efficiency of airplanes and the immense cost of designing and constructing the modern airliner has renewed interest in the theoretical approach to aerodynamics and in particular to the question of shaping wing profiles mathematically.

It is the purpose of the present paper to give a short elementary exposition of the use of conformal mapping in ideal two-dimensional airfoil theories.

2. Basic principles of wing theory [**4**, **5**, **6**]. The basic principle underlying the construction of all flying machines is the property of a body (such as an inclined plane), when moving horizontally through the air, of experiencing a force R which may be decomposed into two components, one called the *drag* D in the direction of the airflow and the other called the *lift* L in a direction perpendicular to the flow. (Fig. 1.) The angle between the direction of the flow and the

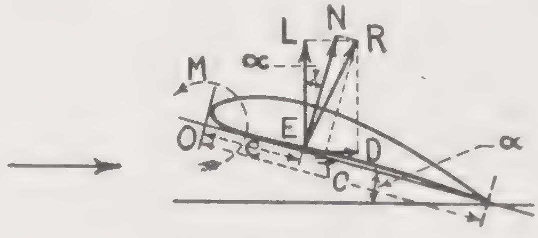

FIG. 1

force R exerted on the body is a function of the geometrical shape and position of the body with respect to its motion. The lift is essential for carrying the airplane but the drag is largely undesired and must be compensated for by the thrust of the propeller.

The drag is largely due to the friction of the air along the airfoil and to the

* Extracted from an illustrated lecture given to the Ohio Section of the Mathematical Association of America, April 8, 1939.

appearance of the lift. The lift is accompanied by the creation of a definite flow pattern in the neighborhood and in the wake of the airfoil and demands a continuous supply of energy. Study of the drag due to friction requires a consideration of the viscosity of the fluid, while the study of the drag due to lift may be made through the application of fluid mechanics to ideal fluids.

An ordinary type of airfoil can be described as having the form of a flat or slightly cambered body usually symmetrical with respect to a medium plane. The section made by the medium plane is called the *profile* of the airfoil.

The force experienced by the airfoil is due mainly to the difference in the pressures acting upon the lower and upper surfaces. For steady motion of a fluid of density ρ along a body at rest, the pressure p at any point of absolute velocity U can be calculated from *Bernoulli's equation*, which in its simplest form is

$$p + \tfrac{1}{2}\rho U^2 = p_0 + \tfrac{1}{2}\rho V^2,$$

where p_0 is the pressure and V is the absolute velocity at "infinity."

For certain purposes the airfoil is taken to extend to infinity on both sides in a direction normal to the airfoil. This assumption simplifies the problem since the effects of wing tips need not be considered, reducing the flow to the two-dimensional case.

The fundamental problem of wing theory is that of determining the flow around a geometrically determined wing or wing system. The distribution of velocities and pressures must be calculated, assuming that a body having the shape of the wing is moving uniformly and with constant velocity through the fluid. For the investigation of the stability of airplanes and for various other problems it is necessary to know the *moment* of the force experienced by the airfoil. A general account of the mathematical theory must furnish formulas for calculating the lifting force and the energy to be expended in producing the flow system.

3. Two-dimensional ideal fluid mechanics. To avoid certain serious difficulties and for reasons of simplicity an ideal fluid is often assumed in the study of fluid mechanics. An *ideal fluid* is one in which the viscosity is zero and the density is constant; the fluid is assumed to be continuous, homogeneous, perfectly mobile and free of shearing movement; and such changes in pressure as may occur are assumed to be such as not to sensibly effect the density.

We consider an ideal fluid of unit depth flowing over the ζ-plane in such a manner that the vector velocity U of the liquid is everywhere horizontal and independent of the depth of the point at which it is measured. (Fig. 2.) Let $\mathcal{C}$ be an arbitrary curve in the ζ-plane passing through the points $A(a, b)$ and $B(c, d)$, and let $\mathcal{S}$ be the cylindrical surface through $\mathcal{C}$ with elements perpendicular to the ζ-plane. Let V_s denote the component of U along $\mathcal{C}$ at the point $P(\xi, \eta)$, and let the ξ and η components of U at (ξ, η) be u and v, respectively. The volume of fluid flowing per unit of time (flux H) across the arc $\widehat{AB}$ of $\mathcal{C}$ through $\mathcal{S}$ is

(1) $$H = \int_A^B - v d\xi + u d\eta .$$

Now H is independent of $\mathcal{C}$ if and only if [8]

(2) $$\frac{\partial(-v)}{\partial\eta} = \frac{\partial u}{\partial\xi} .$$

In case (2) holds, the integral

(3) $$\psi(\xi, \eta) \equiv \int_{(a,b)}^{(x,y)} - v d\xi + u d\eta$$

defines a function $\psi(\xi, \eta)$ known as the *stream function*. If (2) holds, the fluid is *incompressible*.

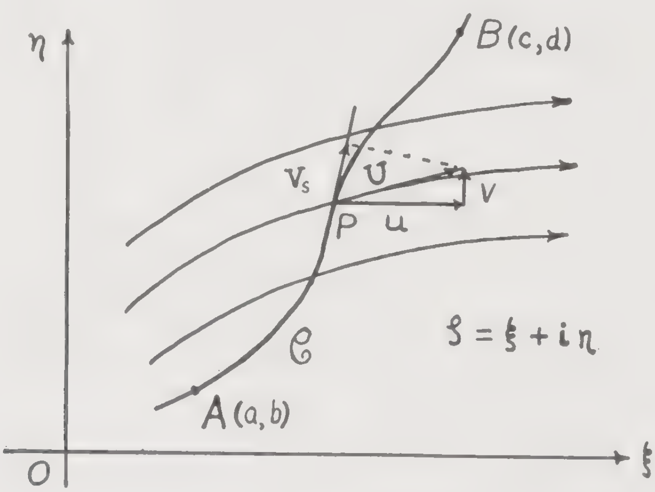

FIG. 2

The *circulation* along $\mathcal{C}$ from A to B is defined to be

(4) $$\int_A^B V_s ds = \int_A^B u d\xi + v d\eta .$$

In case (4) is independent of $\mathcal{C}$,

(5) $$\frac{\partial u}{\partial\eta} = \frac{\partial v}{\partial\xi} .$$

and conversely, and the motion of the fluid is then said to be *irrotational*. When (5) holds the integral

(6) $$\phi(\xi, \eta) \equiv \int_{(a,b)}^{(x,y)} u d\xi + v d\eta$$

defines a function $\phi(\xi, \eta)$, known as the *velocity potential*.

The complex function

$$F = \phi + i\psi = f(\zeta) \tag{7}$$

is known as the *generalized potential function.* Since $\partial\phi/\partial\xi = \partial\psi/\partial\eta = u$, $\partial\phi/\partial\eta = -\partial\psi/\partial\xi = v$, *F is an analytic function of the complex variable* $\zeta = \xi + i\eta$. A plane flow which may be characterized by such a function F is called a *potential flow.* Both ϕ and ψ are solutions of Laplace's equation, that is, $\nabla^2\phi = 0$, $\nabla^2\psi = 0$.

The velocity components may be obtained from $dF/d\zeta$, since

$$\frac{dF}{d\zeta} = \frac{\partial\phi}{\partial\xi} + i\,\frac{\partial\psi}{\partial\xi} = u - iv. \tag{8}$$

The curves of the family $\psi = c_1$, a constant, are called *stream lines*, and the curves of the family $\phi = c_2$, a constant, are known as *potential lines.* These families are mutually orthogonal.

The simple types of generalized potentials are well known. For a fluid flowing with a uniform velocity $Ue^{+i\alpha}$, U and α being real, the potential is $F_R = \phi_R + i\psi_R = Ue^{-i\alpha}\zeta$; for a simple *source* of strength m (real) at ζ_0, the potential is $F_s = (m/2\pi)\log(\zeta - \zeta_0)$; for a *vortex* of circulation Γ (real) and center at ζ_0, the potential outside a circular barrier of radius a and center ζ_0 is $F_V = -(i\Gamma/2\pi)\log(\zeta - \zeta_0)/a$; for a *doublet* whose axis is the line of angle α with the ξ-axis and of moment M at ζ_0, the potential is $F_D = -Me^{i\alpha}/(\zeta - \zeta_0)$.

If $F_1, \cdots, F_n$ are n generalized potentials then any linear combination of these is also a generalized potential, hence by *superposition* of simple potential flows, flow fields of various degrees of complexity can be constructed.

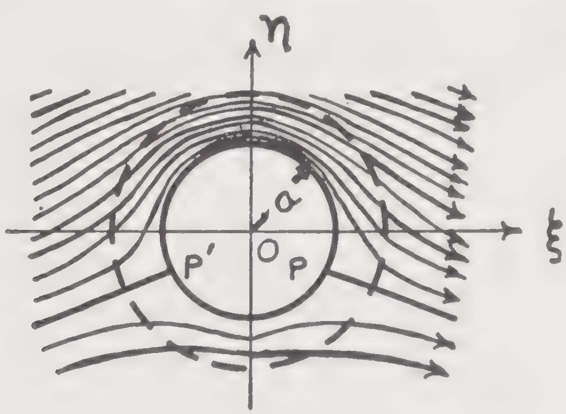

FIG. 3

4. Flow about a circular barrier. A case of particular interest arises from a superposition of a doublet and a vortex, of common center ζ_0, upon a rectilinear flow. A potential $F_c = \phi_c + i\psi_c$ for such a flow is

$$F_c = F_R - F_D - F_V = Ue^{-i\alpha}(\zeta - \zeta_0) + \frac{Me^{i\alpha}}{\zeta - \zeta_0} + \frac{i\Gamma}{2\pi}\log\frac{(\zeta - \zeta_0)}{a}\,. \tag{9}$$

We require that the flow across the boundary of the circle $\zeta-\zeta_0=ae^{i\theta}$ be zero. This means that for ζ on this circle, $\psi_c=0$. From (9), we then find that $M=Ua^2$. The velocity function for this flow is

$$\frac{dF}{d\zeta}=Ue^{-i\alpha}-\frac{Ua^2e^{i\alpha}}{(\zeta-\zeta_0)^2}+\frac{i\Gamma}{2\pi}\frac{1}{\zeta-\zeta_0}\equiv W_\zeta\equiv U_\xi-iV_\eta. \tag{10}$$

The velocity at infinity in this case is $Ue^{i\alpha}$. Figure 3 illustrates this case. Note the barrier circle and barrier lines emanating from the stagnation points P and P'.

From Bernoulli's theorem and the relation

$$R=p_\xi+ip_\eta=\frac{\rho i}{2}\oint W_\zeta^2 d\zeta, \tag{11}$$

the total resultant force R due to pressure on the circular boundary $\zeta=re^{i\theta}$ can be calculated. When $\alpha=0$, we find

$$p_\xi=0,\qquad p_\eta=\frac{\rho\Gamma U}{2}\left(1+\frac{a^2}{r^2}\right). \tag{12}$$

If $a=r$, $p_\eta=\rho\Gamma U$. The circle of radius a may represent the boundary of a solid body of circular cross-section placed in the stream. We conclude that *a circular cylinder of uniform cross-section and infinite length when placed in a field of flow composed of a translation, a vortex motion, and a doublet, will experience per unit length a total force (lift) of measure $\rho\Gamma U$ at right angles to the stream.*

This result is known as the *Kutta-Joukowski theorem.*

It has been shown that this theorem remains true when the circular cylinder is replaced by a cylinder whose cross-section is uniform and simply connected.

These results are not quite in accord with the facts for actual fluids, largely due to the assumption of zero viscosity. If the circulation Γ were zero, the cylinder would experience a turning moment but would have no lift.

5. Potential functions for an arbitrary wing shape. A number of methods have been used for determining the potential functions for a given wing shape. For the two-dimensional case these methods may be roughly divided as follows: (1) superposition of elementary potentials, (2) solution of Laplace's equation, (3) conformal mapping, (4) other methods.

The first method has been illustrated in §4 for the flow past a circular cylinder and consists in a search for a superposition of elementary potentials whose resulting streamline pattern imitates the flow past a wing shape.

The second method is usually very difficult and depends upon finding ϕ and ψ as solutions of Laplace's equation, meeting the required boundary conditions.

The third method involves the use of conformal mapping [**7**, **8**, **13**] and is discussed in the following sections.

6. Principle of application of mapping to wing theory [**5**, **4**]. Independent

of the earlier work of Lord Rayleigh, Kutta (1902) calculated the stream lines around an airfoil using conformal mapping. This method has proved to be of great value, though by the nature of the mathematics used, the method is largely restricted to two-dimensional cases.

The theory of the two-dimensional flow of an incompressible ideal fluid around wing sections is based on the following principles.

We suppose that the potential function for a plane flow about a profile $\mathcal{G}$ of boundary G in the ζ-plane (Fig. 4) is known and is

$$F(\zeta) = \phi(\xi, \eta) + i\psi(\xi, \eta), \tag{13}$$

where on $G, \psi = 0$, since the flow across G must be zero.

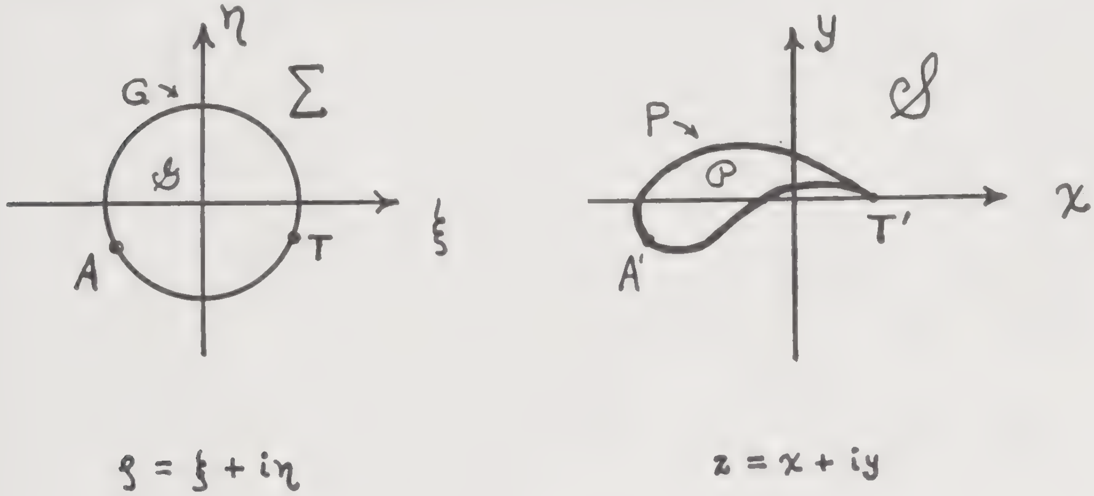

Fig. 4

The procedure consists in mapping by means of a suitable analytic function the profile $\mathcal{G}$ (and the corresponding flow picture) into a profile $\mathcal{P}$ of boundary P resembling airfoils used in aeronautic practice.

We subject the points in the ζ-plane to a transformation

$$\zeta = t(z), \qquad (z = x + iy), \tag{14}$$

which is analytic almost everywhere. We restrict $t(z)$ so that: (a) to each point ζ of G there corresponds a definite point z of the new profile boundary P (and that this correspondence is biunique); (b) dt/dz is finite and not zero over Σ, the whole of the ζ-plane outside of G, and so that Σ will be represented by the domain $\mathcal{S}$ of the z-plane outside of the airfoil $\mathcal{P}$; (c) $t(z)$ is analytic at infinity, and that as $z \to \infty$, $\zeta \to \infty$, and $dt/dz \to K$, a real, finite, non-zero constant; (d) everywhere outside of G, (14) has a single-valued inverse $z = f(\zeta)$.

The new potential function $\overline{F}(z)$ for the flow about $\mathcal{P}$ may be obtained from the potential $F(\zeta)$ and transformation (14), and is given by

$$\overline{F}(z) = F(t(z)) = \Phi(x, y) + i\Psi(x, y). \tag{15}$$

Since for each ζ on G, $\psi(\xi, \eta)=0$, and $F(\zeta)$ is real, it follows from (15) and (a) that $\Psi(x, y)=0$, for all points on P, so that the flow across P is zero.

The velocity function for the flow in the ζ-plane is $dF/d\zeta=u-iv\equiv w_\zeta$, and for the new flow in the z-plane is

$$\frac{d\bar{F}}{dz}=\frac{dF}{d\zeta}\cdot\frac{d\zeta}{dz}=w_\zeta\frac{dt}{dz}=u_z-iv_z\equiv w_z. \tag{16}$$

The restriction (c) is made in order that the velocity w_z of the fluid at infinity in the z-plane be a real constant K times the fluid velocity w_ζ at infinity in the ζ-plane.

Since $\bar{F}(z)=F(\zeta)$ with $\zeta=t(z)$, it follows that for each z and corresponding ζ, $\phi(\xi, \eta)=\Phi(x, y)$ and $\psi(\xi, \eta)=\Psi(x, y)$. Hence, to each equipotential line $\phi=c_1$ in the ζ-plane (c_1 a real constant) there corresponds in the z-plane the equipotential line $\Phi=c_1$; and to each stream line $\psi=c_2$ in the ζ-plane (c_2 a real constant) there corresponds in the z-plane the stream line $\Psi=c_2$.

The restriction in (b) that dt/dz shall be analytic, finite and non-zero over Σ is made in order that the flow pattern in the region Σ formed by the families $\phi=c_1$, $\psi=c_2$ shall be mapped conformally (with preservation of angles) into the flow pattern of the z-plane formed by the families $\Phi=c_1$ and $\Psi=c_2$.

Since $\phi=c_1$, and $\psi=c_2$ are orthogonal, so are $\Phi=c_1$ and $\Psi=c_2$.

The points in the z-plane where $d\bar{F}/dz=0$ are called the critical points of the z-plane; at such points the velocity w_z vanishes. From (16), we see that if dt/dz is finite and non-zero at a critical point A' in the z-plane, then $dF/d\zeta=0$, and the point A in the ζ-plane corresponding to A' is a critical point of flow in the ζ-plane.

If, however, C is a critical point in the ζ-plane and at C, dz/dt is zero, then the right-hand side of (16) is indeterminate. Let C' be the point in the z-plane corresponding to C. We require that the velocity $d\bar{F}/dz$ of flow at C' be finite and non-zero. (The Joukowski condition.) Evidently, C' is not a critical point in the z-plane.

Suppose that T and T', and A and A' are corresponding points (Fig. 4) and that T, A', and A are critical points (and the only ones on G and P). T and T' have the same value of velocity potential ϕ. A short calculation shows that the circulation I along G from T counter-clockwise through A to T is equal to the circulation I' along P from T' counter-clockwise through A' to T', and that in general $I\neq 0$. Furthermore, if one calculates I (or I') around G (or P) starting from some point other than T (or T'), one finds a sudden increase in the value of the velocity potential of the same amount I when crossing T (or T').

We have shown how to obtain from a given potential $F(\zeta)$ and associated profile $\mathcal{G}$ the potential $\bar{F}(z)$ for the flow of an ideal fluid around any profile $\mathcal{P}$ which is generated from $\mathcal{G}$ by a conformal transformation.

7. Application of the method. The usual starting potential is that considered in (9), for the flow around a circular cylinder $\mathcal{G}$ of infinite span. Thus

$$F(\zeta) = Ue^{-i\alpha}(\zeta - \zeta_0) + \frac{Ua^2e^{i\alpha}}{\zeta - \zeta_0} + \frac{i\Gamma}{2\pi} \log \frac{(\zeta - \zeta_0)}{a} \cdot \tag{17}$$

At present no method is known for obtaining the exact value of the circulation Γ. *The only case* in which we are able to calculate Γ in fair accord with experience, is that of a profile with a "sharp trailing edge," with moderate thickness, slight curvature, and small angle of attack. In this case we obtain a definite value of Γ—independent of viscous phenomena in the boundary layer of fluid about the body—by using *Joukowski's assumption of a "smooth flow" at the trailing edge.* According to this assumption, *the circulation has in every case just the amount necessary to secure a finite velocity of the fluid at the trailing edge.*

This restriction to profiles with a sharp trailing edge implies that $z=f(\zeta)$ is not conformal at the point T of G which corresponds to the trailing edge T' of P. Let T be ζ_T and T' be z_T.

If at T the angle τ in the ζ-plane transforms into the angle $(2\pi-\tau)$ at T' in the z-plane, then at T, $z=f(\zeta)$ has the development

$$z - z_T = (\zeta - \zeta_T)^{(2\pi-\tau)/\pi} + \sum_{i=2}^{\infty} a_i(\zeta - \zeta_T)^{i(2\pi-\tau)/\pi}. \tag{18}$$

Since $\tau < \pi$,

$$\left.\frac{dz}{d\zeta}\right]_T = 0. \tag{19}$$

By (16), in order that w_z be finite at the trailing edge T, $dF/d\zeta\,]_T$ must vanish. This is called the *Joukowski condition* for circulation.

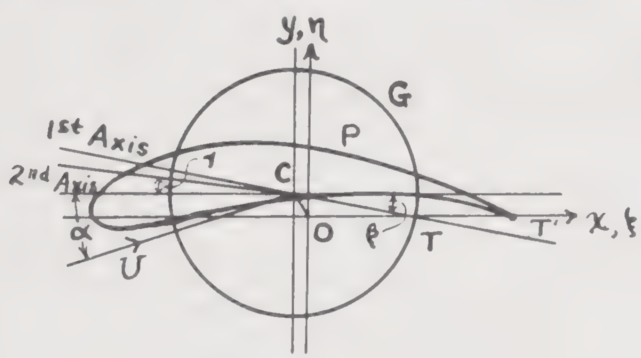

FIG. 5

Let $\zeta_T=\zeta_0+ae^{-i\beta}$. (In Figure 5, the angle β is taken to be positive if T lies on the lower half of G.) From (10) and the Joukowski condition, we have

$$\left.\frac{dF}{d\zeta}\right]_T = Ue^{-i\alpha} - Ue^{i(\alpha+2\beta)} + \frac{i\Gamma}{2\pi a} e^{i\beta} = 0, \tag{20}$$

hence

(21) $$\Gamma = \frac{2\pi a U}{i}\left[e^{i(\alpha+\beta)} - e^{-i(\alpha+\beta)}\right] = 4\pi U a \sin(\alpha+\beta).$$

Expressions for lift and moment. Suppose that for large values of ζ, $z=f(\zeta)$ can be developed in the form

(22) $$z = c + \zeta + \frac{k_1}{\zeta} + \frac{k_2}{\zeta^2} + \frac{k_3}{\zeta^3} + \cdots,$$

which meets the requirements (c) that z has no singularities at ∞, and that at $\zeta=\infty$, $dz/d\zeta=1$. Select the center (ξ_0, η_0) of G so that $c=0$. From (11) the resulting force acting on the airfoil can be calculated; it is equal to $\rho\Gamma U$, and acts perpendicular to $Ue^{+i\alpha}$. Thus from (12) the lift is $L=4\pi\rho U^2 a \sin(\alpha+\beta)$. Here $(\alpha+\beta)$ is the *effective angle of attack*. In Figure 5, the direction of no lift—*the first axis of the profile*—is shown.

The moment of the forces acting on $\mathcal{P}$ with respect to ζ_0 is

(23) $$M_c = \mathcal{R}\left[\frac{\rho}{2}\oint_{\mathcal{P}} (z-\zeta_0) w_z^2\, dz\right],$$
$$M_c = 2\pi\rho U^2 k^2 \sin 2(\gamma-\alpha).$$

The line through ζ_0 with inclination γ to the ξ-axis is known as *the second axis of the profile.* Now

$$M_c = M_F - L\frac{k^2}{a}\cos(2\gamma+\beta-\alpha),$$

where $M_F=2\pi\rho U^2k^2\sin 2(\beta+\gamma)$. If $M_F>0$, the airfoil $\mathcal{P}$ is unstable; if $M_F<0$, $\mathcal{P}$ is stable. For neutral equilibrium, L and M_c must vanish for the same angle of attack.

The system of straight lines representing the lines of action of the lift for different angles of attack envelope a (*metacentric*) parabola. The first axis is the directrix of this parabola. Let the focus be F. Then F lies at a distance k^2/a from ζ_0 on a line CF which makes an angle $2(\beta+\gamma)$ with the first axis and $(\beta+\gamma)$ with the second axis.

8. The Joukowski family of airfoils [3, 4]. *Case* $\tau=0$. *Airfoil with vanishing tail angle.* Consider

(24) $$\frac{z-z_T}{z+z_T} = \left(\frac{\zeta-\zeta_T}{\zeta+\zeta_T}\right)^2.$$

Let T be the point $\zeta_T=b$, where b is a real number. Evidently $z_T=2\zeta_T$ if $dz/d\zeta=1$ at ∞. From (24), we find

(25) $$z = \zeta + \frac{b^2}{\zeta},$$

which (with a properly defined inverse) satisfies the requirements listed in §6. The non-conformal (critical) points L and T of transformation (25) are $\zeta = \pm b$.

Using transformation (25), we find that *every circular arc through T and L in the ζ-plane is represented in the z-plane by an airfoil of circular arc through L' and T'*, where L' and T' are the points $z = \pm 2b$. The circle of radius b and center at $\zeta = 0$ is represented in the z-plane by the line segment (called the *chord*) joining $z = -2b$ and $z = 2b$.

By using different circles passing through L and T in the ζ-plane as generators, we can obtain information concerning the flow about indefinitely thin airfoils of circular shape with different curvatures.

Airfoils of finite thickness and rounded nose can be investigated by using as generators circles passing through T and having L as an interior point. (Fig. 6.) Let the center of the generators be

$$\xi_0 = a\epsilon \cos \beta, \qquad \eta_0 = \frac{h}{2} + a\epsilon \sin \beta, \tag{26}$$

where ϵ is a constant, keeping constant the direction of the first axis. The radius of the generating circle is $a(1+\epsilon)$.

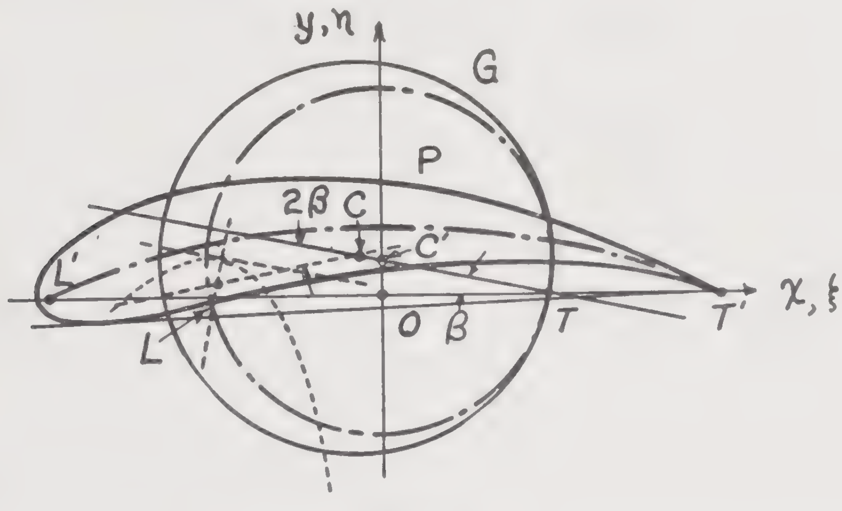

FIG. 6

Airfoils deduced from (25) in this manner are called *the airfoils of the Joukowski family*. The *skeleton* $\widehat{L'T'}$ of a Joukowski airfoil section is the image of the circle whose radius is a and whose center is $(0, h/2)$.

9. Kármán-Trefftz family of airfoils [9]. There are two essential restrictions in the airfoil profiles constructed by the Joukowski process: (1) they all have zero tail angles at the trailing edge, and (2) their thickness is relatively small. These restrictions are of great importance due to strength and stability requirements of practical airfoils. Some of these restrictions can be removed by generalizing (25).

The simplest way to obtain a given tail angle τ is to use

$$\frac{z - z_T}{z + z_T} = \left(\frac{\zeta - \zeta_T}{\zeta + \zeta_T}\right)^\lambda, \qquad \lambda = 2 - \tau/\pi, \tag{27}$$

a transformation which satisfies the general requirements stated in §6. Let $\zeta_T = b$. Then $z_T = \lambda\zeta_T$. *Any circle through* ζ_T *and* $-\zeta_T$ *transforms by* (27) *into an airfoil called a skeleton enclosed by circular arcs with tail angle* τ *in the z-plane.* Furthermore, *circles through* ζ_T *and having* $-\zeta_T$ *as an interior point transform into profiles having round noses and sharp trailing edges of the desired tail angle.*

Airfoils derived in this manner are known as the *Kármán-Trefftz* family. (Fig. 7.)

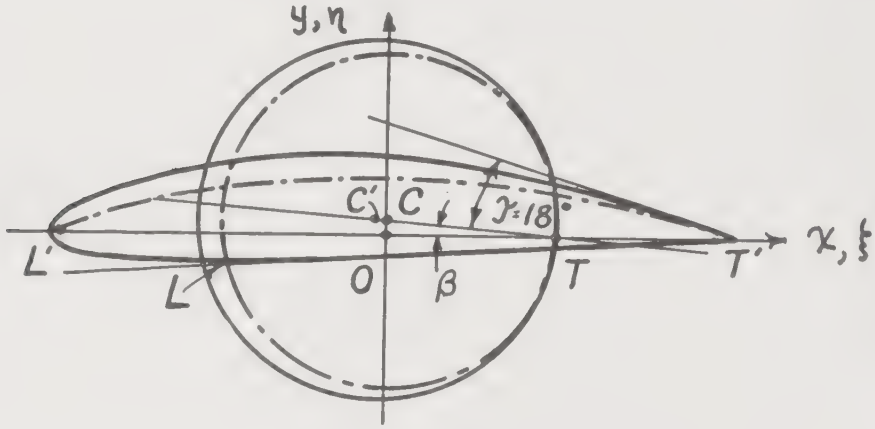

Fig. 7

The Kármán-Trefftz airfoils give aerodynamic characteristics only slightly changed from those of Joukowski airfoils. However, an improvement in regard to velocity and pressure distribution along the airfoil is obtained. All these airfoils are inherently unstable. In spite of this, transformation (27) has proved to be very useful in computing aerodynamic characteristics of given airfoils.

10. Other families of airfoils. By using transformation $z = f(\zeta)$ having four or more critical points inside the generating cylinder $\mathcal{G}$, Mises [**10**] derived a family of stable profiles having sharp trailing edges. Direct methods for finding the aerodynamic characteristics of airfoils of a given shape have been developed by Kármán, Trefftz, . . ., Müller, Höhndorf, Theodorsen, and others. The problem is identical with that of finding the conformal transformation between the given airfoil and a circle.

In a recent paper Piercy, Piper, and Preston [**12**] have discussed a new family of wing profiles which they claim has certain advantages over wing shapes previously discussed by Joukowski and others. The method consists essentially in selecting one branch of a hyperbola as a generating curve in the z_1-plane and applying a transformation $z_1 z_2 = e^{i\beta}$ which produces an inversion and reflection with respect to a point C in the line segment CA, and which results

in airfoils closely resembling those of successful modern wings. By applying a sequence of conformal transformations, the hyperbola is transformed into a circle permitting the application of the method to the potential function (17) for a flow past a circle.

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

A GENERALIZATION OF THE WATER-FETCHING PUZZLE

H. D. GROSSMAN, New York City

The problem to be solved is the following. I go to a well with an empty a-quart and an empty b-quart jug. How can I bring back exactly c quarts of water?

It is understood, of course that a, b, and c are positive integers. Let $a+b=n$. If $c>n$, the problem is unsolvable; if $c=n$, the solution is trivial. Hence we may suppose that $c<n$. If the G.C.D. of a, b, and c is $d>1$, we may take d quarts as the unit and treat the problem as one of a/d, b/d, and c/d units. Hence we may suppose that the G.C.D. of a, b, and c is 1. If the G.C.D. of a and b does not divide c the problem is unsolvable; hence let the G.C.D. of a and b be 1. Finally, $a=b=1$ is trivial, and so we may assume that $a>b$.

If the filling of an x-quart jug be written formally $+x$, and the emptying of a y-quart jug, $-y$, the study of this problem may be reduced to the study of the sums $s_k=t_1+t_2+\cdots+t_k$, where $t_i=a$ or $-b$, each t_i being chosen so that $0\leqq s_k<n$ for $k=1, 2, \cdots$. The choice of t_i is always possible and unique; for $t_1=a$ and if $0\leqq s_{i-1}<b$, then $0\leqq s_{i-1}+a<n$ but $s_{i-1}-b<0$, while if $b\leqq s_{i-1}<n$, then $0\leqq s_{i-1}-b<n$ but $n<s_{i-1}+a$.

Let s_k be the first s to equal any previous s_h. Then $1\leqq h<k\leqq n+1$, since there are at most n distinct possible values of s, namely $0, 1, 2, \cdots, n-1$. Therefore $s_k-s_h=\sum_{h+1}^{k}t_i=0$. Let u be the number of a's, v the number of $-b$'s in this sum. Then $ua=vb$. Since a and b are relatively prime, $u=mb$ and $v=ma$. But $0<u+v=k-h\leqq n$. Therefore $m=1$, $u=b$, and $v=a$. Hence there are n terms in $\sum_{h+1}^{k}t_i$, that is, $n+1\leqq h+n=k\leqq n+1$. Hence $k=n+1$ and $h=1$, so that $s_{n+1}=s_1=a$. Thus the first n s's are distinct, non-negative, and each $<n$; they are therefore a permutation of $0, 1, 2, \cdots, n-1$. If $t_{n+1}=-b$, then $s_n=n$, which is impossible. Therefore $t_{n+1}=a$, and $s_n=s_{n+1}-t_{n+1}=0$. Hence there is a unique $r<n$ for which $s_r=c$.

Interpret s_r as follows. For each $+a$ read "Fill the a-quart jug and pour as much of its contents as possible into the other jug." For each $-b$ read "Empty the b-quart jug and pour into it as much of the contents of the other jug as possible." Omit however the second clause in reading the final term of s_r. The reading of s_r will leave c quarts in the two jugs.

Comments on the solution. 1. Now s_r may evidently be written in a purely mechanical way. We write $+a$, then the smallest number of $-b$'s that will make the sum $<b$, then repeat the process as often as necessary, stopping whenever the sum is c.

2. A solution may also be obtained by using the sums $\sigma_k=\tau_1+\tau_2+\cdots+\tau_k$, where each $\tau_i=b$ or $-a$. An argument similar to that above shows that there is a unique $\rho<n$ such that $\sigma_\rho=c$. Then s_r and σ_ρ each determine solutions of the problem, and since $t_1=a$, $\tau_1=b$, the solutions are distinct. It is easy to show that $r+\rho=n$, and that for any k, $\sigma_k=\bar{s}_{n-k}$, so that the two solutions are in a sense complementary.

3. Since the only non-trivial operations with the jugs are those represented by $\pm a$ and $\pm b$, the solution corresponding to the smaller of r and ρ is the one requiring the fewest operations.

4. Of the following illustrations, the first two are popular puzzles and the others have been taken from the Binet Intelligence Test.

a	b	c	s_r	σ_ρ
5	3	4	$5-3+5-3$	$3+3-5+3$
9	5	7	$9-5+9-5-5+9-5$	$5+5-9+5+5-9+5$
5	3	7	$5-3+5$	$3+3-5+3+3$
7	5	8	$7-5+7-5+7-5-5+7$	$5+5-7+5$
9	4	7	$9-4-4+9-4-4+9-4$	$4+4+4-9+4$

TO TEXT-BOOK WRITERS—AND READERS

A. B. Brown, Queens College

We wish to call attention to certain inaccuracies, errors and procedures open to criticism in American college mathematics texts, mainly in calculus. The reason for this note is that the faults in question seem to appear about as often in recent texts as in older books. We list them individually, in the order in which the student ordinarily meets them, but do not go beyond what is covered in about three semesters of calculus.

1. *Locus and equation in analytic geometry.* The criticism which we shall make here is the most serious of all, yet will be familiar to most readers. It is that in deriving equations of the straight line, circle, parabola, ellipse and hyperbola, most texts merely show that points on the locus have coördinates satisfying the equation. Few prove that conversely if a point has coördinates satisfying the equation it lies on the locus. As the relation between equation and locus is the basis of most of analytic geometry, this particular lapse from logical procedure is regrettable. To the student, who was given a proper introduction to the idea of locus in high school, it is a step backward. He takes his cue from the author, and an obstacle is placed in the way of his understanding the subject.

As a minor matter, most writers, in deriving the equation of the line through

a given point in a given direction, neglect to prove that the coördinates of the given point satisfy the equation.

2. *The asymptotes of a hyperbola.* A few analytic geometry texts "prove" that the lines $y=\pm(b/a)x$ are asymptotic to the hyperbola $x^2/a^2-y^2/b^2=1$, by showing that as the point (x, y) on the hyperbola recedes indefinitely from the origin, say on the upper right-hand branch, the ratio $y/x=(y-0)/(x-0)$ approaches b/a as a limit. The same type of "proof" could be used to show any line $y=(b/a)x+k$ to be asymptotic to the hyperbola!

A correct treatment is given in most analytics texts.

3. *Formula for differentiation of a function of a function.* Calculus texts, in proving the formula

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx},$$

use the equation $\Delta y/\Delta x=\Delta y/\Delta u\cdot\Delta u/\Delta x$. Few mention the fact that Δu may be zero, thus leaving the proof incomplete. A complete proof might well be given in a footnote, or as a second proof in the text. One method is the following:

When $\Delta u\neq 0$, let $\epsilon=\Delta y/\Delta u-dy/du$; when $\Delta u=0$, let $\epsilon=0$. Then ϵ is a continuous function of Δu at $\Delta u=0$. Also, from the definition of ϵ,

$$\Delta y=\frac{dy}{du}\cdot\Delta u+\epsilon\cdot\Delta u \tag{1}$$

whether Δu is zero or not. Now take any $\Delta x\neq 0$, find the corresponding Δu and Δy, write equation (1), and divide by Δx, getting

$$\frac{\Delta y}{\Delta x}=\frac{dy}{du}\cdot\frac{\Delta u}{\Delta x}+\epsilon\cdot\frac{\Delta u}{\Delta x}\cdot$$

Let Δx approach zero, and the desired relation is obtained.

4. *Proof of condition for a maximum or minimum.* The theorem that $dy/dx=0$ at a maximum or minimum is proved ordinarily either by geometric intuition as applied to the graph of the curve, or by use of the theorem that at a point where dy/dx is positive (negative) y increases (decreases) when x increases. However, the latter result is usually taken as intuitively obvious. Now the law of the mean, Taylor's theorem, formulas of differentiation for functions of several variables, and further work depend directly or indirectly on the condition $dy/dx=0$ at a maximum or minimum. Therefore it is desirable to give an analytic proof. One method is this:

At a point where $dy/dx>0$, $\Delta y/\Delta x$ is also positive when $|\Delta x|$ is small. Hence Δy has the same sign as Δx, showing that y increases if x increases by a small amount, and decreases if x decreases by a small amount. The case $dy/dx<0$ is treated similarly. At a maximum at which dy/dx exists and Δx may assume positive and negative values, the results just obtained show that dy/dx can be neither positive nor negative, hence must be zero.

A second method follows: Suppose we have a maximum at a point where dy/dx exists and Δx may assume positive and negative values. Then $\Delta y \leqq 0$. Hence if $\Delta x > 0$, $\Delta y/\Delta x \leqq 0$; and we infer that $dy/dx \leqq 0$. If $\Delta x < 0$, $\Delta y/\Delta x \geqq 0$; and we infer that $dy/dx \geqq 0$. The only value of dy/dx satisfying both relations is $dy/dx = 0$.

5. *Validity of condition for a maximum or minimum.* The texts usually imply, if they do not directly state, that maxima and minima are found by equating derivatives to zero. It should be pointed out, perhaps illustrated by exercises, that at a maximum or minimum on the boundary of the interval (region) of definition, the derivative (partial derivatives) may exist and not equal zero.

6. *The definition of differential.* It seems that many college graduates who majored in mathematics believe that dx and dy are "infinitely small" quantities which yet have a definite ratio. This confusion is due to the fact that most texts have "$dx = \Delta x$" by definition. Now the only Δx's which the student has seen up to this stage have been those which are made to approach zero. Naturally he imagines that dx is also approaching zero.

This confusion can be avoided by refraining from any mention of Δx in the definition. The definition should make dx an *arbitrary quantity*. The text may well mention that dx is *not* made to approach zero in general.

A minor matter in connection with differentials is the statement, "If we take $dx = \Delta x$, then dy is the principal part of Δy." The necessary qualification that dy/dx be not zero at that point is sometimes omitted.

7. *A theorem on differentials.* Many texts neglect to prove or mention that if $dz \neq 0$, the ratio of the differentials dy/dz equals the derivative of y with respect to z, even though dy and dz are determined with x as the independent variable. But all texts use this fact; and it should be proved.

8. *The constant of integration.* Not all texts contain proofs that if $\phi(x)$ is one integral of $f(x)$, then $\phi(x) + C$ gives all integrals. At least one fairly recent text "proves" it as follows: Since $\phi(x) + C$ yields an infinite number of integrals, it yields all integrals!

9. *Integration by substitution.* Some texts perform "integration by substitution" (replacing $\int f du$ by $\int f(du/dx)\, dx$, or *vice versa*) for some time without explanation, before taking it up as a topic with proof.

10. *Integrands involving* $[a^2 + x^2]^{1/2}$, *etc.* In integrating such expressions, the following point is not adequately treated. To simplify $[a^2 - x^2]^{1/2}$, say, with $a > 0$, we set $x = a \sin \theta$. Then

$$[a^2 - x^2]^{1/2} = [a^2 - a^2 \sin^2 \theta]^{1/2} = a[1 - \sin^2 \theta]^{1/2} = a[\cos^2 \theta]^{1/2} = a \cos \theta.$$

Now under the convention that the radical calls for the positive square root, writing $a \cos \theta$ at the end amounts to assuming that $\cos \theta \geqq 0$. The texts do not mention this point. With the above as a model, one can hardly criticize the student for the error of writing $[x^2]^{1/2} = x$ instead of $\pm x$ in algebraic work.

Complete treatment is easy: After observing that $|x| \leqq a$, introduce $\theta = \text{arc} \sin (x/a)$, taking the principal value. Then $\cos \theta \geqq 0$, and the treatment

given in the texts can be used. Note that $\sin\theta = x/a$, $\cos\theta = [a^2 - x^2]^{1/2}/a$, and the other four functions are obtained from these two without any minus sign before the radical.

The case of $[a^2+x^2]^{1/2}$ is treated similarly. However, in the case of $[x^2-a^2]^{1/2}$, if $x < -a$, $\theta = \text{arc sec }(x/a)$ must be taken in the *third* quadrant to make $\tan\theta$ positive. Otherwise the treatment is similar.

11. *Applications of the definite integral.* The subject here is the definite integral,

$$\int_a^b f(x)dx \qquad \text{or} \qquad \lim_{\substack{n\to\infty \\ \Delta x_k \to 0}} \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

The seven procedures described below are used in applications. We illustrate by the case of the formula for the moment of inertia about the y-axis of a straight wire of variable density $\rho(x)$, stretched along the x-axis from a to b. The interval from a to b is divided into parts $\Delta x_1, \Delta x_2, \cdots, \Delta x_n$. Let x_k be the coördinate of the left end of the interval Δx_k (the right end with some authors). Let $\bar{x}_k$ be the coördinate of a point of Δx_k of average density for Δx_k. Let $\bar{\bar{x}}_k$ be the coördinate of a point (usually unique) of Δx_k such that a particle whose mass equals the mass of Δx_k, placed at $\bar{\bar{x}}_k$, will have the same moment of inertia as Δx_k. The procedures:

(1) The "element of moment of inertia" is $\rho\cdot x^2\cdot\Delta x_k$ (sometimes $\rho\cdot x^2\cdot dx$). Hence the moment of inertia $I = \int_a^b \rho\cdot x^2\cdot dx$.

Criticism: The definition of definite integral has not been used correctly, since it is not stated for what value of x the density ρ is taken, nor what value of x is raised to the second power. However, it is useful to show the students this method as a quick way of obtaining the formula when they have forgotten it.

(2) The moment of inertia due to Δx_k is approximately $\rho(x_k)\cdot x_k^2\cdot\Delta x_k$. (Or, replace x_k by $\bar{x}_k$ or by $\bar{\bar{x}}_k$.) As the number of intervals is taken larger and larger, the errors grow smaller and smaller. Hence the limit of the error is "clearly" zero, and $I = \int_a^b \rho(x)\cdot x^2 dx = \int_a^b \rho\cdot x^2 dx$.

Criticism: Yes, the errors grow smaller, but their number increases without limit. Hence it is not clear that the limit of the total error is zero. The student may imagine the answer to be incorrect by a small per cent.

(3) By a careful use of inequalities, the formula is derived.

Criticism: This method cannot be applied in the present case unless $\rho(x)$ is monotonically increasing for x positive, and decreasing for x negative.

(4) Write

$$I = \lim_{n\to\infty} \sum_{k=1}^{n} \rho(\bar{x}_k)\cdot\bar{\bar{x}}_k^2\cdot\Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \rho(\bar{x}_k)\cdot\bar{x}_k^2\cdot\Delta x_k$$

since $\lim_{n\to\infty} [\rho(\bar{x}_k)\cdot\bar{\bar{x}}_k^2\cdot\Delta x_k]/[\rho(\bar{x}_k)\cdot\bar{x}_k^2\cdot\Delta x_k] = 1$, and

$$I = \int_a^b \rho(x) \cdot x^2 dx = \int_a^b \rho \cdot x^2 dx.$$

Criticism: This derivation uses the so-called Duhamel's theorem. One objection is that the *hypotheses* of the theorem require that the ratio of corresponding infinitesimals approach 1 uniformly. Hence in each application the student would have to check the uniformity. He could not very well do that because (a) he does not understand the concept; (b) the checking may be difficult; (c) the hypotheses may not be satisfied.

In our example above, in case $\rho(x)=0$ at some points, or if $a<0<b$, it is not clear that the hypotheses of Duhamel's theorem are satisfied.

(5) A new independent variable $m=m(x)$ is introduced, where $m(x)$ is the mass from a to x. Then the moment of inertia due to Δx_k is $\bar{\bar{x}}_k^2 \cdot \Delta m_k$; hence the moment of inertia of the wire is

$$\int_0^M x^2 dm = \int_a^b x^2 \cdot \rho dx, \qquad (M = \text{total mass})$$

by a change of variable, since $dm/dx=\rho$.

Criticism: This method is correct, but cannot be applied, for example, in finding the length of a space curve (nor for the length of a plane curve given parametrically) when length is defined as the limit of the length of an inscribed broken line. (Compare treatment (2) for item 12, arc length.)

(6) The moment of inertia due to Δx_k is $\rho(\bar{x}_k) \cdot \bar{\bar{x}}_k^2 \Delta x_k$. Since $\rho(x)$ and x^2 are continuous, $a \leqq x \leqq b$ [alternative: since $\rho(x) \cdot y^2$ is continuous, $a \leqq x \leqq b, a \leqq y \leqq b$], the total moment of inertia is

$$\int_a^b \rho(x) \cdot x^2 dx = \int_a^b \rho \cdot x^2 dx.$$

Criticism: This method, based on a theorem of Bliss,* is the quickest complete method among those described here. The proof of Bliss's theorem takes only a few lines.

Until the general form of Bliss's theorem is needed, the special case

$$\lim \sum_{k=1}^n f(\bar{x}_k) \cdot g(\bar{\bar{x}}_k^2) \Delta x_k = \int_a^b f(x) \cdot g(x) dx$$

is preferable, since the notation of the general theorem would be confusing to the students at an early stage in their work.

(7) The formula is derived correctly by giving the complete proof of Bliss's theorem for the case at hand.

Criticism: Unnecessary labor.

12. *Arc length.* This topic has been placed beyond the usual position where

* G. A. Bliss, A substitute for Duhamel's theorem, Annals of Mathematics, second series, vol. 16, 1914–15, pp. 45–49.

first encountered in the study of calculus, because of the treatment.

It seems that calculus texts derive the formula

$$ds^2 = dx^2 + dy^2$$

for curves in the plane, and a corresponding formula for curves in space, before defining arc length. Some do not define arc length at all. This is accomplished by use of the property that "limit of arc/chord $=1$." The property is assumed without proof as applying to the undefined arc length. The student is familiar with the property for the special case of a circle, so is likely to pass over this omission without noting it.*

Two procedures can be followed to meet the difficulty in question.

(1) Seek a definition of arc length having the property that limit of arc/chord $=1$. The usual derivation then gives the formula $(ds/dx)^2=1+(dy/dx)^2$, from which the formula $\int_a^b[1+(dy/dx)^2]^{1/2}dx$ or $\int_{t_0}^{t_1}[(dx/dt)^2+(dy/dt)^2]^{1/2}dt$ for the length is derived; and corresponding formulas for the case of a curve in space. The formula is taken as the definition of arc length. Then the assumed property should be proved valid, by use of the formula just derived.

(2) We prefer the following method. The case of a curve in space in parametric form is taken:

$$x = f(t), \qquad y = g(t), \qquad z = h(t),$$

where f, g, h are given continuous and with continuous first derivatives, $c \leq t \leq d$. Define arc length as the limit, if it exists and is unique, of the length of the inscribed broken line determined by the values $t_0=c$, t_1, t_2, $\cdots$, $t_n=d$, $(t_k<t_{k+1})$, as n becomes infinite and the largest (t_k-t_{k-1}) approaches zero. Letting $\Delta t_k=t_k-t_{k-1}$, and Δx_k, Δy_k, Δz_k the corresponding changes in x, y, z respectively, we have

$$\lim_{n\to\infty}\sum_{k=1}^{n}[\Delta x_k^2+\Delta y_k^2+\Delta z_k^2]^{1/2} = \lim\sum\left[\left(\frac{\Delta x_k}{\Delta t_k}\right)^2+\left(\frac{\Delta y_k}{\Delta t_k}\right)^2+\left(\frac{\Delta z_k}{\Delta t_k}\right)^2\right]^{1/2}\cdot\Delta t_k$$

$$= \lim\sum\{[f'(u_k)]^2+[g'(v_k)]^2+[h'(w_k)]^2\}^{1/2}\cdot\Delta t_k,$$

$$(t_{k-1} < u_k,\ v_k,\ w_k < t_k)$$

by the law of the mean of the differential calculus, and thus by Bliss's theorem

$$= \int_c^d \{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2\}^{1/2}dt$$

$$= \int_c^d \left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right\}^{1/2}dt.$$

* At least one text gives a proof of the property for the case of an arc in the plane with no inflection, using an implied, but not stated, definition of arc length. But an arc to which the property applies may have an infinite number of points of inflection—an objection of little importance, some will say. However, nothing is proved for the case of a curve in space.

Next prove that limit of arc/chord = 1. The formula $ds^2 = dx^2 + dy^2 + dz^2$ can then be proved either by the usual method, or by use of the integral formula for length.

If either method (1) or method (2) is used, and the formula for ds^2 is derived in the usual place in the calculus course, reference must be made to the future work in the integral calculus for the definition of arc length and the proof that limit of arc/chord = 1.*

COMMENTS ON THE PRECEDING PAPER

E. J. Moulton, Northwestern University

Catching errors in text-books or printed articles is an instructive and amusing game.† Reflection on the question of accuracy in mathematical writing leads me to make the following comments.

To obtain completeness and rigor seems impossible. Moreover, in elementary college mathematics, vigor should not be sacrificed for an attempt at complete rigor. At times we probably strive too hard for rigor, especially in the first courses in calculus, in which we must be brief and *we cannot be rigorous*. A text-book writer or a teacher has other things than rigor to consider, and it is, in a sense, unfair to be severe in criticizing his occasional lapses. It is more interesting to look for lapses by the critics, and this I shall do in an effort to show what I mean by the statement that *we cannot be rigorous*.

To illustrate my point I shall criticize Professor Brown's critical presentation of an application of the definite integral to a "moment of inertia" problem where he favors the use of Bliss's substitute for Duhamel's theorem. Some of my criticisms would apply to somewhat similar discussions by W. F. Osgood, R. L. Moore, and G. A. Bliss, which appeared in the *Annals of Mathematics* about thirty years ago, and to others which have appeared more recently. For simplicity of exposition, I shall comment in some detail on a succession of sentences in Brown's paper, section 11, first paragraph.‡

He refers to "a straight wire of variable density $\rho(x)$, stretched along the x-axis from a to b." No definition is given for "variable density," and no postulates are stated. It will be hard to argue rigorously under these conditions. We also observe for future reference that a and b are *points*, and *not numbers*.

* The following additional items, while not serious may be worth mentioning. In deriving the the formula $\tan(A/2) = \sin A/(1+\cos A)$ by use of radicals, several trigonometry texts drop a "plus or minus" sign at the last step, without explanation. Analytics texts contain the statement that if a right circular cone is cut by a plane parallel to an element (omitting the necessary restriction that it be parallel to only one element), the intersection is a parabola. Calculus texts give the formula $\int dx/x = \log x + C$, which is meaningless to the student when x is negative.

† It tickles me to find that Professor Brown, in the preceding, interesting article, has called my attention to an error which I have carelessly made for years in teaching freshman mathematics.

‡ For emphasis I shall try to make out a bad case against Professor Brown, but I want to say in advance that I consider his paper stimulating and valuable for its criticisms.

"The interval from a to b is divided into parts $\Delta x_1, \cdots$." Thus Δx_k is a *line segment*, and is *not a number.*

"Let x_k be the coördinate of the left end of the interval Δx_k." Thus x_k is a *number*, and it is *emphasized that* Δx_k *is an interval*, or *segment*, and not a number.

"Let $\bar{x}_k$ be the coördinate of a point of Δx_k of average density for Δx_k." Thus $\bar{x}_k$ is a number, and presumably Δx_k is again a segment made up of points. The term "average density" is used without definition or set of postulates; hence it will be impossible to argue rigorously about it. An adequate definition of average density, if based on a previous concept of density, would appear to require the use of the definite integral formula for total mass, which is too complicated to pass over lightly. Moreover, we note that it says "average density for Δx_k"; apparently it is average density for a line segment instead of average density for a piece of wire, or else Δx_k is now a piece of wire instead of a line segment. Also, observe that it says, "let $\bar{x}_k$ be $\cdots$"; before we may proceed logically to use $\bar{x}_k$, *we must prove that there is such a number.* Since such a proof is not given, the whole argument is lacking in rigor. Furthermore, it was not assumed that the density function is continuous, and without more hypotheses than are stated it would not be possible to prove the existence of $\bar{x}_k$ under the usual definition of average density.

"Let $\bar{\bar{x}}_k$ be the coördinate of a point (usually unique) of Δx_k such that a particle whose mass equals the mass of Δx_k, placed at $\bar{\bar{x}}_k$, will have the same moment of inertia as Δx_k." Here $\bar{\bar{x}}_k$ is first a number, then later it is a point; and Δx_k is first a line segment, then later a piece of wire. We have a particle (undefined), the particle has a mass (undefined), and it has a moment of inertia (undefined). There is said to be an equality between the mass of the particle and the mass of Δx_k (which now is a piece of wire); and also an equality between the (undefined) moment of inertia of the particle and the (undefined) moment of inertia of Δx_k (*presumably* now a piece of wire). The problem of supplying a definition of the moment of inertia of the piece of wire, which is logically necessary at this stage, is most satisfactorily met by taking the final answer of the whole problem as the desired definition; but if this is done, why bother to be so rigorous, for we do not need to prove a definition! The matter of supplying a definition of the moment of inertia of the wire therefore presents a dilemma. Suppose that we succeed in solving it without using the formula we are trying to establish. As a vital part of the logic, we then must prove by use of this definition that there is a number (and point) $\bar{\bar{x}}_k$ satisfying the stated conditions. Obviously, to make the argument rigorous, much remains to be done in connection with the sentence quoted at the beginning of this paragraph.

In section 11, paragraph (6), the author writes, "The moment of inertia due to Δx_k is $\rho(\bar{x}_k)\bar{\bar{x}}_k^2\Delta x_k$." We note that previously it was "moment of inertia of Δx_k," and now it is "moment of inertia due to Δx_k," an intriguing difference. Moreover, we may now wonder whether Δx_k is a piece of wire, a line segment, or a number. We started out to find the moment of inertia of a piece of wire; hence, presumably, "due to Δx_k" indicates that now Δx_k is a piece of wire (which

is, we suppose, quite different from a geometrical line segment, or a number, since, for example, the piece of wire possesses "variable density"). As we have seen, however, $\bar{x}_k$ and $\bar{\bar{x}}_k$ are (or were!) numbers associated with a segment Δx_k, and we may therefore fear that in the quotation Δx_k must mean this segment. But at the end of the quotation we are required to multiply Δx_k by a number; hence Δx_k must be a number, and not a piece of wire or a line segment, since we have not defined products for such things. Since we no longer know what Δx_k means, what chance is there for rigorous thinking when the symbol is used?

A little greater care on the part of the writer might have made the exposition less susceptible of criticism, but the fundamental difficulty which comes from a lack of really *careful definition of* what is meant by the *moment of inertia of a wire of variable density*, and the *logically necessary proof of the existence of* $\bar{x}_k$ *and* $\bar{\bar{x}}_k$ is far from simple. I fear that such a proof, if rigorous, would involve quite a lot of fundamental point set theory, and we might even get caught on some snag in the foundations of mathematics which is still a subject for debate.

THE ALTITUDE QUADRIC OF A TETRAHEDRON

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1. Introduction. The four altitudes of a general tetrahedron belong to one regulus of a quadric surface, which we shall call the altitude quadric. The orthocentric lines and the normals to the faces through their orthocenters belong to the other regulus.* This quadric is a hyperboloid of one sheet, for the lines of a regulus of a hyperbolic paraboloid are all parallel to a plane, and this is impossible with the four altitudes of a tetrahedron.

2. The center. Each ruling of a regulus of the hyperboloid is parallel to one line of the other regulus. The points midway between these two parallels lie on a line which passes through the center of the quadric and is a generator of the asymptotic cone. If we recall Mannheim's theorem to the effect that a normal to a face midway between the foot of the altitude and the orthocenter of the same face passes through the Monge point, we may state the following:

Theorem 1. *The center of the altitude quadric of a tetrahedron is the Monge point.*

3. The altitude property. It is known that altitudes from three vertices of a tetrahedron have the property that a plane through one parallel to the common perpendicular of the other two passes through the orthocentric line from the fourth vertex.† We shall call this the altitude property, and prove next the following:

Theorem 2. *A unique tetrahedron can be constructed having any four lines of one regulus of an altitude quadric as the altitudes.*

* Altshiller-Court, Pure Solid Geometry, p. 66.

† *Ibid.*, p. 69.

Proof. Let the quadric be determined by three altitudes r_1, r_2, r_3 of a tetrahedron. The orthocentric line t_4 through the fourth vertex is of course determined. Any point of this line may now be used as a vertex of a new tetrahedron having the three original rulings as altitudes. The fourth altitude r_4 must be a line of the same quadric, for three skew lines determine the quadric. We may proceed as before and construct a tetrahedron using r_2, r_3, r_4 as altitudes and include an additional arbitrary r_5 as the fourth altitude. In two more steps, we have a tetrahedron with arbitrary rulings r_4, r_5, r_6, r_7 as its altitudes. Referring to the other regulus as consisting of t rulings, we may remark that any quadruple of t rulings may be reflected into a quadruple of r rulings. Hence the announced result is valid for any four rulings of either regulus.

COROLLARY. *Any three rulings of the same regulus of an altitude quadric have the altitude property.*

4. Application to the asymptotic cone. Let r_1', r_2', r_3', t_4' be generators of the asymptotic cone parallel to three altitudes and the orthocentric line they determine on the quadric. We see at once that t_4' is the orthocentric line of the trihedron r_1', r_2', r_3'. Since this is true for any three rulings, we may state the following:

THEOREM 3. *The orthocentric line of three generators of the asymptotic cone of an altitude quadric is itself a generator of this cone.*

The fact that any one of the four lines r_1', r_2', r_3', t_4' is the orthocentric line of the trihedron consisting of the other three shows us how a third altitude r_3 may be found when two altitudes r_1, r_2 and the orthocentric line t_4 are given. We see readily that r_3 is parallel to the t ruling in which the plane through r_2 parallel to the common perpendicular of r_1 and t_4 cuts the surface.

5. Tetrahedrons with three given altitudes. From what has been said we conclude that tetrahedrons sharing three altitudes have their Monge point in common. If the vertex on t_4 is D, then the directions of AD, BD, CD are fixed since they are common perpendiculars of r_2, r_3, r_1 taken in pairs. When D moves uniformly along t_4, A, B, C move uniformly along r_1, r_2, r_3, respectively. Thus the centroid of the tetrahedron moves uniformly along a line. Since the centroid is midway between the Monge point and the circumcenter, we can summarize in the following:

THEOREM 4. *All tetrahedrons having three lines as common altitudes have the same Monge point, and their centroids and circumcenters lie on parallel lines.*

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

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REVIEWS

Three Copernican Treatises. The *Commentariolus* of Copernicus, The *Letter against Werner*, The *Narratio Prima* of Rheticus. Translated with an introduction and notes by Edward Rosen. (Records of Civilization: Sources and Studies, no. 30.) New York, Columbia University Press, 1939. 11+211 pages. $3.00.

Copernicus's great work *De revolutionibus orbium caelestium* was published in 1543, immediately before the author's death, its printing being supervised in part by his student, George Joachim Rheticus, and in part by the Lutheran clergyman Osiander. At an earlier date Copernicus had written a brief account of his astronomical system, without the elaborate calculations required for the larger work, which circulated in manuscript during his life-time but which was not published until 1878. In 1524 he wrote also a criticism of a paper on the *Motion of the Eighth Sphere* by John Werner, which also was known in manuscript but was first printed in 1854. These were the only works written by Copernicus on astronomy, except his *magnum opus*. In 1540 his admiring student Rheticus, partly to make known his master's discoveries and partly to test the reception that his innovations might expect from the learned world, published a survey of the new astronomy called *Narratio prima*. These are the three treatises which Mr. Rosen has here presented for the first time in English translation, believing quite rightly that they form an introduction to Copernicus's work more readily comprehensible to the non-technical reader than his larger book.

Translation of this sort is extremely difficult, for it requires a combination of linguistic, historical, and scientific competence which can be developed only

by the systematic study of the history of science. So far as the reviewer is able to judge Mr. Rosen has met all the requirements admirably. His English version is clear, readable, and apparently accurate. He has supplied in his notes an excellent commentary on the more difficult passages and on the necessarily loose use of terms that occurs in any writer before the growth of a settled technical vocabulary. In his introduction he gives a history of the texts that he translates, a brief but adequate account of the famous controversy about "hypotheses" which was occasioned by Osiander's misleading preface to Copernicus's principal work, and a clear statement of the elements of the new astronomy, in comparison with the rival Ptolemaic system.

In his preface Mr. Rosen states that he intends later to publish a translation of the *De revolutionibus*, which also has never appeared in an English version. It is greatly to be hoped that he will be able to carry out this intention. Works of this sort, by adequately trained scholars, are essential to bring the sources of the history of science within the reach of scientists, who can rarely have the historical knowledge needed to read a work of the sixteenth century, and of historians, who can rarely have the scientific knowledge needed to understand precisely what was achieved. They contribute also to the general reader's grasp of modern science as the great intellectual achievement of our civilization.

G. H. Sabine

Report of the Fifth Annual Research Conference on Economics and Statistics, July 3 to 28, 1939. Colorado Springs, Colorado: Cowles Commission for Research in Economics. 93 pages.

The increasing use of mathematics in the development of economic theory is illustrated by this report. It gives abstracts of the 38 lectures which were presented during the fifth annual research conference on economics and statistics of the Cowles Commission.

Mathematical methods were applied in 14 of the papers presented, or more than one-third of the total number. There were 3 lecturers who used mathematical notation but did not draw on mathematical theory. There were 7 papers in which well known mathematical theory played an essential rôle in the development of economic theory. There were 4 papers which presented new results in mathematical statistics.

The following titles of these 14 papers indicate their general nature; brief abstracts are available in the conference report.

1. *Mathematical Notation Used.*

The General Problem of Durable-Goods Demand, Charles F. Roos.

The Adaptation of Index-Number Construction to Punched-Card Equipment, Francis McIntyre.

The Influence of Price on Exports, J. B. D. Derksen.

2. *Mathematical Theory Applied.*

Problems in the Theory of Business Cycles, Harold T. Davis.

A Simplified Economic System with Dynamic Elements, Francis W. Dresch.

The Use of Fourier Series in the Analysis of Seasonal Variation, Alexander Sturges.

Some Remarks on the Dynamic Theory of Production, Gerhard Tintner.

Individual and National Income and Consumption, Jakob Marschak.

Sources for Demand Analysis: Market, Budget, and Income Data, Jakob Marschak.

Relationships between the Distribution of Income and Total Real Income, Harold T. Davis.

3. *Results in Mathematical Statistics.*

The Fitting of Straight Lines if Both Variables are Subject to Error, Abraham Wald.

Statistical Testing of Dynamic Systems if the Series Observed are Shock Cumulants, Trygve Haavelmo.

On the Measurement of the Degree of Inequality of Income Distributions, Horst Mendershausen.

The Problem of Estimating the Length of the Cycles Created by the Moving Average and by Other Graduation Processes, Edward L. Dodd.

This report provides a sample of modern econometrics and mathematical statistics which will be of interest to the mathematician as well as to the economist or statistician.

M. M. FLOOD

Mathematics for Actuarial Students. By Harry Freeman. Part II: Finite Differences, Probability, and Elementary Statistics. Published for the Institute of Actuaries by the Cambridge University Press, 1939. 13+339 pages. 25 s.

Since changes have been made in the syllabus for the examinations of the Institute of Actuaries, Harry Freeman has revised his *An Elementary Treatise on Actuarial Mathematics.* This was first published in 1932 in England and since that time has been a standard text for students preparing for actuarial examinations. In 1936 a fourth edition was published and now the *Treatise* has been enlarged and published in two parts: *Mathematics for Actuarial Students*, Part I: Elementary Differential and Integral Calculus, 8+184 pages; Part II: Finite Differences, Probability, and Elementary Statistics, 13+340 pages.

Although this text is designed to meet the need of students preparing for the examinations of the Institute of Actuaries, it will be found useful in preparing for Parts II and III of the Associateship examinations given jointly by the Actuarial Society of America and the American Institute of Actuaries.

The section on finite differences which includes interpolation, summation, and approximate integration covers about two-thirds of the text. The first chapter on definitions and fundamental formulas stresses the use of the symbolic operators Δ and $E\equiv 1+\Delta$. In the next three chapters, Newton's formula, Newton's divided difference formula, Lagrange's interpolation formula and the central difference formulas of Gauss, Stirling, Bessel, and Everett are considered.

In Chapter V, the practical methods of successive approximation and the elimination of third differences are used to solve problems in inverse interpolation. The application of finite differences to the summation of series is given in Chapter VI. In Chapter VII, "Miscellaneous Theorems," and Chapter VIII, "Modern Extensions and Special Devices," Freeman deals much too briefly with certain modern methods of interpolation and summation. The remaining chapter in this section is on approximate integration.

The section on probability is very good. Thirty-one rather interesting problems are fully discussed in the text and a list of ninety-six problems is given at the end of the chapter.

In the two chapters on elementary statistics the topics briefly considered are: averages (mean, median, and mode), probable value, expectation, the range, mean deviation, standard deviation, semi-inter-quartile deviation, relative measures of dispersion and probable error. To the reviewer, this is an inadequate treatment of elementary statistics for actuarial students.

In the last chapter of the text the calculus is applied to the solution of certain problems in probability.

An excellent feature of the book is the unusually large number of examples illustrating the various formulas and methods discussed in the text. A large number of problems is given at the end of each chapter and a miscellaneous collection of one hundred problems is added at the end of the book. Answers to all problems are given in a special section. These problems and their answers will be helpful to students preparing for the actuarial examinations.

C. H. Graves

Aufgabensammlung der Planimetrie. By F. Gonseth and P. Marti. Zürich and Bern, Orell Füssli, 1939. 154 pages, 33 figures. Fr. 3.10.

This book is apparently designed as a supplementary text to two texts on plane geometry by the same authors, published by the same firm. It contains over 1500 problems. It could be used, as suggested in the preface by the authors, over a period of several years without fear of much duplication of problems. Most of the problems are devoted to the topics that are familiar to American students studying plane geometry. In addition, the book contains problems involving the use of the rectangular coördinate system, the linear function $y=ax$, theorems on the "power" of a point, and simple perspectivity relations. The problems are well selected, interesting, and stimulating. The mechanical make-up of the book is good.

W. O. Gordon

Denksport-Aufgaben. Die Mathematik an der Schweiz. Landesausstellung Zürich 1939. By E. Trost. Zürich, Orell Füssli, 1939. 32 pages.

The first few pages of this short tract are devoted to a discussion of the mathematical exhibit at the Swiss Exposition of 1939 at Zürich. The discussion is

presented in the same style as the original lectures to the visiting public at the exhibition.

This exhibit consisted of several plaques, pictures, graphs, and models designed to acquaint the general public with some of the concepts treated in the various fields of mathematics. It contained, to mention only three items, soap film models of minimal surfaces, an illustration of a method for constructing a curve through every point of the interior of a square, and examples of one- and two-sided surfaces.

The next few pages are devoted to a discussion of the nature and content of 22 problems which were presented to the general public for solution. These problems presupposed no special mathematical training. They might be characterized as mathematical puzzles. Several of these, or problems similar to them, have appeared in various American newspapers and magazines.

The remaining pages are devoted to the statement and solution of these 22 problems.

W. O. Gordon

College Algebra. By L. M. Reagan, E. R. Ott, and D. T. Sigley. New York, Farrar and Rinehart, Inc., 1940. 18+445 pages. $2.50.

The general scope and content of a text on college algebra are pretty well prescribed. Minor variations in sequence of topics are permitted, and rather wider variations in selection of material may be exercised, based on the purpose of the particular course in view. There is no such thing as a best book on algebra, since the general aims and purposes of different books may be widely divergent and yet equally meritorious.

The present book is the outcome of the efforts of three different teachers, all of considerable experience in widely different environments, to produce a unified book capable of providing sufficient instruction in algebra for all ordinary college purposes.

The book is rather large; it contains more material than is used in such a course. It does not include any work on statistics, certainly a welcome chapter in a well-balanced course at the present time. It does include an acceptable chapter on determinants, one on series, on partial fractions, on the solution of the general cubic and the general quartic equations, on inequalities, as well as a full, well-written chapter on the theory of equations.

Of the 209 paragraphs of the text proper, 30 are marked for possible omission in an abbreviated course. The plan provides for a five-hour one-semester course, or for a three-hour course, if the omissions are made. Each new chapter has one distinct feature, that of being approached inductively. In the hands of an alert and sympathetic teacher, this procedure has bright possibilities. This is particularly true of the chapters on choice and chance, which come unusually early in the book. A very large number of exercises appear throughout the book, many of them requiring careful thought.

Virgil Snyder

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.

CLUB REPORTS, 1938-39

Continuing with the plan of club topics as presented in the last issue of this Monthly, we have selected from remaining club reports some subjects or topics heretofore unlisted. These with the lists which have gone before should form a working basis for choosing subjects for papers and discussion. As in the case of the other listings of club topics we should greatly appreciate a bibliography, from either club files or individuals, which would prove helpful in the development of the subject-matter.

Pi Mu Epsilon, Hunter College of the City of New York

Central topic: Linear inequalities*

Review of linear equations
Necessary and sufficient conditions for the inconsistency of linear inequalities
Applications of the theorems
History of linear inequalities, application to proportional representation
Mixed systems of linear equations and inequalities
Geometrical solutions and applications of the theorem

Mathematics Club, University of Kansas

Mathematical aspects of chemistry
The four-color problem
Non-euclidean geometries
Mathematical relationships in physical chemistry
Mathematics in the theory of light

Mathematics Club, Northwestern University

Mathematics in America
Connected ordered groups
Introduction to the theory of numbers
Continued fractions
From Euclid to Einstein
Sporting goods, production problems and mathematics
Power series in which each exponent is twice the preceding
Series of linear inequalities
A brachistochrone problem

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* W. B. Carver, Systems of linear inequalities, Annals of Mathematics, vol. 23, no. 3, March, 1922.

L. L. Dines, Systems of linear inequalities, Annals of Mathematics, vol. 20.

H. Minkowski, Geometrie der Zahlen, Leipzig, 1910, pp. 40–45.

W. V. Lovitt, Preferential voting, this Monthly, vol. 23, 1916, pp. 363–366.

R. W. Stokes, A geometric theory of solution of linear inequalities, Transactions of the American Mathematical Society, vol. 33, no. 3, pp. 782–805.

H. M. Schlauch, Mixed systems of linear equations and inequalities, this Monthly, vol. 39, no. 4, April, 1932.

Mathematics Club, Milwaukee-Downer College

The great pyramid, its mathematical significance and some superstitions regarding the relationship of various dimensions to important events

The logarithmic spiral and its approximation to the curve of life

The mathematical relationships found in music and the application of mathematics to music in such fields as acoustics

The construction and use of the telescope

Pi Mu Epsilon, University of California at Berkeley

Geometrical optics
Uniform squares
Geometrical representation of critical phenomena
The application of mathematics to certain biological developments
Bernoulli numbers
Demonstration of string models of ruled surfaces

Pi Delta Theta, University of Denver

Oddities in mathematics
The perpetual calendar
The mathematics of the Hoover Dam

Phi Chi Mu, Washington and Jefferson College

Number writing in the binary system
Functions in polar coördinates

Mathematics Club, Oshkosh State Teachers College

Mathematics used in the construction of the automobile
Contributions of the Chinese to mathematics
Mathematics and general culture
Treatment of individual differences in teaching high school mathematics

Kappa Mu Epsilon, Athens College

Relation of mathematics and psychology, chemistry, business, science, religion, biology
Relation of mathematics to life

Aphesteon, Montclair State Teachers College

Fermat's last theorem
The teaching of junior high school mathematics
Organizing high school mathematics clubs
The four-color problem
Linkages
Five little solids and how they grew

Mathematics Club, Stanford University

The problem of finding a point, the sum of whose distances from two fixed points is a minimum
The problem of finding the triangle of minimum perimeter inscribed in a given triangle
On the theory of some mathematical games

The classical converse of Lehmus-Steiner: If the bisectors of the base angles of a triangle are equal, then the triangle is isosceles*

Three fundamental inequalities: Cauchy's, Minkowski's, Hölder's

* Mathematical Gazette, vol. 22, no. 251.

School Science and Mathematics, vol. 39, pp. 561–572; includes bibliography

Pi Mu Epsilon, University of Nebraska

The theory and application of the slide rule (The speaker, Mr. Hush, displayed his collection of rules which included straight and round rules, a sixty foot Thatcher rule, a forty-five foot rule, and other miscellaneous rules.)

The solution of chemical equations by determinants

A demonstration of the experimental determination of pi

Kappa Mu Epsilon, Nebraska State Teachers College at Wayne

The duodecimal number system

Intersections

The challenge of the slow pupil

Mathematics in mechanical engineering

Pythagoras and the Pythagoreans

Mathematics Club, Ball State Teachers College

Mathematics and architecture

Mathematics as applied to astronomy

Mathematics in insurance

Biography of one famous mathematician presented at each meeting

Mathematics Round Table, University of Illinois

The index of a closed curve in a direction field

Linear functional equations

Normal families and Picard's theorem

Some contributions to the theory of centroids

Almost periodic functions

Alpha Mu Gamma, St. Lawrence University

Mathematical analysis of simple card tricks

Chemical calculations

The fourth dimension

Extra-sensory perception experiments at Duke University

Chance in gambling

Pi Mu Epsilon, Washington University

The use of actuarial analysis in the determination of the average life of a telephone plant

First principles of groups of substitutions

The elementary principles of relativity

Some elementary ideas underlying vector algebra, ordinary and projective

Some elementary problems in geometric probability

Buffon's needle problem

The determination of the real roots of a certain eighth degree equation that enters in astronomy

Kappa Mu Epsilon, University of New Mexico

The golden section

Remarks concerning a conjecture of Minkowski

Classical development of our number system

Pi Mu Epsilon, University of Georgia

The present status of high school mathematics in the Progressive Education Program

Use of the parabola and the catenary in finding necessary length of a cable to allow a given sag

Hydraulics—The Panama Canal

Kappa Mu Epsilon, Emporia State Teachers College

Perpetual calendars
The applications of mathematics to psychology and statistics
Mathematics research in Poland

Euclid's Circle, Mount St. Scholastica College

The mathematical basis of life insurance
Some mathematical considerations of investments
Short cuts in interest and discount
Relationship of philosophy to mathematics
Sidelights of the Cardan-Tartaglia controversy

Echols Mathematics Club, University of Virginia

Fallacies and paradoxes
The meaning of distance
Surfaces and paper cutting
Mathematical oddities

Pi Mu Epsilon, University of California at Los Angeles

Spectacular mathematics
Fibonacci's series
Kronecker delta

Newtonian Society, Lehigh University

Elasticity
History of mathematics in America
Determinants
Algebraic fallacies

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 425. *Proposed by W. C. Rufus, University of Michigan.*

A man sold cows at a price per head equal to the number. With the proceeds he bought an odd number of calves at \$10 each, and a pig with the remainder (less than \$10). If a pig cost one-sixth as much as a cow, how many calves did he buy?

E 426. *Proposed by V. Thébault, Le Mans, France.*

Find the locus of a point whose polar planes with respect to four given spheres are concurrent, and the locus of the point of concurrence.

E 427. *Proposed by A. Gloden, Luxembourg.*

Find a palindromic pentagonal number greater than 22.

E 428. *Proposed by Ruth Mason Ballard, Chicago.*

It can be shown simply that there is only one way of replacing the asterisks by the integers from 1 to 7 so as to make

$$\begin{matrix} * & 9 & * \\ * & * & * \\ 8 & * & * \end{matrix}$$

a magic square, and that this square cannot be uniquely determined by fewer than two of the nine numbers. Give an analogous scheme for a 4×4 square, using the numbers 1, 7, 11, 16 (and twelve asterisks). Is it possible to determine the 4×4 square by fewer than four of the sixteen numbers?

E 429. *Proposed by Emma Lehmer, Berkeley, Calif.*

Prove that

$$\sum_{k=0}^{[n/3]} \binom{n}{3k} = \frac{2^n + r}{3},$$

where $r = (-1)^n 2$ or $(-1)^{[(n+1)/3]}$ according as n is or is not divisible by 3.

E 407 [1940, 110]. *Correction.* The last sentence should read: "Prove that the points D, E, F and D', E', F' lie respectively on two parallel lines, perpendicular to the Euler line."

E 409 [1940, 110]. *Correction.* By considering a particular hexagon, E. P. Starke has shown that the stated result is false. The proposer and editor accordingly beg leave to withdraw the problem.

SOLUTIONS

E 391 [1939, 597]. *Proposed by James Travers, Harrow, England.*

If P is a point inside a square $ABCD$, so situated that $PA:PB:PC=1:2:3$, calculate the angle APB. Use only the methods of Euclid, Book I.

Solution by V. W. Graham, Dublin, Ireland.

Construct N outside the square so as to make angle $CBN=ABP$ and $BN=BP$. Join PN, CN. Applying Pythagoras' theorem to the triangle PBN, we have

$$PN^2 = 2BP^2 = 8AP^2.$$

Since triangles ABP, CBN are congruent, $CN=AP$. Hence

$$PN^2 + CN^2 = 9AP^2 = CP^2,$$

and CNP is a right angle. But since the right triangle PBN is isosceles, PNB is half a right angle. Therefore the required angle, being equal to angle CNB, is $1\frac{1}{2}$ right angles.

Corollary. The segment CP is bisected by the line through B perpendicular to AP.

Also solved by L. R. Chase and the proposer. A similar problem was published in Haugh's *Higher Arithmetic*, about 1880.

E 392 [1939, 597]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Determine the locus of a point from which the four normals to a given ellipse form a harmonic pencil.

Editorial Note. We know that the feet of the normals from any point (x_1, y_1) to the ellipse $x^2/a^2+y^2/b^2=1$ lie on the "Apollonius' hyperbola" $c^2xy+b^2y_1x-a^2x_1y=0$. Since this hyperbola goes through (x_1, y_1) and the origin, the four normals have the same cross-ratio as the lines joining the origin to their feet, namely the lines

$$(c^2xy)^2 = (b^2y_1x - a^2x_1y)^2(x^2/a^2 + y^2/b^2)$$

or

$$(b^4/a^2)y_1^2x^4 - 2b^2x_1y_1x^3y + (a^2x_1^2 + b^2y_1^2 - c^4)x^2y^2 - 2a^2x_1y_1xy^3 + (a^4/b^2)x_1^2y^4 = 0.$$

Now, the roots of the general quartic equation $px^4+4qx^3+6rx^2+4sx+t=0$ are harmonic if one root of the "reducing cubic" is zero, *i.e.*, if

$$\begin{vmatrix} p & q & r \\ q & r & s \\ r & s & t \end{vmatrix} = 0.$$

(See, for instance, Durell and Robson, *Advanced Algebra*, London, 1937, p. 314.) In the present case this means that

$$\begin{vmatrix} 6(b^4/a^2)y_1^2 & -3b^2x_1y_1 & (a^2x_1^2+b^2y_1^2-c^4) \\ -3b^2x_1y_1 & (a^2x_1^2+b^2y_1^2-c^4) & -3a^2x_1y_1 \\ (a^2x_1^2+b^2y_1^2-c^4) & -3a^2x_1y_1 & 6(a^4/b^2)x_1^2 \end{vmatrix} = 0.$$

Hence the locus of (x_1, y_1) is

$$54a^2b^2x^2y^2(a^2x^2+b^2y^2-c^4) - 54(a^2b^4x^2y^4 + a^4b^2x^4y^2) - (a^2x^2+b^2y^2-c^4)^3 = 0$$

or

$$(a^2x^2+b^2y^2-c^4)^3 + 54a^2b^2c^4x^2y^2 = 0.$$

This curve has the same general appearance as the evolute

$$(a^2x^2+b^2y^2-c^4)^3 + 27a^2b^2c^4x^2y^2 = 0;$$

it has the same cusps, but otherwise lies entirely inside the evolute, *i.e.*, in the region from whose points the four normals are all real.

E 393 [1939, 597]. *Proposed by V. Thébault, Le Mans, France.*

Find two perfect squares, of five digits each, which together contain all the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. How many solutions are there?

Solution by G. W. Wishard, Norwood, Ohio.

Since the sum of the ten digits is divisible by 9, the sum of the two squares must be divisible by 9. Since $(3m+1)^2+(3n+2)^2$ is never divisible by 9, each square must be divisible by 9, and its root by 3. By referring to Barlow's Tables, we see that eighteen such squares with five distinct digits occur, namely:

$117^2 = 13689$,	$186^2 = 34596$,	$228^2 = 51984$,
$126^2 = 15876$,	$189^2 = 35721$,	$252^2 = 63504$,
$144^2 = 20736$,	$195^2 = 38025$,	$267^2 = 71289$,
$147^2 = 21609$,	$198^2 = 39204$,	$273^2 = 74529$,
$153^2 = 23409$,	$213^2 = 45369$,	$282^2 = 79524$,
$174^2 = 30276$,	$219^2 = 47961$,	$309^2 = 95481$.

Among these, the following eight solutions are found:

(15876, 23409),	(20736, 51984),	(30276, 51984),	(38025, 47961),
(15876, 39204),	(20736, 95481),	(30276, 95481),	(63504, 71289).

Also solved by W. E. Buker, L. R. Chase, Daniel Finkel, D. D. Leib, K. W. Miller, E. P. Starke, W. R. Talbot, C. W. Trigg, Alan Wayne, B. C. Zimmerman, and the proposer.

E 394 [1939, 597]. *Proposed by N. A. Court, University of Oklahoma.*

If the lines AM, BM, CM joining any point M to the vertices A, B, C of a tetrahedron $ABCD$ meet the respective opposite faces in the points P, Q, R, and if the lines DM, DP, DQ, DR meet the face ABC in the points S, X, Y, Z, prove that (both in magnitude and in sign)

$$\frac{DM}{MS} = \frac{1}{2}\left(\frac{DP}{PX} + \frac{DQ}{QY} + \frac{DR}{RZ}\right).$$

Solution by L. M. Kelly, Boston University.

If M is the centroid of masses a, b, c, d at A, B, C, D, then P will be the centroid of b, c, d; and X of b and c. Hence

$$DP/PX = (b + c)/d,$$

and similarly $DQ/QY=(c+a)/d$, $DR/RZ=(a+b)/d$. Therefore

$$\frac{1}{2}\left(\frac{DP}{PX} + \frac{DQ}{QY} + \frac{DR}{RZ}\right) = \frac{a + b + c}{d} = \frac{DM}{MS}.$$

Also solved by W. E. Buker, T. C. Esty, V. W. Graham, and the proposer.

E 395 [1939, 597]. *Proposed by E. P. Starke, Rutgers University.*

In high school geometry texts and elsewhere one frequently meets the statement that the reason for the straightness of the crease in a folded piece of paper is that the intersection of two planes is a straight line. This is fallacious. What is the correct reason?

Solution by L. R. Chase, Rogers High School, Newport, R. I.

Let P, P' be two points of the paper that are brought into coincidence by the process of folding. Then any point A of the crease is equidistant from P, P', since the lines AP, AP' are pressed into coincidence. Hence the crease, being the locus of such points A, is the perpendicular bisector of PP'.

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3963. *Proposed by V. Thébault, Le Mans, France.*

In an orthocentric tetrahedron the first sphere of twelve points is the locus of points the sum of whose powers, with respect to the spheres having as diameters the edges (or bimedians), is zero. Generalize.

Note. The bimedians are the straight line segments joining the midpoints of opposite edges.

3964. *Proposed by V. Thébault, Le Mans, France.*

The sum of the powers of the vertices of a tetrahedron, with respect to the Monge sphere of the circumscribed ellipsoid of Steiner, is equal to the negative of half the sum of the squares of the edges.

SOLUTIONS

3878 [1938, 389]. *Proposed by V. Thébault, Le Mans, France.*

A convex quadrilateral is circumscribed about a circle. Show that there exists a straight line segment with ends on opposite sides dividing both the perimeter and the area into equal parts. Show that the straight line passes through the center of the inscribed circle. Consider the converse.

Editorial Note. The proposer gave the following indications of a solution: Let the consecutive sides of the quadrilateral be $AB=a$, $BC=b$, $CD=c$, $DA=d$; and let the segment EF divide its perimeter into equal parts where its ends are on AB and CD, respectively. We find easily that $EB+CF=d$ and $AE+FD=b$. It then follows that EF must pass through the center of the inscribed circle, if it divides the area also into equal parts. The values of $x=EB$ and $y=CF$ may be obtained by considering the triangle AMD formed by the prolongations of the sides AB and CD. In addition, EF is tangent to a hyperbola and to a parabola which are easily determined; and the desired line EF is a tangent passing through the center of the circle. The same considerations apply to the opposite sides BC and DA. A discussion of the possibility of the construction is interesting.

In what follows will be supplied two constructions for the desired line, if it exists, using straight lines and circles, one of which corresponds perhaps to the parabola mentioned by the proposer. Since the sides are formed by eight tangents to the circle equal in pairs, the sum of the lengths of two opposite sides is

equal to s, half of the perimeter, or $a+c=b+d=s$. Let P, Q be points on AB, CD, respectively; then the straight line PQ divides the perimeter into equal parts if and only if $PB+CQ=d$. The area of $ABCD$ is Rs, where R is the radius of the inscribed circle (O); and the area of the quadrilateral $PBCQ$ is equal to the area of $PBCQO$ and that of OQP, or $Rs/2$ plus the area of OQP. Hence PQ bisects the area if and only if the area of OQP is zero, or what is the same, if and only if PQ passes through O. If PQ satisfies the condition $PB+CQ=d$, then P and Q are corresponding points in two projective ranges on the lines of AB and CD as bases. If $P'Q$ passes through O we have also projective ranges on the two bases, and we can then set up projective ranges P, P' on the same base, the line of AB; and the problem is solved by finding the self-corresponding points. Take Q_3 on CD, or its prolongation, so that $CQ_3=d$, and P_2 on BA, or its prolongation, so that $BP_2=d$. Thus on CD the three points ∞, C, Q_3 correspond respectively to ∞, P_2, B on AB in the first projectivity. Let the parallel to CD through O cut AB in P_1'; then, in the second projectivity, to the above three points on CD, there correspond on AB the points P_1', P_2', P_3', where CP_2', Q_3P_3' pass through O. In the plane of the figure let S be any chosen point on any conveniently selected auxiliary circle, which may be the inscribed circle (O); and let the parallel to AB through S cut this auxiliary circle in S_1, while SP_2 and SB cut it in S_2 and S_3. Similarly, let SP_1', SP_2', SP_3' cut it in S_1', S_2', S_3'. Let $(S_2S_3', S_2'S_3)$ denote the intersection of the indicated chords, or their prolongations. Then $(S_2S_3', S_2'S_3)$, $(S_1S_3', S_1'S_3)$, $(S_2S_1', S_2'S_1)$ lie on a straight line by the Pascal theorem for the hexagon $S_3S_2'S_1S_3'S_2S_1'$; and, if there is a real self-corresponding point, this line cuts the auxiliary circle in two points, which may be coincident. Let the straight lines through S and these two intersections cut AB in P_0 and P_0'; and let P_0O, $P_0'O$ cut CD in Q_0, Q_0'. Then we have one, two, or no solutions according as both end-points of P_0Q_0, $P_0'Q_0'$ lie on the segments of the corresponding sides or do not. In a similar manner we consider the opposite sides BC and DA.

In the first projectivity the envelope of PQ is a conic tangent to the two bases AB and CD; and, since from the above one element, P_1Q_1, is the line at infinity, the conic must be a parabola. This is probably the parabola mentioned by the proposer. If AB and CD are parallel, P_2BCQ_3 is a parallelogram with a center O', and the lines PQ which bisect the perimeter of $ABCD$ pass through O'. Hence there are self-corresponding lines P_0Q_0 for these two sides if and only if O' is at O. If O' is at O, there are two types. In one type P_2 is at A and Q_3 is at D, and $ABCD$ has sides of equal length; and any line through O furnishes a solution, that is it bisects both the perimeter and the area. In the second type $ABCD$ is an isosceles trapezoid, and we may assume that CD is the smaller of the two parallel sides. If Q is any point on the segment CD, the line QO furnishes a solution. As for the other two, non-parallel, sides BC, DA, it will be shown later that there are two distinct self-corresponding lines; but they do not furnish a solution, for it will be shown that there is no line through O with its end-points on the segments BC, DA which bisects the area. The proof of this last statement

is as follows: The parallel to AB through O cuts BC, DA in M_b, M_d; and obviously ABM_bM_d has an area greater than one-half the area S of $ABCD$. The line AO cuts the segment BC in K_b, while BO cuts DA in K_d and cuts CD prolonged in D'. The area of triangle ABK_b exceeds $S/2$ by the area of triangle $DD'K_d$, as is shown by dissection. The line $D'M_d$ cuts AB in A'; $A'D'$ is parallel and equal to BC. Let R, R' be points on the segment AM_d so that $AR<AR' \leqq AM_d$; then RO, $R'O$ cut $A'D'$ in $\overline{S}$, $\overline{S}'$ and cut BC in S, S' on M_bK_b, where $M_bS'<M_bS\leqq M_bK_b$. The area of ROR' exceeds the area of SOS' by the area of $R\overline{S}\overline{S}'R'$. Hence the area of $ABSR$ increases from that of ABK_b to that of ABM_bM_d as R passes from A to M_d. From symmetry we see that as R passes from M_d to K_d the area decreases to that of ABK_d, which is the same as the area of ABK_b. Thus ABM_bM_d has the maximum area, and the above statement is proved.

Suppose now that AB and CD intersect in the finite point M so that the points A, B, M have the order as written, while BC and DA may or may not be parallel; and let the points of tangency of AB and CD with the parabola be T_a and T_c. Then to M on AB corresponds T_c on CD, and thus $MT_c=MC+MB+d$; and, similarly, $MT_a=MB+MC+d$. Hence MT_a and MT_c have equal lengths, and the axis of the parabola is along MO, the perpendicular bisector of the chord T_aT_c, with the midpoint N. The vertex V of the parabola is the midpoint of MN; the perpendicular at V to MN is the tangent at the vertex cutting the tangent MT_c in M_v. The perpendicular at M_v to MT_c cuts the axis OM in the focus F. If the circle (OF) on OF as diameter cuts the line VM_v in two real points, then the two straight lines through O and these two points of intersection are the self-corresponding lines P_0Q_0, $P_0'Q_0'$.

If BC and DA are perpendicular to MO, $ABCD$ is an isosceles trapezoid with BC as the smaller of the two parallel sides, and $CD=(BC+DA)/2$. Here $CM_v=DA/2$, and hence MV is greater than MO. Thus for the two non-parallel sides AB, CD there are always two self-corresponding lines, but as shown above neither gives a solution. Let C_1, in this last figure, lie on the segment MC and B_1 on AB so that B_1C_1 is tangent to (O) giving the circumscribed quadrilateral AB_1C_1D; then, since $\angle B_1OC_1=\angle BOC$, $\angle C_1OC=\angle B_1OB$. Hence $BB_1/C_1C =OB_1/OC_1$, and hence $C_1C>BB_1$. Thus, for DA fixed, as C_1 moves from C to M, MV decreases. If $MO>(\sqrt{2}+1)R$ and C_1 is at M, the vertex V is at a point V_0 such that $MO>MV_0$. Thus for such a triangle AMD there exist circumscribed quadrilaterals AB_1C_1D such that there are no real self-corresponding lines for AB_1 and C_1D; and there are no solutions for these sides.

A complete discussion appears to be lengthy and would involve tedious computations.

3882 [1938, 482]. *Proposed by J. R. Musselman, Western Reserve University.*

The pedal circle of the centroid G of a triangle $A_1A_2A_3$ passes through the centers of the hyperbolas of Kiepert and Jerabek. These hyperbolas are considered in Casey's *Analytic Geometry of the Point, Line, and Circle*, pp. 442–448.

Solution by Otto J. Ramler, Catholic University of America.

Bobillier's theorem states that the pedal circle with respect to a triangle inscribed in a rectangular hyperbola of a point P on the hyperbola passes through the center of the hyperbola. The locus of points isogonally conjugate to the points on the Brocard diameter OK, where K is the symmedian point, is Kiepert's hyperbola. The centroid G and K are isogonally conjugate. Hence G lies on Kiepert's hyperbola, and the pedal circle of G passes through the center of Kiepert's hyperbola. Now the pedal circle of G is the same as the pedal circle of K since G and K are isogonally conjugate. Moreover the pedal circle of K passes through the center of the rectangular hyperbola which is the locus of points isogonally conjugate to OG, the Euler line; this locus is Jerabek's hyperbola. Hence the pedal circle of G and K passes through the centers of the Kiepert and Jerabek hyperbolas.

Solved also by R. Goormaghtigh.

3883 [1938, 482]. *Proposed by J. R. Musselman, Western Reserve University.*

The orthopole of the Euler line of a triangle $A_1A_2A_3$ as to the same triangle is the center of the hyperbola of Jerabek; the orthopole of the Brocard line of the triangle as to the same triangle is the center of the hyperbola of Kiepert. Orthopole is defined in Johnson's *Modern Geometry*, p. 247.

Note by R. Goormaghtigh, Bruges, Belgium.

The considered properties are special cases of the following theorem: The counter-point conic of a circumdiameter t is the circumscribed rectangular hyperbola having as center the orthopole of t. See f.i. Gallatly, *The Modern Geometry of the Triangle*, p. 82.

Solved also by Otto J. Ramler.

Editorial Note. Goormaghtigh's solution of 3882 was similar to the one above; and he stated that Bobillier's theorem (*Annales de Gergonne*, 1829, p. 349) was generalized by T. Lemoyne (*Nouvelles Annales de Mathématiques*, 1904, p. 400) in the theorem: The pedal circles of the points of a straight line Δ have the same power with respect to the orthopole π of Δ; the power is twice the product of the distances of Δ to π and to the circumcenter. He referred also to Gallatly, *The Modern Geometry of the Triangle*, p. 51; and for properties of the orthopole to his paper, *The orthopole*, Tohôku Mathematical Journal, 1926, pp. 77–125. Ramler's solution of 3883 was based upon the theorems of Lemoyne and Bobillier.

These references may not be available to some readers, as is the case in the preparation of this note, so we shall give proofs of the general theorems upon which the solutions are based. Let A_0, B_0, C_0 be the midpoints of the sides of the triangle ABC; let O be the center of the circumcircle (O); and let N be the center of the nine-point circle (N) passing through the above midpoints. The four circles (OA), (OB), (OC), on the indicated diameters OA, OB, OC, and (N) are equal. Let any chosen diameter s of (O) cut the first three circles in A', B', C',

the orthogonal projections of A, B, C on s. Since A_0C_0 is the common chord of (N) and (OB), the reflection of B' in this chord is a point S on (N). The arcs $\widehat{A_0B'}$ and $\widehat{A_0S}$ are symmetric with respect to this chord. Considering (OB) and (OC), the arcs $\widehat{A_0B'}$ and $\widehat{A_0C'}$, where the senses of rotation are the same, are equal. Hence $\widehat{A_0C'}$ and $\widehat{A_0S}$ are equal but of opposite sense; and therefore S is the reflection of C' in the common chord A_0B_0 of (N) and (OC). This suffices to show that S is the orthopole of the diameter s.

The isogonal conjugates of points on the diameter s describe a rectangular hyperbola Γ through the vertices of ABC; and, since H, the orthocenter, is the isogonal conjugate of O, it lies also on Γ. Let s' be the diameter of (O) symmetric to s with respect to OA_0; then it is easily shown that the asymptotic directions of Γ are along the sides of the right angle subtended at A by the diameter s'. The center S' of Γ may be constructed in at least two ways; we shall use the asymptotic directions. On CA as hypotenuse construct a right angled triangle whose other two sides have the asymptotic directions; then S' lies on the straight line joining the vertex of the right angle with B_0. On BC construct a similar straight line through A_0; then the intersection of these two straight lines is S'. We prove next that S' is the same as S. When s rotates about O through an angle θ, s' rotates through an angle $-\theta$; the sides of the right angle through A and C rotate through $-\theta/2$, and thus the straight line B_0S' rotates through $-\theta$. So also does A_0S', and hence S' lies on a circle with the chord A_0B_0. If s passes through C_0, we shall prove that C_0 is the center of the isogonal conjugate of s. Here s bisects the arc $\widehat{AB}$ of (O) subtended by C, say in $\overline{C}$, whose reflection in OA_0 is $\overline{C}'$. Then $\angle CA\overline{C}' = \angle A\overline{C}'\overline{C}$; and hence S' is on B_0C_0. With AB, or BC, as hypotenuse, S' lies on a similar straight line passing through C_0; and therefore C_0 is the center. A second proof, which is easy and familiar, is obtained by showing that the tangents to Γ at A and B are parallel. Thus the circle locus of S' passes through C_0, and so it must be (N); and A_0S' turns through $-\theta$. We may take $\theta = \angle C_0OB' = \angle C_0A_0B'$. When s passes through C_0 both S and S' are at C_0; and it has now been shown that S' is the reflection of B' in A_0C_0; and therefore $S' \equiv S$. From the definition of the two special cases of Γ in the solution of 3882, we see that this gives the solution of 3883.

Let P be any given point on a rectangular hyperbola Γ through the vertices of the triangle ABC. Then Γ must be the isogonal conjugate of a diameter of the circumcircle (O) of ABC. This diameter is along OP', where P' and P are isogonal conjugates with respect to ABC. Then P and P' are the foci of a conic K tangent to the sides of ABC, and the common pedal circle of P and P' is the auxiliary circle for the focal diameter of K. Let I and J be the circular points at infinity; then PI and PJ are tangents to K, cutting Γ in Q and R, respectively. Since ABC and PQR are inscribed in Γ, there exists a conic tangent to their six sides; and, since K is tangent to five of these sides, this conic must be K. Let S be the projection of P on the tangent QR to K; then S must lie on the above pedal circle. We shall show that S is the center of Γ by showing that it is

the pole of the line at infinity. Since PS is perpendicular to QR at S, $S(I, Q, J, P)$ is a harmonic pencil. If IJ cuts Γ in X and Y, the range I, X, J, Y is harmonic, since the asymptotes are perpendicular. Hence I is the pole of SJ. Let IR and PS intersect in U; then in the complete quadrilateral formed by P, Q, R, U the straight lines PR, QU, SJ are concurrent; and therefore the point of concurrency must be J, U lies on Γ, and IJS is a self-polar triangle. Hence the polar of S is IJ; S is the center of Γ, while SX and SY are the asymptotes. We now have the theorem:

If A, B, C, P are points on a rectangular hyperbola, the common pedal circle of P and of its isogonal conjugate P' with respect to the triangle ABC passes through the center S of the hyperbola.

This theorem gives the solution of 3882. If P, P', O are not collinear, the usual case, and Γ' with the center S' is the isogonal conjugate of OP, then S and S' are distinct points on (N), SS' is the common chord of (N) and the common pedal circle of P and P', and S and S' are the orthopoles of OP' and of OP, respectively. This result will be referred to in another problem.

The solutions of 3797 [1938, 483] and of 3831 [1939, 454] contain matters related to the above two problems. In Rouché and Comberousse's *Géométrie*, Part 2, eighth edition, page 617, is the interesting remark that Kiepert's hyperbola was encountered for the first time in Kiepert's study (*Nouvelles Annales*, 1869, p. 41) of the problem:

On the sides of the triangle ABC as bases similar isosceles triangles are constructed P_aBC, P_bCA, P_cAB, all directed either toward the interior or the exterior of ABC. The straight lines AP_a, BP_b, CP_c are concurrent; and, when the base angle P_aBC varies, the point of concurrency describes a conic through the vertices of ABC, the centroid G, and the orthocenter H.

It is then verified that this conic locus is Kiepert's hyperbola defined as the rectangular hyperbola which is the isogonal conjugate of OK, where K is the Lemoine point, the isogonal conjugate of the centroid G, and O is the circumcenter of ABC. For P_b and P_c lie respectively on OB_0 and OC_0, and hence BP_b and CP_c are corresponding rays in two projective pencils with centers at B and C. Hence their point P of intersection describes a conic through B and C; when the base angle is A, 0, $\pi/2$, P is at A, G, H respectively. Then, if we use C and A for centers of the pencils, we must get the same conic; hence the lines AP_a, BP_b, CP_c are concurrent, and the conic is the isogonal conjugate of OK.

Immediately preceding this in Rouché and Comberousse are simple proofs of the following: The isogonal conjugate of a straight line d which cuts (O) in two distinct points is a hyperbola. This hyperbola is rectangular if and only if d passes through O. It is a parabola if d is tangent to (O), and an ellipse in all other cases. Also there are proofs of some of the facts used in this note.

3884 [1938, 482]. *Proposed by H. S. M. Coxeter, University of Toronto.*

Prove that the points of contact of the real bitangents to a plane quartic (of genus 3) are its points of intersection with 1, 2, 4, or 7 conics.

Solution by the proposer.

By a well known result of Zeuthen (*Mathematische Annalen*, vol. 7, 1874, p. 411), the plane quartic has 4, 8, 16, or 28 real bitangents. These arise by Geiser's projection from the 3, 7, 15, or 27 real lines on the cubic surface (Schläfli, *Quarterly Journal of Mathematics*, vol. 2, 1858, p. 117), together with the tangent plane at the point of projection. When only 3 of the 27 lines are real, they form a triangle; therefore the corresponding 4 bitangents are a syzygetic tetrad (*i.e.*, their 8 points of contact lie on a conic). Other syzygetic tetrads arise from "double-two's," *i.e.*, sets of four lines of the cubic surface which form two intersecting pairs (XX). When 7 of the 27 lines are real, they consist of three intersecting pairs and one line meeting all the six. Such a configuration can be regarded (in three ways) as a triangle and a double-two; therefore the corresponding 8 bitangents can be regarded (in three ways) as two syzygetic tetrads. When 15 of the 27 lines are real, they can be taken to be c_{ij}, $(i, j=1, 2, \cdots, 6; i<j)$, in Schläfli's notation. The corresponding 16 bitangents are ij and 78, in the Hesse-Cayley notation. When these are tabulated in the form

24	13	23	14
45	46	35	36
15	16	26	25
12	34	56	78

each row is a syzygetic tetrad. (Incidentally, so is each column.)

Finally, the whole set of 28 bitangents can be divided into 7 syzygetic tetrads by cyclically permuting the symbols 1, 2, 3, 4, 5, 6, 7, beginning with the tetrad

18 27 36 45.

This is most easily seen by arranging the numbers $1, 2, \cdots, 7$ symmetrically around a circle, and observing that 27, 36, 45 are three unequal chords. (For completeness, the number 8 might be placed at the center.) The fact that the 28 bitangents can be divided into 7 syzygetic tetrads was first remarked by Du Val (see Baker, *Principles of Geometry*, vol. 6, 1933, p. 14).

3886 [1938, 482]. *Proposed by V. Thébault, Le Mans, France.*

A parallelogram is inscribed in an ellipse and a point P is chosen arbitrarily on the ellipse. Two straight lines are drawn from P parallel to the sides of the parallelogram cutting them in four points. A third straight line is drawn from P parallel to one of the diagonals of the parallelogram cutting the tangents at the ends of this diagonal in two points. Show that the six points thus obtained are the vertices of a hexagon whose consecutive sides are parallel to two conjugate diameters of the ellipse, and that the area of the hexagon is the same as that of the parallelogram. See 3861 [1938, 122].

Editorial Note. Since the theorem of this problem is a special case of 3861 [1940, 118], it seems that the proposer desires an independent proof. It will

suffice to replace the ellipse and its inscribed parallelogram by a circle (O) and an inscribed rectangle $(A)=A_1A_2A_3A_4$. Let M be a point on (O) whose projections on the sides A_iA_{i+1} are B_i. We shall show that the quadrilateral (B) with vertices B_i has its opposite sides perpendicular; that, as M moves on (O), the intersection of one pair of opposite sides of (B) moves on one diagonal of (A) and the intersection of the other pair moves on the other diagonal; and that for all positions of M on an arc of (O) between consecutive vertices of (A) the angles of (B) remain constant.

Suppose first that M lies on the arc of (O) from A_4 to A_1, where the vertices of (A) are in positive order, so that B_1, B_3 are outside of (O) and the other two vertices B_2, B_4 are inside. Let O_{41}, O_{12}, O_{23}, O_{34} be the midpoints of sides B_4B_1, B_1B_2, B_2B_3, B_3B_4; these midpoints are centers of rectangles with these sides as diagonals. The reflections of A_2, B_2, A_3 in the line of $O_{12}O_{23}$ are B_1, M, B_3, and therefore $\angle B_3B_2B_1=\angle A_2MA_3=\angle A_4A_3A_1=\delta$. The reflections of B_1, A_1, A_2 in $O_{41}O_{12}$ are M, B_4, B_2, and therefore $\angle B_2B_1B_4=\angle A_1MA_2=\angle A_1A_4A_2=\pi/2-\delta$. Similarly, reflection in $O_{23}O_{34}$ gives $\angle B_4B_3B_2=\pi/2-\delta$. Thus B_1, B_2, B_3, B_4 is an orthocentric set of points. Let A_1A_3 and B_1B_2 intersect in A_{13}, and A_2A_4 and B_2B_3 in A_{24}. A circle passes through B_1, A_1, A_{13}, B_4, from the equality of angles at the first two points, and this circle must pass also through M. Hence $\angle B_4A_{13}B_1=\pi/2$, and therefore B_3, B_4, A_{13} are collinear. Similarly, a circle passes through B_3, A_4, A_{24}, B_4, and hence $\angle B_3A_{24}B_4=\pi/2$, and then it follows that B_1, B_4, A_{24} are collinear. We also have $\angle MA_{24}A_2=\angle MA_{13}A_1=\pi/2$. It is obvious that the area of (B) is one-half of the area of (A). If M is on the arc from A_1 to A_2 the proof is similar; the angles at B_2 and B_4 are now each equal to δ while the one at B_3 is $\pi/2-\delta$; and in both cases M is the Miquel point for the complete quadrilateral (B).

Let the projections of M on the tangents to (O) at A_2 and A_4 be T_2 and T_4; then $A_{24}A_4T_4M$ and $A_{24}MT_2A_2$ are rectangles. A circle (A_2M), on A_2M as diameter, passes through A_2, M, B_1, T_2, and it goes also through B_2 and A_{24}. Since B_1B_2 and T_2A_{24} are diameters, $\angle B_2T_2B_1=\angle T_2B_1B_4=\pi/2$. Thus $B_1T_2B_2A_{24}$ is a rectangle. Also A_4, M, B_3, T_4 lie on a circle (A_4M) which passes through B_4 and A_{24}; and $\angle T_4B_3B_2=\angle B_4T_4B_3=\pi/2$, since $A_{24}T_4$ and B_3B_4 are diameters. Thus $T_4B_3A_{24}B_4$ is a rectangle. Since the areas of triangles $B_2A_{24}B_1$ and $B_1T_2B_2$ are equal, and similarly for $A_{24}B_3B_4$ and $T_4B_4B_3$, the area of hexagon $B_1T_2B_2B_3T_4B_4$ is equal to that of (A), and its consecutive sides are perpendicular.

In a similar manner the tangents at the ends of the diagonal A_1A_3 yield the two points T_1 and T_3, and the right triangles $T_1B_1B_4$ and $T_3B_3B_2$, the sum of whose areas is one-half the area of (A). We now have the octagon $B_1T_2B_2T_3B_3T_4B_4T_1$, which has the re-entering angle $T_4B_4T_1$ if M is on the arc A_4A_1 and $T_1B_1T_2$ if M is on the arc A_1A_2. The area of this octagon is three-halves of the area of (A); its opposite sides are perpendicular; and its four vertices T_1, T_2, T_3, T_4 lie on the cardioid with its cusp at M and with the diameter of (O) through M as its axis; and the other four vertices lie within the cardioid.

3888 [1938, 554]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let P_1, P_2, P_3, P_4 be the projections of a point P on the faces of a tetrahedron $A_1A_2A_3A_4$. The intersection of the edge A_1A_2 with the plane passing through A_3A_4 and perpendicular to the median corresponding to P in the triangle $P_1P\ P_2$ and the five other similar intersections are in a plane π. The plane π is perpendicular to the straight line joining the isogonal point P' of P to the point having as barycentric coördinates in the pedal tetrahedron of P' the squares of the normal coördinates of P in the tetrahedron $A_1A_2A_3A_4$.

Solution by the Proposer.

Let $P_1'P_2'P_3'P_4'$ be the pedal tetrahedron of P', and $P'\alpha_{12}$ the symmedian corresponding to P' in the triangle $P_1'P'P_2'$; then

$$P_1'\alpha_{12}:\alpha_{12}P_2' = \overline{P'P_1'}^2:\overline{P'P_2'}^2.$$

It follows that the plane $P_3'P_4'\alpha_{12}$ and the five similar planes are concurrent at a point Q having in the tetrahedron $P_1'P_2'P_3'P_4'$ as barycentric coördinates $\overline{P'P_1'}^{-2}$, $\overline{P'P_2'}^{-2}$, $\overline{P'P_3'}^{-2}$, $\overline{P'P_4'}^{-2}$.

Consider further an inversion, P' being the pole; then P_1', P_2', P_3', P_4' are transformed into P_1'', P_2'', P_3'', P_4'', and $P'\alpha_{12}$, for instance, passes through the midpoint of $P_1''P_2''$.

If $B_1B_2B_3B_4$ is the antipedal tetrahedron of P' with respect to $P_1''P_2''P_3''P_4''$, $P_1'P_2'$ is transformed, by the considered inversion, into the circle $P'P_1'\ P_2''$, placed in a plane perpendicular to B_3B_4, the point α_{12}' corresponding to α_{12} being the point where the plane through B_3B_4 perpendicular to $P'\alpha_{12}$ meets $P'\alpha_{12}$.

The plane $P_3'P_4'\alpha_{12}$ is transformed into a sphere passing through P', P_3'', P_4'', α_{12}' and the planes perpendicular at P_3' , P_4'', α_{12}' to $P'P_3''$, $P'P_4''$, $P'\alpha_{12}'$ meet on B_1B_2 at the image Q_{12} of P' through the center of that sphere. As the considered sphere and the five other similar spheres are concurrent at the inverse of Q, the points similar to Q_{12} belong to a plane perpendicular to $P'Q$.

The theorem considered in the proposed question will then follow if it is noted that the figure $A_1A_2A_3A_4PP_1P_2P_3P_4$ is homothetic to $B_1B_2B_3B_4P'P_1''P_2''P_3''P_4''$.

Editorial Note. The above solution may be set in a different form to prove the following theorem:

Let P and P' be finite isogonal conjugate points with respect to the nondegenerate simplex $S\equiv A_1A_2\cdots A_{n+1}$, in n dimensions, $n\geqq 2$; and let $S_p\equiv P_1P_2\cdots P_{n+1}$ be the pedal simplex of P with respect to S. The intersection of the edge A_iA_j with the plane through the remaining vertices of S perpendicular to the median for P in the triangle $P_iP\ P_j$ is a point Q_{ij}. The $(n+1)n/2$ points Q_{ij} lie in a plane π which is perpendicular to the straight line joining P' with that point Q which has for barycentric coördinates with respect to S_p', the pedal simplex of P', the squares of the normal coördinates of P with respect to S.

If C is the midpoint of PP', the sphere (C) with center C and radius CP_i is

the common pedal sphere of P and P' with respect to S. If $\overline{P}_i$ is the other end of the diameter of (C) through P_i', we obtain the simplex $\overline{S}\equiv\overline{P}_1\overline{P}_2 \cdots \overline{P}_{n+1}$, the symmetric with respect to C of S_p'; and $PP_i\cdot P\overline{P}_i=k$, a constant, for $i=1, 2, \cdots, n+1$. Thus P_i, $\overline{P}_i$ are inverse points with respect to the sphere (P) with center P and with a radius whose square is k. From this inversion we see that P_iP_j and $\overline{P}_i\overline{P}_j$ are antiparallel with respect to the angle P_iPP_j; and, if M_{ij} is the midpoint of P_iP_j and if PM_{ij} cuts $\overline{P}_i\overline{P}_j$ in α_{ij}, $P\alpha_{ij}$ is the isogonal conjugate of the median for P in the triangle $\overline{P}_iP\overline{P}_j$. Hence $P_i\alpha_{ij}/\alpha_{ij}\overline{P}_j=(P\overline{P}_i/P\overline{P}_j)^2$, and the plane $[\alpha_{ij}\overline{S}_{ij}]$ through α_{ij} and the vertices of $\overline{S}$ omitting $\overline{P}_i$ and $\overline{P}_j$ has the barycentric equation with respect to $\overline{S}$, $x_i/x_j=(P\overline{P}_i)^{-2}/(P\overline{P}_j)^{-2}$. The $(n+1)n/2$ planes $[\alpha_{ij}\overline{S}_{ij}]$ have in common the point $\overline{Q}$ with barycentric coördinates with respect to $\overline{S}$, $(P\overline{P}_1)^{-2}$, $(P\overline{P}_2)^{-2}$, $\cdots$, $(P\overline{P}_{n+1})^{-2}$. With respect to (P) the poles $\overline{Q}_{ij}$ of $[\alpha_{ij}\overline{S}_{ij}]$ lie in π, the polar plane of $\overline{Q}$; also S and $\overline{S}$ are polar reciprocals. The polar of $\overline{S}_{ij}$ is the straight line A_iA_j; and the polar of α_{ij} on $\overline{P}_i\overline{P}_j$ is a plane perpendicular to PM_{ij} passing through the polar of $\overline{P}_i\overline{P}_j$, which is S_{ij}. Thus the pole $\overline{Q}_{ij}$ of $[\alpha_{ij}\overline{S}_{ij}]$ is the intersection of A_iA_j with the plane perpendicular to PM_{ij} passing through S_{ij}; and hence $\overline{Q}_{ij}$ is the Q_{ij} in the theorem. The plane π, containing the points Q_{ij}, is perpendicular to $P\overline{Q}$, and to $P'Q$, its symmetric with respect to C. The barycentric coördinates of Q with respect to S_p' are $(P'P_i')^{-2}$, or $(PP_i)^2$, since P and P' are isogonal conjugates with the respective normal coördinates PP_i, $P'P_i'$ with respect to S. This completes the proof.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Ill.

The honorary degree of Doctor of Laws was conferred by Duke University upon Dean L. P. Eisenhart of Princeton University.

Professor Rufus Oldenburger of Armour Institute of Technology will deliver a series of lectures on mathematics under the auspices of the University of Mexico this summer.

The degree of Doctor of Laws was conferred on Professor Maria M. Roberts by the Iowa State College on June 10, 1940, "in recognition of fifty years of devoted and distinguished service to the college in the department of mathematics, as dean of the Junior College, and as administrator of the student loan funds."

Dr. R. G. Sanger was awarded one of the three $1000 awards given at the University of Chicago to selected teachers for excellence in undergraduate teaching, and has been promoted to an assistant professorship.

Professor E. F. Weinberg of Rollins College was awarded the Rollins Decoration of Honor for distinguished service to the college.

National Research Fellowships in mathematics for 1940–41 have been awarded to Dr. H. T. Muhly to work at Princeton University, and to Dr. C. E. Shannon to work at the Institute for Advanced Study.

Dr. Joseph Allen, professor of mathematics at the College of the City of New York, will retire September 1, 1940. He has been a member of the faculty since 1897.

Dr. Marjorie H. Beaty has been appointed to an assistant professorship at the University of South Dakota.

At the University of Illinois Associate Professors H. R. Brahana and W. J. Trjitzinsky have been promoted to professorships, and Dr. Josephine H. Chanler has been made an associate in mathematics.

Professor C. T. Bumer of Kenyon College has been appointed Peabody Professor of Mathematics.

Associate Professor Jewell Hughes Bushey has been elected chairman of the department of mathematics at Hunter College for the coming year.

Professor Abraham Cohen has retired after forty-five years on the faculty at Johns Hopkins University.

Dr. Myrtie Collier of Immaculate Heart College, Los Angeles, was made head of the department of mathematics in January.

Professor W. B. Ford has retired after forty years at the University of Michigan. Included in this period is his service as editor-in-chief of the MONTHLY for the years 1923–26 and as president of the Association for the years 1927–28.

At the university of Notre Dame, the Rev. J. H. Kenna, Dr. A. N. Milgram, and Dr. P. M. Pepper have been promoted to assistant professorships.

Dr. Norman Levinson of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Dr. Hans Lewy of the University of California has been promoted to an associate professorship.

Professor Lao G. Simons, chairman of the department of mathematics at Hunter College, has retired after forty-five years of association with the college. Her friends and colleagues gave expression to their appreciation of her long and distinguished service at a reception and dinner in her honor at Hotel Madison, New York, on June 18th.

After twenty-nine years of service as Johnson Professor of Mathematics at the College of Wooster, Professor B. F. Yanney has retired with the title Johnson Professor of Mathematics Emeritus.

The following appointments to instructorships are announced:
University of Alabama: R. E. Gaskell
Allegheny College: J. A. Joseph
Brooklyn College: Dr. Moses Richardson
Eastern Washington College of Education: R. F. Bell
Massachusetts Institute of Technology: Dr. Eric Reissner, Dr. D. C. Spencer
University of Notre Dame: Dr. C. V. Robinson
Princeton University: A. D. Wallace, part-time
Purdue University: R. C. Davis
University of Virginia: P. A. White
University of Washington: Dr. H. S. Zuckerman
Woodrow Wilson Junior College, Chicago: Dr. Bernard Friedman.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va. April 20; Grove City, Pa., November 2.
Illinois, Bloomington, May 10–11.
Indiana, Richmond, May 3–4.
Iowa, Mt. Vernon, April 19–20.
Kansas, Wichita, March 31.
Kentucky, Lexington, April 27.
Louisiana-Mississippi Oxford, Miss., March 8–9.
Maryland-District of Columbia-Virginia, Richmond, Va., May 11.
Michigan, Ann Arbor, April 26–27.
Minnesota, Mankato, May 4.
Missouri, Warrensburg, April 19.
Nebraska, Omaha, May 9–11.
Northern California, Berkeley, January 27.
Ohio, Columbus, April 5.
Oklahoma, Oklahoma City, February 16.
Philadelphia, November 23 or 30.
Rocky Mountain, Fort Collins, Colo., April 19.
Southeastern, Athens, Ga., March 29–30.
Southern California, Compton, March 2.
Southwestern, Tucson, Ariz., April 22–23.
Texas, Dallas, March 29–30.
Wisconsin, Milwaukee, May 4.

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics.

MATHEMATICAL ASSOCIATION OF AMERICA

The following sixty-nine persons and three institutions have been elected to membership on applications duly certified:

To Institutional Membership

D'YOUVILLE COLLEGE, Buffalo, N. Y.

LOS ANGELES CITY COLLEGE, Los Angeles, Calif.

THE UNIVERSITY OF TORONTO, Toronto, Ontario, Canada.

To Individual Membership

G. J. ADAMS, JR. Engineering Aide, Alabama State Highway Dept., Thomaston, Ala.

DANIEL ARANY. Formerly Prof. of Math. in the Royal School for Industry, Budapest, Hungary.

H. F. BOHNENBLUST, Ph.D.(Princeton) Asso. Prof., Princeton Univ., Princeton, N. J.

ELIZABETH F. BROWN, M.S.(J.B.Stetson Univ.) Teacher, Arsenal Jr. High School, Pittsburgh, Pa.

H. E. BURNS, A.M.(Indiana) Instr., Indiana Univ. Extension, East Chicago, Ind.

C. L. BUXTON, M.S.(Case School) Instr., Clarkson Coll. of Tech., Potsdam, N. Y.

C. L. CARROLL, JR., A.M.(North Carolina) Instr., Georgia School of Tech., Atlanta, Ga.

RICHARD CEBULA, A.B.(Mt. Angel Coll.) Instr., St. Martin's Coll., Lacey, Wash.

L. G. CHELIUS. Student, Univ. of Utah, Salt Lake City, Utah.

ANNIE M. H. CHRISTENSEN (Mrs. L. C.), A.M.(Duke) Asst., Louisiana State Univ., University, La.

J. C. CURRIE, A.M.(Mississippi) Northeast Jr. Coll., Monroe, La.

R. B. DEAL, JR. Student, Univ. of Oklahoma, Norman, Okla.

CARL DENBOW, Ph.D.(Chicago) Instr., Ohio Univ., Athens, Ohio.

I. C. FISCHER, M.S.(Marquette) Prof., Math. and Mech., Milwaukee School of Eng., Milwaukee, Wis.

W. C. FLAHERTY, A.B.(Georgetown Univ.) Asst. Prof., Georgetown Univ., Washington, D. C.

M. W. FLECK, M.S.(New Mexico) Asst. Prof., Biology, Eastern New Mexico Jr. Coll., Portales, N. M.

GRACE A. FULLER, A.B.(Fresno State Coll.) Teacher, Union High School, Madera, Calif.

W. A. GAGER, M.S.(Pennsylvania State) Head of Dept., St. Petersburg Jr. Coll., St. Petersburg, Fla.

L. L. GARNER, A.M.(North Carolina) Asst. Prof., Univ. of North Carolina, Chapel Hill, N. C.

E. L. GODFREY, A.M.(Indiana) Lecturer, Indiana Univ. Extension, East Chicago, Ind.

EDISON GREER, M.S.(Kansas State Coll.) Instr., Univ. of Wichita, Wichita, Kans.

Sister MIRIAM HAND, A.B.(Immac. Heart Coll.) Instr., Immaculate Heart Coll., Hollywood, Calif.

R. J. HANNELLY, Ph.D.(Colorado) Director of Math., Phoenix Jr. Coll., Phoenix, Ariz.

T. F. HICKERSON, A.M.(North Carolina) Prof., Appl. Math., Univ. of North Carolina, Chapel Hill, N. C.

Mrs. FRANCES L. C. HINDS, A.M.(U.C.L.A.) Asst. Prof., George Pepperdine Coll., Los Angeles, Calif.

MARY H. HODGDON, A.B.(Bates) Teacher, Dover High School, Dover, N. H.

J. V. HOWELL, A.M.(North Carolina) Asso., Mars Hill Coll., Mars Hill, N. C.

H. M. HUGHES, A.M.(Texas) Instr., Univ. of Tennessee, Knoxville, Tenn.

R. N. JOHANSON, Ph.D.(Chicago) Instr., Bradley Poly. Inst., Peoria, Ill.

P. S. JONES, A.M.(Michigan) Instr., Edison Inst. of Tech., Dearborn, Mich.

MARK KAC, Ph.D.(John Casimir Univ., Lwow) Instr., Cornell Univ., Ithaca, N. Y.

SOLOMON KULLBACK, Ph.D.(George Washington) Lecturer in Stat., George Washington Univ., Washington, D. C.

MARJORIE M. LEY, A.M.(Stanford) Head of Dept., San Benito Co. Jr. Coll., Hollister, Calif.

NATALIE M. LONGFELLOW, A.M.(Chicago) Head of Dept., The Chapin School, Ltd., New York, N. Y.

W. A. MARTIN, A.M.(Alabama) Instr., Alabama Poly. Inst., Auburn, Ala.

W. J. MAYS, A.M.(Vanderbilt) Prin., Masseyville School, Bethel Springs, Tenn.

G. J. MCCUTCHEON, B.S.(Minnesota) Teacher, High School, New Ulm, Minn.

F. A. MCMAHON, A.M.(Brooklyn Coll.) Tutor, Evening Session and Exten. Div., Brooklyn Coll., Brooklyn, N. Y.

EMANUEL MEHR, A.B.(Brooklyn Coll.) Grad. student, Brooklyn Coll., Brooklyn, N. Y.

Sister M. MERCEDES. Prof., Coll. of St. Scholastica, Duluth, Minn.

L. L. MERRILL, Ph.D.(Rensselaer) Asst. Prof., Rensselaer Poly. Inst., Troy, N. Y.

E. R. C. MILES, Ph.D.(Rice) Asst. Prof., Duke Univ., Durham, N. C.

W. B. MOYE, A.M.(Mercer Univ.) Asst. Prof., Georgia Teachers Coll., Collegeboro, Ga.

W. D. MUNRO, A.B.(Colorado) Teaching asst., Univ. of Minnesota, Minneapolis, Minn.

MARGARET OLMSTED, A.M.(Illinois) Asso. Prof., Latin and Math., Augustana Coll., Rock Island, Ill.

ISAAC OPATOWSKI, D.Eng.(R. Inst. of Eng., Turin), D.Math.(R. Univ., Turin) Instr., Math. and Mech., Univ. of Minnesota, Minneapolis, Minn.

J. F. PAYDON, A.M.(Northwestern) Asst., Northwestern Univ., Evanston, Ill.

C. R. PETTIS, Ph.D.(Michigan) Head of Dept., Mississippi State Coll., State College, Miss.

C. R. PHELPS, A.M.(Harvard) Instr., Georgia School of Tech., Atlanta, Ga.

EDRIS P. RAHN, A.M.(California) Teacher, Union High School, Hayward, Calif.

P. W. A. RAINE, A.M.(Virginia) Teacher, Shawnee High School, Louisville, Ky.

L. M. RAUCH, Ph.D. (California) Head of Dept., Seton Hall Coll., South Orange, N. J.

Sister MARY L. REPS, A.M.(Notre Dame) Instr., Our Lady of Cincinnati Coll., Cincinnati, Ohio.

W. C. ROBERTS, A.M.(U.C.L.A.) Teacher, Glendale Jr. Coll., Glendale, Calif.

ALEXANDER RUSS, A.M.(Duke) Prof., Mississippi Woman's Coll., Hattiesburg, Miss.

VERYL G. SCHULT, A.M.(George Washington) Instr., Wilson Teachers Coll., Head of Dept., Washington High Schools, Washington, D. C.

NATHAN SCHWID, Ph.D.(Wisconsin) Asst. Prof., Texas Coll. of Mines, El Paso, Texas.

D. W. SNADER, A.M.(Columbia) Asst. Prof. and Supervisor of Math., N. Y. State Coll. for Teachers, Albany, N. Y.

H. W. STEINBACH. Student, George Washington Univ., Washington, D. C.

J. A. STRAW, A.M.(Michigan State Coll.) Instr., Speed Scientific School, Univ. of Louisville, Louisville, Ky.

C. R. SUMMERS, A.M.(State Coll. of Washington) Teacher, Rogers High School, Spokane, Wash.

E. L. SWANSON, A.M.(Colorado Coll.) Teacher, Primero High School, Segundo, Colo.

B. P. TAYLOR, A.M.(U.C.L.A.) Instr., Exten. Div., Univ. of California at Los Angeles, Roscoe, Calif.

EADY M. TUBB, M.S.(Chicago) Teacher, High School, Smithville, Miss.

E. P. VANCE, Ph.D.(Michigan) Instr., Univ. of Nevada, Reno, Nev.

G. B. VAN SCHAACK, Ph.D.(Harvard) Instr., Michigan State Coll., East Lansing, Mich.

R. W. WAGNER, Ph.D.(Michigan) Instr., Oberlin Coll., Oberlin, Ohio.

BERTRAM YOOD, M.S.(California Inst. of Tech.) Grad. Student, Brown Univ., Providence, R. I.

JANET R. YOUNG, A.B.(Pomona) Teacher, Scott Rancho School, Azusa, Calif.

W. D. CAIRNS, *Secretary-Treasurer*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Catholic University of America, Washington, D. C., on Saturday, December 9, 1939. This meeting was held as part of the fiftieth year jubilee of the University. The chairman of the Section, Dr. L. S. Dederick, presided over both sessions, morning and afternoon. Four papers were read at the morning session and three at the afternoon session. The Section gave a rising vote of thanks to the Catholic University of America for its generous hospitality.

There were seventy-three in attendance, including the following twenty-seven members of the Association: O. S. Adams, C. C. Bramble, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, E. J. Finan, Michael Goldberg, J. L. Kelley, L. M. Kells, A. E. Landry, Florence P. Lewis, S. B. Littauer, Sister Thomas Marie Maloney, Carol V. McCamman, T. W. Moore, F. D. Murnaghan, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, J. P. Smith, F. W. Sohon, J. L. Stearn, G. C. Vedova, C. H. Wheeler III, G. T. Whyburn, John Williamson.

After an address of welcome by the Right Reverend Monsignor McCormick, Vice Rector of the Catholic University of America, the following papers were read:

1. "A non-associative matrix theory and some applications" by J. L. Stearn, U. S. Coast and Geodetic Survey.
2. "On triangles having a common mean" by Professor O. J. Ramler, Catholic University of America.
3. "Hyperspace analogs of circular functions" by Dr. G. E. Alrich, University of Maryland, introduced by the Secretary.
4. "Non-planar linkages" by Michael Goldberg, U. S. Bureau of Ordnance, Navy Department.
5. "On groups of subtraction and division" by Professor E. J. Finan, Catholic University of America.
6. "On the definition of variance" by Dr. S. B. Littauer, U. S. Naval Academy.
7. "An elementary treatment of the problem of rencontre" by Professor C. C. Bramble, Post Graduate School, U. S. Naval Academy.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. A new matrix theory using a column by column multiplication was discussed. The Cayley associative product $A\cdot(B\cdot C)=(A\cdot B)C$ is replaced by a non-associative product $A\cdot(B\cdot C)=A\cdot I(C\cdot B)=A\cdot IC\cdot B$, where I is the unit matrix. Mr. Stearn proved a theorem on the non-associative matrix product and showed how this matrix product applies to the solution of n simultaneous linear algebraic equations. A rapid method for determining the inverse of the coefficient matrix of the given system follows easily from the equation

$A_{cm} \cdot A_{cm}^{-1} = I$, requiring the determination of $n(n+1)/2$ additional quantities. Mr. Stearn also showed that in the case where the product matrix $B \cdot C = A_{2m}$ is axi-symmetrical the problem reduces to the solution of the normal equations of the method of least squares.

2. This paper was published in full in the March, 1940, issue of the MONTHLY.

3. Dr. Alrich exhibited certain functions of several variables which have properties analogous to those of the ordinary sine and cosine. While the methods and conclusions are general, he presented only the case of four functions of three variables each. The functions are defined by the equations

$$\sum_{j=1}^{4} r^{(j-1)(2k-1)} f_j(\theta_1\theta_2\theta_3) = \exp \sum_{1}^{3} r^{j(2k-1)}\theta_j, \qquad (k = 1, \cdots, 4),$$

$$r = \cos \pi/4 + i \sin \pi/4.$$

The first point of analogy with circular functions is the identity

$$\Delta(f) = \begin{vmatrix} f_1 & -f_4 & -f_3 & -f_2 \\ f_2 & f_1 & -f_4 & -f_3 \\ f_3 & f_2 & f_1 & -f_4 \\ f_4 & f_3 & f_2 & f_1 \end{vmatrix} = 1,$$

which is the analog of

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1.$$

The second point of analogy is found in the derivative properties; *viz.*, that the second column of the determinant $\Delta(f)$ is the derivative of the first with respect to θ_1, the third column is the derivative of the first with respect to θ_2, *etc.*, so that no new functions are formed by differentiation. Thirdly, the formulas for functions of the differences of two arguments offer precise analogy with those for the ordinary sine and cosine.

4. A closed chain of n hinged links has, in general, $n-3$ degrees of freedom when the linkage is planar, but only $n-6$ degrees of freedom, in general, when the linkage is non-planar. There exist, however, special cases of less than seven links which, nevertheless, possess degrees of freedom. These cases have not been completely determined. Mr. Goldberg exhibited working models of several different types of hinged chain linkages. In particular, he showed a seven-bar linkage from which, by the vanishing of various parameters, he derived six hitherto unknown linkages; namely, two new five-bar hinged chain linkages and four new six-bar hinged chain linkages.

5. Professor Finan presented new theorems pertaining to the following problem: If one starts with the indeterminate θ and applies the operators O_1 and O_2, where $O_1(x) = 1/x$ and $O_2(x) = 1-x$, one obtains six distinct functions of θ which form a group commonly called the Cross-Ratio Group. The six functions or

operators are θ, $1-\theta$, $1/\theta$, $1/(1-\theta)$, $(\theta-1)/\theta$, and $\theta/(\theta-1)$. If one replaces O_1 and O_2 by the more general operators $R_1(x)=r/x$ and $R_2(x)=s-x$ respectively, where r and s are now any complex numbers, it is conceivable that other groups may be generated. This problem was studied by G. A. Miller, who obtained several interesting results.

6. Dr. Littauer asked for comment on the following definition of the variance σ^2 of a population obtained from a sample of n elements: Consider $s^2=\Sigma(\text{residuals})^2/n$ as a measure of inaccuracy of the mean, and s^2/n as such a measure of the mean of means of samples of size n. Iterating the notion of means of means, the total measure of inaccuracy of the mean determined from the original sample is defined as

$$\sigma^2 = \lim_{i\to\infty} \frac{\Sigma\,(\text{residuals})^2}{n}\left(1+\frac{1}{n}+\frac{1}{n^2}+\cdots+\frac{1}{n^i}\right)$$
$$= s^2\,\frac{n}{(n-1)} = \frac{\Sigma\,(\text{residuals})^2}{(n-1)}\,.$$

7. The problem of rencontre was first treated in the case of n elements all different. The number of permutations giving m "rencontres" or "hits" was estimated by a simple use of symmetric functions. Then Professor Bramble extended the procedure to the case of the Banker's Clock Problem and the Jeu de Treize. He gave two methods for estimating the number of permutations giving m strikes for s similar sets of n things and indicated the extension to the case of several sets of p, q, r, *etc.*, elements.

S. B. Littauer, *Secretary*

THE SEVENTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The seventeenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the University of Mississippi, Oxford, Mississippi, March 8–9, 1940. Sessions were held on Friday afternoon and Saturday morning. A joint dinner with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was held on Friday evening. The chairman of the Section, Professor V. B. Temple, presided at all sessions.

The attendance was about seventy-five, including the following thirty-five members of the Association: T. A. Bickerstaff, H. E. Buchanan, W. B. Carver, Mrs. L. C. Christensen, W. A. Cordrey, G. F. Cramer, J. C. Currie, Virginia I. Felder, H. T. Fleddermann, B. E. Gatewood, F. C. Gentry, Alfred Hume, H. S. Kaltenborn, H. T. Karnes, Z. L. Loflin, Dorothy McCoy, Janet MacDonald, B. E. Mitchell, S. B. Murray, I. C. Nichols, Arthur Ollivier, W. V. Parker, C. R. Pettis, H. L. Quarles, F. A. Rickey, S. T. Sanders, H. F. Schroeder, P. C. Scott, C. D. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker, Marelena White, R. C. Yates.

At the business session on Saturday morning it was voted to hold the 1942 meeting at Jackson, Mississippi. The following officers were elected for the year 1940–41: Chairman, C. D. Smith, Mississippi State College; Vice-Chairman for Mississippi, Janet MacDonald, Hinds Junior College; Vice-Chairman for Louisiana, G. F. Cramer, Tulane University; Secretary, W. V. Parker, Louisiana State University.

The Section was honored to have Professor W. B. Carver of Cornell University as visiting speaker. He spoke on "Thinking versus manipulation in mathematics" at the dinner on Friday evening, and on "The polygonal regions into which a plane is divided by n straight lines" at the Saturday morning session. These two addresses contributed much to the value of the meeting.

Professor Carver's first address was about the same as the one he gave at the meeting at Duke University in December, 1936; see this MONTHLY, vol. 44, 1937, p. 359.

Professor Carver's second paper presented one of the unsolved problems of real projective geometry. The projective plane is divided by n straight lines (no three of which are concurrent) into polygonal regions. Let α_i, $(i=3, 4, \cdots, n)$, be the number of regions having i vertices. It is readily seen that the total number of regions is

$$\sum_{i=3}^{n} \alpha_i = (n^2 - n + 2)/2, \tag{1}$$

and that the total number of vertices of all the regions is

$$\sum_{i=3}^{n} i\alpha_i = 2n(n - 1). \tag{2}$$

For $n=3, 4$, the Diophantine equations (1) and (2) determine the α's uniquely; but for higher values of n most of the solutions of (1) and (2) in non-negative integers do not correspond to any actual division of the plane into polygons. Thus for $n=5$, the equations have four solutions, but only one of these solutions, namely $\alpha_3=5$, $\alpha_4=5$, $\alpha_5=1$, corresponds to a possible division of the plane by 5 lines. For $n=6$, the equations have 19 solutions, but only four of these correspond to the four possible ways in which the plane may be divided by 6 lines. Some results (but not a complete solution) are known for $n=7$; but for $n>7$ almost nothing is known about the problem. One general result is that, for any $n \geqq 3$, the n lines may be taken so that $\alpha_3=n$, $\alpha_4=(n^2-3n)/2$, $\alpha_n=1$, and $\alpha_5=\alpha_6=\alpha_7=\cdots=\alpha_{n-1}=0$.

The dual problem, that of dividing the projective plane of lines into "line-regions" by n points, was also discussed briefly.

In addition to the addresses by Professor Carver, the following papers were presented:

1. "Analytic geometry of the triangle" by Professor F. C. Gentry, Louisiana Polytechnic Institute.

2. "A relation between successive values of a certain numerical function" by Professor G. F. Cramer, Tulane University.

3. "The equality of two measure functions" by Professor H. T. Fleddermann, Loyola University.

4. "Some higher plane curves" by Professor V. B. Temple, Louisiana College.

5. "Algebraic analogs" by Professor H. S. Kaltenborn, Louisiana Polytechnic Institute.

6. "The construction of an alignment chart to represent a function of the type $y=at^{bz}$" by W. G. O'Regan, Southern Forestry Experiment Station and Loyola University, introduced by Professor Fleddermann. (By title.)

7. "A peculiar transformation" by Professor F. A. Rickey, Louisiana State University.

8. "Some applications of mathematics" by Professor J. F. Thomson, Tulane University.

9. "Thermal stresses in a long cylindrical body of M concentric materials" by Professor B. E. Gatewood, Louisiana Polytechnic Institute.

10. "The story of the parallelogram" by Professor R. C. Yates, Louisiana State University.

11. "The annual meeting of the mathematical organizations at Louisiana State University in December 1940" by Professor S. T. Sanders, Louisiana State University.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Gentry discussed the derivation of equations for many of the lines and circles connected with the triangle by means of normal trilinear coördinates. Most of the properties of these lines and circles may be proved in this manner.

2. Let $A_n=a^{2^n}+1$, where a is any constant, real or imaginary, and $n=0, 1, 2, \cdots$. By use of mathematical induction, Professor Cramer proved that $A_n=(a-1)\Pi_{i=0}^{n-1}A_i+2$. The special case when a is an even integer furnished a simple proof that the number of primes is infinite. It is clear that, when $a=2$, the number A_n is the Fermat number F_n.

3. Professor Fleddermann showed that if a set is regular in the sense of Carathéodory, *i.e.*, has unit density, using Carathéodory linear measure, at almost all of its points, then the Carathéodory measure and the measure obtained by replacing convex sets by circles in Carathéodory's definition, are equal.

4. Professor Temple gave definitions of and presented a chart showing the graphs and equations of four higher plane curves, which, it is believed, have not been previously defined. The definitions were based on certain geometric conditions relating to the points of intersection of a secant line on a circle and on parallel tangents to the circle. He named three of these curves as differential axle, tennis rackets, and cane knife, as suggested from their general form.

5. Professor Kaltenborn compared algebra to a game, to a tool, and to sym-

bolic language. Various instances where these comparisons may be helpful in teaching different topics in algebra were mentioned.

6. In forest research it is often necessary to represent complicated mathematical expressions graphically in order to secure easy mechanical solutions. Mr. O'Regan showed the method followed in reducing one type of these expressions to a simple alignment chart.

7. If any right cylinder is developed, the curve of intersection of the cylinder and a plane will yield a plane curve. A method of obtaining the equation of such curves was given by Professor Rickey and some unusual geometric applications were obtained.

8. Professor Thomson briefly discussed the following topics: Heaviside's calculus in the study of the behavior of electrical circuits and telephone lines; Maxwell's equations in the development of radio; the effect of perturbations on the paths of planets in the discovery of new planets; computation of orbits of comets; some conic sections used in locating an invisible enemy gun; Dr. Carrel's method of treating deep wounds; predicting the outcome of an election in the city of New Orleans by the study of a previous election.

9. Professor Gatewood reduced the problem of finding the thermal stresses in a long circular cylinder of M concentric materials to obtaining a particular integral for the Poisson equation $\nabla^2 V = kT$, where T is the temperature distribution and k is a constant, and to solving the biharmonic equation $\nabla^4 U = 0$ in the cross-section of the cylinder. The conditions were that the forces on the surface of the body be zero and that the displacements and normal stresses be continuous at the junction of the materials. The functions U and V were related through the boundary conditions. The biharmonic equation was solved by use of two analytic functions (Muschelisvili, *Bulletin de l'Academie des Sciences de Russie*, VI Serie, vol. 13, 1919; *Mathematische Annalen*, vol. 107, 1933). The special problem of three concentric materials was solved by use of the theory, the stresses in this case being calculated for the temperature distribution as a function of the radius of the cylinder.

10. Illustrated with numerous models and devices, Professor Yates discussed the rôle of the parallelogram in the geometry of Steiner, in the theory of dissection, and in the study of linkages.

W. V. PARKER, *Secretary*

THE ANNUAL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held at Southern Methodist University in Dallas, on Friday afternoon, March 29, and Saturday morning, March 30, 1940.

Among the fifty people attending the meeting were the following twenty members of the Association: E. F. Beckenbach, J. H. Binney, Myrtle C. Brown, J. E. Burnam, Nat Edmonson, Jr., H. J. Ettlinger, E. H. Hanson, G. B. Huff, H. A. Luther, E. D. Mouzon, Jr., C. A. Murray, G. A. Newton, W. L. Porter,

Maxwell Reade, P. K. Rees, W. A. Rees, C. R. Sherer, F. W. Sparks, Jennie L. Tate, Earl Thomas.

During the business session, Professor J. E. Burnam of Hardin-Simmons University and Dr. E. F. Beckenbach of the Rice Institute were elected, respectively, Chairman and Vice-Chairman for the 1941 meeting. Those present voted to accept the invitation of the North Texas State Teachers College of Denton to hold the 1941 meeting at that institution on March 28 and 29.

The following papers were read:

1. "A geometry associated with Cremona's equations" by Professor G. B. Huff, Southern Methodist University.

2. "Space analogs of function-theoretic results" by Dr. E. F. Beckenbach, The Rice Institute.

3. "A note on transforms of Fuchsian groups" by Professor P. K. Rees, Southern Methodist University.

4. "Mean-value surfaces" by Maxwell Reade, The Rice Institute.

5. "Further properties of matrix solutions of linear systems of ordinary differential equations" by Professor H. J. Ettlinger, University of Texas.

6. "On the Fourier series of a class of continuous functions" by J. P. Nash, The Rice Institute, introduced by Dr. Beckenbach.

7. "Prerequisites and requirements for mathematics teaching in high schools from the view-point of the colleges and universities" by Professor C. A. Murray, West Texas State Teachers College.

8. "Teaching freshman mathematics from a scientific standpoint" by Professor C. R. Sherer, Texas Christian University.

9. "Teacher training requirements in mathematics from the view-point of the State Department of Education" by Dr. E. H. Hereford, State Department of Education, introduced by Professor Ettlinger.

10. "Practice and background in the training of high school teachers of mathematics" by Professor Hob Gray, University of Texas, introduced by Professor Ettlinger.

11. "Improving the teaching of high school mathematics" by Elizabeth Dice, North Dallas High School, introduced by Professor Ettlinger.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. A *complete* and *regular* system $\Sigma_{p,d}$ of plane curves of *dimension* d, with the generic curve having *genus* p, is defined by its order x_0 and its multiplicities $x_1, x_2, \cdots, x_\rho$ at a set of ρ prescribed base points. The *characteristic* $x=\{x_0;\ x_1,\ x_2, \cdots, x_\rho\}$ of a system satisfies the Cremona equations $x_1^2+x_2^2+\cdots+x_\rho^2-x_0^2=1-d-p$, $x_1+x_2+\cdots+x_\rho-3x_0=-1-d+p$. If a system $\Sigma_{p,d}$ is subjected to a Cremona transformation with F-points as the base points of the system, the characteristic x' of the transformed system may be obtained from x by means of a linear transformation. Professor Huff studied the solutions of Cremona's equations and the groups of these transformations as points and collineation groups in a projective space S_ρ. Some earlier results

were essentially simplified and additional information was obtained for $\rho > 9$. The principal new result is that each elliptic characteristic E of $d=0$, $p=1$ defines an infinite Abelian group which leaves E invariant.

2. This paper appeared in full in the April, 1940, issue of the MONTHLY.

3. Professor Rees stated and proved a theorem which gives the necessary and sufficient condition that k be a minimum in the equation $kr_s = r_t$, where T is any transformation of a Fuchsian group, G is a Fuchsian transformation, $S = GTG^{-1}$, and r_s and r_t are the radii of the isometric circles of S and T, respectively.

4. If $x_j(u, v)$, $(j=1, 2, 3)$, are the coördinate functions of a surface S, then the functions

$$A_{j,\rho}(u, v) \equiv \frac{1}{\pi\rho^2}\iint_{\xi^2+\eta^2=\rho^2} x_j(u+\xi, v+\eta)\,d\xi d\eta, \qquad (j = 1, 2, 3), \tag{1}$$

are said to define a mean-value surface S_ρ associated with S. A typical result of Mr. Reade's was as follows: If the functions $x_j(u, v)$, $(j=1, 2, 3)$, map a simply connected domain D isothermically on a surface S that lies on a sphere of finite non-null radius, such that circles are mapped on circles, then each mean-value surface associated with S lies on a surface of revolution which is given in isothermic representation by (1).

5. Professor Ettlinger continued the discussion of the solution of general linear systems of ordinary differential equations by matrix methods, to the case where the coefficients of the system are general integrable functions of the independent variable. Properties of the adjoint system were obtained, the Wronskian was evaluated by direct and simple properties of matrices and determinants, and functions defined by such systems were discussed.

6. Mr. Nash defined a class of functions which forms an extension of the class of functions of *écart fini* of Hadamard. A generalization of H. E. Bray's result for functions of *écart fini* (*Comptes Rendus*, vol. 190, p. 1371) was obtained, giving a sufficient condition in terms of Fourier coefficients that a given function be of this class. He showed that continuous functions of this class whose moduli of continuity satisfy a slight restriction have uniformly convergent Fourier series, and exhibited examples of such functions which satisfy none of the classical conditions for uniform convergence of their Fourier series.

7. Professor Murray presented data gathered from twenty-five Texas colleges and universities and thirty and thirty-four like institutions outside of Texas showing thirty to fifty per cent failures in first semester freshman mathematics for the 1938–39 session. Prerequisites for mathematics teaching in high schools, as given by the speaker, were essentially those set forth by the committee report of the Mathematical Association of America made in 1935. For minor teaching in mathematics it was urged that teachers should have at least a college minor in preparation. The speaker believes mathematics teachers should read professional magazines and participate in the activities of professional organizations.

9. The State Department of Education has attempted to put into operation the minimum requirements as set up by the special committee on teacher education for the Southern states. It requires all classified and affiliated high schools to assign teachers to their major or minor field of preparation. Through its bulletin, "Teaching Mathematics in Junior and Senior High Schools," it provides teaching procedure, subject-matter, charts and units, and the relation of mathematics to other subjects and its function in life.

10. One principal difficulty in mathematics teaching is the fact that in the same school are (a) those who are going to college; (b) those who are not going to college. Subject-matter in courses is frequently unsatisfactory, often lacking connection with life after graduation. Professor Gray believes selection instead of elimination should be the guide in carrying out the mathematics program in the schools.

11. Miss Dice advised supplying the grade schools with mathematics teachers who understand and believe in mathematics, who will not promote incompetent pupils, but will teach the basic principles and provide informational mathematics for pupils who retard those who, if not hampered, can benefit themselves and mankind by studying pure mathematics.

NAT EDMONSON, JR., *Secretary*

THE FOURTH ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The fourth annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, on Monday and Tuesday, April 22 and 23, 1940, in conjunction with the annual meeting of the Southwestern Division of the American Association for the Advancement of Science. Professor J. W. Branson, chairman of the Section, presided over the three sessions.

The attendance was forty, including the following thirteen members of the Association: A. W. Boldyreff, J. W. Branson, C. D. Firestone, R. J. Hannelly, H. D. Larsen, H. B. Leonard, Roy MacKay, L. E. Mehlenbacher, C. V. Newsom, E. J. Purcell, Nathan Schwid, P. M. Swingle, Charles Wexler.

At the business meeting the following officers were elected for next year: Chairman, E. J. Purcell, University of Arizona; Vice-Chairman, Roy MacKay, Eastern New Mexico College; Secretary (four years), H. D. Larsen, University of New Mexico. It was voted to hold the 1941 meeting at a time and place to be selected by the officers of the Section.

On Tuesday, April 23, a luncheon was held for members of the Section and their guests. Professor S. B. Talmadge, New Mexico School of Mines, was invited to speak on "Evolution, plus and minus."

The following papers were presented during the three sessions:

1. "A series of space Cremona involutions" by Professor E. J. Purcell, University of Arizona.

2. "On the euclidean n-simplex" by Professor Roy MacKay, Eastern New Mexico College.

3. "A simple formula for approximating the yield of a bond" by Professor H. D. Larsen and Marvin Roberts, University of New Mexico.

4. "Summation of pth powers of numbers, and explicit formulas for Bernoulli numbers of any order" by Professor A. W. Boldyreff, University of Arizona.

5. "Some trisectible angles" by Professor Charles Wexler, Arizona State Teachers College, Tempe.

6. "Cyclic fields of order 8" by Professor J. B. Shaw, University of Illinois, introduced by the Secretary.

7. "Some identities in the calculus of finite differences" by Gerald Harrison, New Mexico State College, introduced by Professor Branson.

8. "The asymptotic representation of functions of the Bessel type" by Professor C. V. Newsom, University of New Mexico.

9. "Some generalizations of the preceding paper" by Abraham Franck, University of New Mexico, introduced by Professor Newsom.

10. "Alternate proofs of some fundamental theorems in advanced calculus" by Professor A. W. Boldyreff, University of Arizona.

11. "An approximation to the solution of a non-linear differential equation of heat conduction" by Dr. Nathan Schwid, Texas College of Mines and Metallurgy.

12. "Concerning the implication relation" by C. D. Firestone, class of 1941, University of New Mexico.

13. "On general groups" by Professor J. B. Shaw, University of Illinois, introduced by the Secretary.

14. "A basis for tabulating all angles with rational cosines which may be trisected by compasses and straight edge" by Professor C. A. Barnhart, University of New Mexico.

15. "Certain types of connected sets" by Professor P. M. Swingle, New Mexico State College.

16. "The general education movement and mathematics" by Professor C. V. Newsom, University of New Mexico.

17. "Experiments with the college mathematics curriculum" by Professor Charles Wexler, Arizona State Teachers College, Tempe.

18. "General courses in mathematics" by Professor E. J. Knapp, Texas College of Mines and Metallurgy, introduced by the Secretary.

Abstracts of some of these papers follow, the numbers corresponding to those in the list of titles:

1. Professor Purcell considered a skew curve C_n of order n having $n-2$ points on a fixed line d. A generic point P determines two transversals, one intersecting C_n in A and d in B, and the other intersecting C_n in C and d in D. He defined P, the correspondent of P in the involution, to be the intersection of AD and BC, and showed the order to be $n-1$, where n is any integer greater than one.

2. Professor MacKay discussed some interesting theorems concerning a point on the Euler line of a nondegenerate n-simplex.

3. Professor Larsen and Mr. Roberts presented a simple formula for approximating the interest rate earned on an ordinary bond purchased on a dividend date at P. If r be the dividend rate per period, n the number of dividends to maturity, F the face of the bond, then to a high degree of accuracy it was shown that the yield rate per period is approximately

$$r - \frac{3(P - F)}{2P + F}\, a_{\overline{n}\, r}^{-1}.$$

4. Professor Boldyreff derived a modification of Abel's formula for the sum of the pth powers of numbers in A.P. This was used to establish a number of explicit formulas for the Bernoulli number of order m. A number of other results were obtained at the same time, including a simple general formula for Stirling numbers of the first kind. Applications of the above results and methods were indicated.

6. Professor Shaw defined a cyclic field of order 8 as follows: Let a, p, q, r, s, m, n be numbers in a field F; $a=p^2+q^2$ not a square in F; $\alpha=(r^2+s^2)(n^2+m^2a)$. Let $y^2=\alpha(a+q\sqrt{a})$, y determining a cyclic quartic field; then

$$y^2 = [r^2 + s^2][n^2 + m^2a]\left[\left(\frac{p + q + \sqrt{a}}{2}\right)^2 + \left(\frac{p - q - \sqrt{a}}{2}\right)^2\right] = P^2 + Q^2.$$

Let A be in the field of $\sqrt{a}$, and meeting certain conditions. Then the most general octic field will be given by $x^2=A(Y^2+Q\sqrt{y})$. If $\theta=(\sqrt{a}-q)/p=y'/y$, accents indicating that in $f(w)$ we change the sign of $\sqrt{a}$ giving $f'(w)$, then

$$\frac{x'}{x} = \frac{\theta}{AP}(Pz + Q - y)\sqrt{\frac{AA'}{1 + z^2}}, \qquad x^{(\mathrm{IV})} = - x.$$

The radical means that $AA'=t^2(1+z^2)$. The number z is determined from the equation

$$Q' = \frac{\theta}{1 + z^2}[- 2zP + (z^2 - 1)Q].$$

If $z=\theta$ we have cases that Albert has studied (*Transactions of the American Mathematical Society*, vol. 35, 1933, pp. 949–964). If $z=1$, or $(s+r\theta)/(r-s\theta)$, we have the other cases.

7. Mr. Harrison derived in a simple manner certain identities which are special cases of the hypergeometric function, $F(a, b, c, 1)$. Other identities, some of which are thought to be new, were developed by elementary methods.

8. Professor Newsom reported upon the asymptotic representation of entire functions wherein the coefficient of the general term of the power series expansion is the reciprocal of the product of two gamma functions. The study represented an extension upon some work reported at a previous meeting of the Section.

9. Mr. Franck, a student of Professor Newsom's, showed how the previous paper could be further extended to the case where the coefficient of the general term was the reciprocal of the product of four gamma functions, and gave further indications of the possibility of extending an earlier theorem of Newsom's to the asymptotic representation of other entire functions.

10. Professor Boldyreff presented simple alternate proofs for such theorems as transformation of coördinates in double integrals, differentiation of an integral with respect to a parameter, and vanishing of the Jacobian and functional dependence of two independent variables.

11. Dr. Schwid considered the non-linear equation for the conduction of heat in one dimension,

$$c p \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial u}{\partial x}\right),$$

where c, p, and k represent the specific heat, density, and thermal conductivity of the medium. An approximation to the solution was obtained for the case where c, p, and k are linear functions, $a+bu$, with the ratio of b to a small. An application is to the conduction of heat in a slab of finite width, infinite in extent, with the same initial temperature throughout and the faces at each end kept at 0°C.

12. Mr. Firestone reported upon some studies which are being made in an attempt to generalize the concept of implication to "implication with a degree." He discussed some of the difficulties involved in the construction of a set of axioms, and also showed what type of theorems may be expected.

13. Professor Shaw defined a general group as follows: Consider a finite set of vids, $A_1, A_2, \cdots, A_n$, subject to a law of combination F, such that $F(A_x A_y) = A_z$ which is such that if A_y or A_x is given, the other is uniquely determined by a process related to F, as $F_r(A_y A_z) = A_x$, $F_l(A_x A_z) = A_y$. Also consider the process $F_{lr}(A_y A_x) = A_z$, $F_{rl}(A_z A_y) = A_x$, $F_{rr}(A_z A_x) = A_y$. Since the number of vids is finite we may write a square table of results for each F, which will be a latin square: that is, no row or column will contain the same letter twice; such a set of six tables constitutes a *general group*. We can extend the term to cover twenty-four related cubical tables arising from $F(A_w A_x A_y) = A_z$. Professor Shaw considered certain cases in which the rows and columns arise by permuting the vids in the rows and columns of F.

14. Professor Barnhart discussed a theorem of Dickson's which formed the basis for determining whether or not an angle with a rational cosine can or cannot be trisected with straight edge and compasses.

15. Professor Swingle defined certain new types of connected sets, and presented several examples and theorems which result.

16. Professor Newsom called special attention to the Junior College and the General College Movement in this country, and to such investigations as the one conducted by the American Association for the Advancement of Science and those of the Coöperative Study in General Education. It was the belief of the

author that there is a distinct tendency in the direction of newtype introductory courses in mathematics upon the college level, and he urged mathematicians to observe the new movement carefully.

H. D. LARSEN, *Secretary*

THE SEVENTEENTH ANNUAL MEETING OF THE MICHIGAN SECTION

The seventeenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Friday, April 26, and Saturday, April 27, 1940. The Friday meeting was held in conjunction with the Mathematics Conference of the Schoolmasters Club of Michigan. The chairman of the Mathematics Conference of the Schoolmasters Club, Professor J. W. Foust, presided at this session at which the attendance was about two hundred. The chairman of the Michigan Section, Professor A. E. Lampen, presided at the regular Saturday morning, luncheon, and afternoon sessions of the Association.

The attendance on Saturday was about ninety. The following forty-six members of the Association attended the meetings: N. H. Anning, W. L. Ayres, J. W. Baldwin, W. D. Baten, F. A. Beeler, W. M. Borgman, J. W. Bradshaw, J. B. Brandeberry, Angeline J. Brandt, C. J. Coe, A. H. Copeland, Max Coral, C. C. Craig, P. S. Dwyer, J. P. Everett, Peter Field, K. W. Folley, R. E. Gaskell, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, L. C. Karpinski, D. K. Kazarinoff, A. E. Lampen, Ruth O. Lane, Theodore Lindquist, D. C. Morrow, A. L. Nelson, H. L. Olson, J. K. Peterson, H. H. Pixley, L. C. Plant, J. E. Powell, Gladdis E. Richards, T. R. Running, Raleigh Schorling, E. R. Sleight, A. G. Swanson, G. B. Van Schaack, Fern Welker, E. T. Welmers, R. L. Wilder, J. B. Winslow, Margarete C. Wolf.

The following officers were elected for the coming year: Chairman, K. W. Folley, Wayne University; Secretary, P. S. Dwyer, University of Michigan. After some discussion it was decided not to have a Section meeting this fall. The time of the annual spring meeting was also discussed.

The following papers were read:

1. "Report on the place of mathematics in secondary education" by Dr. Ruth O. Lane, University High School, Iowa City, Iowa.

2. "Interpretation of the report from the high school point of view" by Professor Raleigh Schorling, University of Michigan.

3. "Interpretation of the report from the college point of view" by Professor E. R. Sleight, Albion College.

4. Discussion.

5. "Least square methods applied to a latin square" by Professor W. D. Baten, Michigan State College.

6. "Some remarks on the space l_p, $(0<p<1)$" by Dr. E. W. Paxson, Wayne University, introduced by Professor Nelson.

7. "Defining relations for some groups of Mathieu" by J. K. Peterson, Lawrence Institute of Technology.

8. "Dualism underlying certain intersection and alignment nomograms" by Professor R. A. Beth, Michigan State College, introduced by Professor Grove.

9. "Jacobi's equation for parametric variation problems" by Dr. Max Coral, Wayne University.

10. "Mathematical Reviews" by Professor W. L. Ayres, University of Michigan.

11. "The 'follow through' from high school to freshman mathematics" by C. H. Nordstrom, Michigan State College, introduced by Professor Grove.

12. "Advanced mathematics from an elementary view-point" by Professor L. S. Johnston, University of Detroit.

13. "Some graphical methods used in aeronautical engineering" by Professor Peter Altman, Director, Department of Aeronautics, University of Detroit, introduced by Professor Johnston.

Abstracts of these papers follow, numbered in accordance with their place on the program:

1. Dr. Lane presented the report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, which is published as the Fifteenth Yearbook of the Council. A preliminary report was published in 1938 for the purpose of securing construtive criticisms. While individual members worked primarily on those aspects of the report on which they could contribute most, no chapter can be thought of as the contribution of a single member. This report, she said, aims to bring up to date the epoch-making 1923 report of the National Committee, and is prepared for all those interested in the mathematics curriculum from the seventh grade to the junior college without ignoring the special interests of the classroom teachers. The first four chapters were written after the Commission had studied the present philosophy and conditions pertaining to the secondary schools. The two curriculum plans which follow are neither revolutionary nor arbitrary, and should not thwart adaptation and experimentation. The Commission believes that the lack of providing for both flexibility and continuity has been the great stumbling block in the development of a national program of instruction. It is recommended that no single year be regarded as a "terminal year." The essentials of a general program include number and computation, geometric forms and space perception, elementary analysis, graphic representation, logical thinking, relational thinking, and symbolic representation and thinking. There are separate chapters on the problems of retardation and acceleration, the junior college, evaluation of progress of students, and education of teachers. The first appendix contains an analysis of mathematical needs in a form readily useful to the teacher. Five more appendices include transfer of training, terms, symbols and abbreviations, equipment, and grade placement charts.

2. Among the desirable features listed by Professor Schorling were the chap-

ter on acceleration and retardation, the endorsement of a social mathematics course as an elective for the late years of the senior high school, and a somewhat more effective classification of mathematical aims and outcomes. The speaker was disposed to think that the effects of certain undesirable recommendations should be minimized. Included in the weak spots are the following: (1) Failure to make a clear-cut distinction between the place of mathematics in general education, and the rôle of mathematics in the training of boys and girls who are preparing for rigorous work in science and mathematics. (2) The inclusion of a chapter showing year by year what the Commission recommends as specific topics in subject-matter. There is no reliable evidence to support such arbitrary grade placement, particularly in grades seven and eight. (3) The statement about transfer of training. The 1923 Report by the National Committee on Mathematical Requirements included very early in its pages a courageous and scholarly statement of disciplinary values. In contrast, the present report relegates the brief and superficial discussion of transfer of training to the appendix. (4) Excessive reliance on the opinions of educationists. The speaker concluded by pointing out the two-fold task that now confronts every teacher of high school mathematics: First, to make certain that every constructive suggestion is widely applied in our schools; and second, to minimize the effects of undesirable recommendations.

3. Professor Sleight expressed appreciation for the fact that the teachers of college mathematics have become interested in the problems of the teachers of secondary mathematics. From the college point of view the report of the Commission is significant for the following reasons: (a) it represents a closer relationship among all teachers of mathematics; (b) it discusses the general trends in mathematics, showing the necessity for thorough foundation, adequate use of skills, modern application and the aims; (c) the stress upon definite standards determined for the average, not for those of low I.Q.; (d) the necessity for ability to read understandingly.

4. Dr. Lane made use of the discussion period to make additional comments on points which had been raised by the other speakers. The Commission gave recognition to the 1923 report, Miss Lane said, and did not attempt to quote at length from it. The chairman, Professor K. P. Williams, for six years devoted a large portion of his time to efficient leadership of the Commission, and study of secondary education, without financial reward. Both the proposed curriculum plans include second course algebra. While the Commission may be open to criticism for not providing sufficient concrete helps for the teacher in the evaluation of outcomes, she concluded, there was no intention of advising teachers against the use of standard tests.

5. Professor Baten explained a five by five latin square, often used as an agricultural design for testing for significance between "treatment" means. Each plot in the square was assumed to be a linear function of the variables R_i, C_j, z_k, w, and x as follows:

$$y_{ij} = \sum_{i=1}^{5} r_i R_i + \sum_{j=1}^{5} c_j C_j + \sum_{k=1}^{5} t_k z_k + mw + bx + e_{ij},$$

where $R_i = 1$ for plots in the ith row and zero elsewhere, $C_j = 1$ for plots in the jth column and zero elsewhere, $z_k = 1$ for plots in the kth treatment and zero elsewhere, $w = 1$ for each plot, $x = x_{ij}$ or the stand or number of plants in the ith row and the jth column, y_{ij} = the yield of the plot in the ith row and jth column, and e_{ij} = the residual errors. It was also assumed that the sums of the constants pertaining to rows, columns, and "treatments" were equal to zero, or

$$\sum_{i=1}^{5} r_i = \sum_{j=1}^{5} C_j = \sum_{k=1}^{5} t_k = 0.$$

By the least squares method Professor Baten showed that the value of b in the equation

$$y'_{ij} = y_{ij} - b(x_{ij} - \bar{x}),$$

used for adjusting yields for stands, came from the error line in the analysis of variance table. It was also shown that the standard error of estimate obtained from the error line was the experimental error to use for testing for significance between "treatment" means.

6. Dr. Paxson discussed the space of all infinite sequences of numbers, the sum of whose absolute pth powers $(0 < p < 1)$ converges. The precise form of a weakened triangular inequality was stated and the lack of local convexity emphasized. The form of the most general linear functional on the space was given, thus violating the consistency between the spaces l_p and L_p that one normally expects, since in L_p no non-zero functionals exist (*cf.* M. M. Day, *Bulletin of the American Mathematical Society*, Abst. 46-3-165).

7. Simple complete sets of defining relations, in terms of several generators, were obtained for the Mathieu group of degree 23 and for its maximal subgroups of degrees 22 and 21. Mr. Peterson pointed out that the quadruply transitive groups of Mathieu are generated by two operators which satisfy, for $p = 23$ and $p = 11$, the relations

$$s^p = t^2 = (st)^q = (s^2t)^{2(p-1)/3} = (s^q t)^q = (s^{-1}tst)^2 = 1,$$

where $q = (p-1)/2$. These groups are being studied for arbitrary prime values of p.

8. Dr. Beth stated that, following good usage, the term *nomogram* may be defined to include intersection charts as well as alignment charts. Quantities represented by points on one are represented by lines on the other. The dualism which exists between the two types permits a common mathematical formulation for all possible linear transformations of either. This, when interpreted graphically as a certain projective transformation, offers an alternative to the usual methods (similar triangles or determinant transformations) of designing alignment charts which permits a better consideration of all possibilities in mak-

ing the choice of the particular form to be drawn. Moreover, the same method may be used for intersection charts. It is particularly useful when the number of variables to be represented, N, is greater than three, the number of imposed relations, $N-2$, being greater than one.

9. Dr. Coral considered the general double integral variation problem in parametric form. For this problem, he filled in a hitherto existing gap in the theory by computing explicitly the form of the Jacobi differential equation. The coefficients of the equation were found to have a comparatively simple form, involving the coefficients of the first fundamental form for the spherical representation of the minimizing surface.

10. Professor Ayres gave a progress report on the new journal, *Mathematical Reviews*, which is being subsidized in part by the Association. At the time the March number was mailed there were 1058 subscribers to the *Reviews*. The desirability of having the *Reviews* available in every college library was stressed.

11. A survey was made by Mr. Nordstrom of grades received in seven different courses by 1012 students representing 240 Michigan high schools and 70 out-of-state high schools in the fall of 1939. The purpose was to obtain information which would help guide the preparation of high school students and which would help the staff of the mathematics department to greater teaching efficiency. The results show that there is no correlation between grade and size of high school, with the exception of the group having a high school population of 100 or less, which group had an average considerably lower than that of other groups. However, this was probably due to the fact that this group offers as a rule but two years of mathematics, and it is not due to inferior teaching nor to inferior students, since in two courses where the minimum entrance requirement is two years, students having only the minimum preparation received respectively 68% and 83% of the failures. With entrance to Michigan State College specifically in mind, Mr. Nordstrom concluded with recommendations as to the high school mathematics program.

12. All sorts of methods have been devised to solve the problem of stimulating and maintaining interest in high school mathematics. Quite a few which have been incorporated in texts are mere concoctions, giving to mathematics an air of triviality, almost of futility. Professor Johnston believes that the best stimulus is mathematics itself, mathematics available to and assimilable by the high school student, which will open to him a realization of his own powers in some phases of so-called "advanced" mathematics. He quoted from his own experience in placing before high school classes such topics as maxima and minima, alligation, curve fitting, the Euler and Simson lines, inversion of curves, linkages, stereographic projections, partial fractions, the theorem of Desargues, perspectivity and projectivity, and several others not usually encountered until well along in college mathematics. He demonstrated several theorems to show that they are entirely within the high school province, and mentioned a list of topics which can be either incorporated in the regular schedule or formally offered as an eighth semester course in high school mathematics for those ex-

pecting to major in engineering or mathematics.

13. Professor Altman indicated the importance of the application of the fundamentals of engineering and mathematical analysis in aeronautical engineering problems. Emphasis was placed on a good combination of analytical and experimental methods in arriving at the most satisfactory design. The talk was illustrated with the use of slides. A number of graphical applications in integration, differentiation and the solution of some differential equations were shown. The graphical integration included the solution for the thrust force produced by a propeller blade, the solution for the centrifugal force and twisting moment produced by a rotating body of variable sections such as a propeller blade, and the determination of the geometric properties of irregular sections such as the moment of inertia, product of inertia, *etc.* The graphical differentiation was applied to the solution of the velocity and the acceleration of a falling body such as in the required drop tests of an airplane shock absorbing system. A method to determine the bending moment, the value of the shear and other important structural factors for beams subjected to axial compression or tension loads in combination with transverse loads, where the effect of the secondary moments due to deflections were important, was connected with the solution of a second order differential equation by the use of polar diagrams.

P. S. DWYER, *Secretary*

THE SEVENTEENTH ANNUAL MEETING OF THE INDIANA SECTION

The seventeenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 3 and 4, 1940, at Earlham College, Richmond, Indiana.

Forty-five registered at the meetings, including the following nineteen members of the Association: Emil Artin, Juna Lutz Beal, P. D. Edwards, B. C. Getchell, G. H. Graves, Cora B. Hennel, L. P. Hutchison, M. W. Keller, Neil Little, Florence Long, H. A. Meyer, H. R. Pyle, L. S. Shively, D. R. Shreve, W. O. Shriner, M. S. Webster, Agnes E. Wells, K. P. Williams, A. J. Zanolar.

At the business session on Saturday morning, the following officers were elected for next year: Chairman, Cora B. Hennel, Indiana University; Vice-Chairman, H. R. Pyle, Earlham College; Secretary, P. D. Edwards, Ball State Teachers College. The eighteenth annual meeting will be held at Butler University.

The annual dinner was held on Friday evening in Earlham Hall. Professor L. S. Shively served as toastmaster and introduced Dean H. R. Pyle who welcomed the visitors. Following the dinner the first session of the Section was held, at which time Dr. Karl K. Darrow of the Bell Telephone Laboratories gave a popular lecture on "Least time and least action." The lecture was open to the public and was well attended.

Dr. Darrow introduced the subject "Least time and least action" by saying

that he would illustrate the principles bearing those names by a single example, the law of refraction. After giving a historical account of the earliest experimental data bearing on this law, he deduced it in the ways invented by Fermat (principle of least time), by Descartes (concept of particles), by Huygens (concept of waves), and by Maupertuis (principle of least action applied to concept of waves). He then unified these several derivations by invoking the concept of beats formed by superposed wave trains of nearly identical frequencies, the principle of least action being obeyed by the beats interpreted as corpuscles and the principle of least time being obeyed by the waves; he also mentioned the extension of these concepts and principles to electricity and matter.

At the two sessions on Saturday the following program was presented:

1. "Fibonacci sequences and certain irrational numbers" by Professor L. S. Shively, Ball State Teachers College.

2. "Conformal mapping on surfaces" by Professor H. R. Pyle, Earlham College.

3. "The drawing of quadric surfaces" by Neil Little, Purdue University.

4. "The transits of Mercury" by Professor K. P. Williams, Indiana University.

5. and 6. "Diagnostic mathematics testing in Purdue University" by Dr. M. W. Keller and Dr. D. R. Shreve, Purdue University.

7. "The real shape of plane cubics" by Professor G. H. Graves, Purdue University.

8. "Methods of teaching irrational numbers" by Dr. R. J. Duffin, Purdue University, introduced by Professor Graves.

9. "Generators for commutative algebras" by Dr. G. W. Whaples, Indiana University, introduced by Professor Artin.

Abstracts of papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Shively pointed out that the well known sequence of Fibonacci which appeared in his *Liber Abaci* satisfies the recursion formula, $u_n+u_{n+1}=u_{n+2}$. Sequences F_k, F_k', and F_k'', whose general terms are u_n, v_n, and w_n respectively, were defined for every positive integer k as follows:

$$u_1 = 0, \qquad u_2 = 1, \qquad u_n + 2ku_{n+1} = u_{n+2};$$

$$v_n = u_{n+1} - ku_n; \qquad w_n = v_{n+1} - kv_n.$$

Some of the properties of these sequences which have been known were stated and others were established. It was then shown, by means of these sequences, how rational fractions whose values are alternately greater and less than $\sqrt{k^2+1}$ can immediately be written down, and that they rapidly approach the latter as a limit. The discussion was then extended to the case in which k is any positive rational number. In this case the terms of the sequences are not all integers and $\sqrt{k^2+1}$ is not always irrational.

2. Given two surfaces, S_1 and S_2, with general form of the matrices

$$S_1: \quad ds^2 = \omega(\alpha dx^2 + \beta dy^2 + 2\epsilon dxdy),$$

where ω, α, β, ϵ are functions of x, y, and

$$S_2: \quad ds^2 = \bar{\omega}(\bar{\alpha}du^2 + \bar{\beta}dv^2 + 2\bar{\epsilon}dudv),$$

where $\bar{\omega}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\epsilon}$ are functions of u, v; Professor Pyle assumed a relation $u=\phi(x, y)$, $v=\psi(x, y)$ with the usual assumptions as to continuity and the existence of the first and second partial derivatives of ϕ and ψ. Necessary and sufficient conditions for conformality were obtained in terms of $\Delta_1\phi$, $\Delta_1\psi$, and $\Delta_2(\phi, \psi)$, the well known differential parameters of the first order. Generalized forms of the Cauchy-Riemann equations which are valid on any surface were given. The Laplace equation was generalized in terms of $\Delta_2\phi$ and $\Delta_2\psi$, the differential parameters of the second order. Conditions for conformality in polar coordinates and in elliptic geometry were given as special cases of the general forms.

3. The purposes of Mr. Little's paper were, first, to point out certain inaccuracies which exist in the usual drawings of quadric surfaces, and second, to describe a mechanical method whereby correct and accurate drawings can be made. Using four sample drawings, copies of which were distributed to the audience, a number of inaccuracies were pointed out on two of the drawings. The other two were drawings of the same surface accurately done, and the method of drawing them was described.

4. Professor Williams gave a complete discussion of the transits of Mercury, making use of the tables of Newcomb. Two solutions of the problem were made. In one solution the actually observed times of contact were used; in the other solution the times were corrected for "fluctuation" as determined in Jones' recent discussion of lunar occultations. Tidal retardation was introduced as one of the unknowns to be determined. The solution that made use of the corrections for fluctuation of time gave decidedly the best results, the residuals being uniformly small. The non-gravitational shift of the perihelion was reduced by about $3''$ below the result obtained by Newcomb.

5. and 6. This was a progress report on the diagnostic testing which has been inaugurated at Purdue University by the authors in coöperation with Dr. H. H. Remmers of the Division of Educational Reference. Dr. Keller outlined the purpose of this program, the construction of the tests, and their administration. He also presented briefly some of the facts which were obtained from the first of the three tests. Dr. Shreve gave some of the more important findings from the third of these tests with some suggestions for changes in construction in accordance with those findings. A portion of the paper appears in this issue of the MONTHLY.

7. Professor Graves exhibited a set of graphs of the 78 species of plane cubics as classified in Newton's *Enumeratio Linearum Tertii Ordinis*, with remarks on reduction of the general equation to Newton's four type forms.

8. Dr. Duffin discussed several methods of demonstrating the existence of irrational numbers suitable for presentation to students in elementary algebra. In particular, a little known geometric method was given.

9. Dr. Whaples showed that if A is a commutative algebra with unit element over field F, such that F has an infinite number of elements and the semi-simple part of A is a direct sum of separable extensions of F, then A can be written as the set of all polynomials in n elements, where n is the number of generators of the radical of A, considered as A-ideal; n is the smallest such number. In particular, A is a principal ideal ring if and only if it is expressible as the set of polynomials in a single element.

P. D. Edwards, *Secretary*

THE EIGHTH ANNUAL MEETING OF THE WISCONSIN SECTION

The eighth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Mount Mary College, Milwaukee, on Saturday, May 4, 1940. The meeting was presided over by the chairman, Sister Mary Felice of Mount Mary College. The members of the Association and guests were graciously welcomed by Sister Dominic, Dean of Mount Mary College.

There were fifty-three persons present, including the following twenty-four members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, H. H. Conwell, L. A. V. DeCleene, Fannie Hopkins, R. C. Huffer, M. L. Jautz, Elizabeth E. Knight, R. E. Langer, Peter Luteyn, C. C. MacDuffee, Morris Marden, Sister Mary Felice, E. A. Nordhaus, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, P. L. Trump, M. J. Turner, J. I. Vass.

Sessions were held in the morning and afternoon with a luncheon in the main dining hall of Mount Mary College at 12:15. The business meeting was held at 2:00 P.M. at which the following officers were elected: Chairman, W. W.Bigelow, Beloit College; Secretary, G. A. Parkinson, University of Wisconsin Extension Division; Program Committee, I. S. Sokolnikoff, University of Wisconsin, and R. E. Norris, Milwaukee State Teachers College. Invitations to hold the 1941 meeting at Beloit College and the 1942 meeting at Oshkosh State Teachers College were accepted by unanimous vote. Appreciation for the hospitality extended to the Section by Mount Mary College was expressed by a rising vote.

At the business meeting Professor R. E. Langer of the University of Wisconsin reported on the national reorganization of the Association. Professor Langer suggested that the members of the Section should avail themselves of the opportunity presented by the new organization to participate more actively in the affairs of the national organization. Following this, Professor P. L. Trump of the University of Wisconsin High School reported on the meeting of the National Council of Teachers of Mathematics to be held in Milwaukee on July 1–3, 1940, in connection with the National Education Association meetings.

At the morning session the following papers were presented:

1. "A plea for natural logarithms" by the Reverend L. A. V. DeCleene, St. Norbert College.

2. "Applications of lattice theory" by Dr. E. A. Nordhaus, University of Wisconsin Extension Division.

3. "The centroids of photographic images of stars" by Professor R. C. Huffer, Beloit College.

The afternoon session was devoted to a panel discussion on "Vitalizing mathematics teaching." The leaders and their subjects were as follows: Professor H. P. Pettit, Marquette University, "Restatement of the objectives of mathematics teaching"; Professor Ethelwynn R. Beckwith, Milwaukee-Downer College, "Mathematics as a social science"; Professor P. L. Trump, University of Wisconsin High School, "Enriching the mathematics course"; and Fannie Hopkins, Waukesha High School, "A high school teacher's view-point on our teacher training courses in mathematics."

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. DeCleene said that little attention is allotted to natural logarithms, often with only the remark that there is another kind besides common logarithms. Because of this, students enter chemistry, physics, and engineering without a facility of handling natural logarithms and e. Tables of natural logarithms should be included with those of common logarithms.

2. Dr. Nordhaus outlined the axiomatic treatment of the theory of partially ordered sets and lattices as developed by Birkhoff, Ore, and others. He indicated the manner in which a Boolean algebra may be obtained from a distributive lattice and how a modular lattice satisfying a chain condition may give rise to a projective geometry. The introduction of a dimension function into the lattice and the work of von Neumann on continuous geometry were discussed.

3. Professor Huffer discussed the methods of measuring positions of stellar images on photographic negatives, with particular reference to close visual binaries. Formulas were derived for the centroids of pairs whose images are blended, in terms of the magnitudes and the distance between the centers of the stars, on the basis of various assumptions with respect to the properties of the photographic plate.

G. A. PARKINSON, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the Teachers College, Mankato, Minnesota, on Saturday, May 11, 1940. A morning session, held at 10:30 o'clock, was followed by luncheon and an afternoon session at 2:15 o'clock. Professor C. S. Carlson of St. Olaf College presided at the morning session, and Professor W. H. Bussey of the University of Minnesota at the afternoon session.

Sixty-eight persons attended the meeting, including the following thirty-seven members of the Association: Mae R. Anderson, H. A. Arnold, C. J. Blackall, R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson,

S. Elizabeth Carlson, Sister M. Claudette, H. H. Dalaker, Arthur Danzl, J. H. Daoust, Brother Louis De La Salle, Gladys Gibbens, C. H. Gingrich, J. S. Hickman, C. J. Kirchen, W. H. Kirchner, Fulton Koehler, Margaret P. Martin, W. R. McEwen, Sigurd Mundhjeld, F. J. Polansky, A. R. Poole, G. C. Priester, Inez Rundstrom, M. G. Scherberg, Ole Schey, C. Grace Shover, R. R. Shumway, H. P. Thielman, Ella Thorp, H. L. Turrittin, A. L. Underhill, K. W. Wegner, Marion A. Wilder.

At the business session officers were elected for the coming year as follows: Chairman, A. J. Strane, Duluth Junior College; Secretary, A. L. Underhill, University of Minnesota; Executive Committee, K. W. Wegner, College of St. Catherine, Brother Louis, St. Marys College, and Sister M. Claudette, College of St. Benedict.

The following eight papers were presented:

1. "On the functional equation $f(x+1)=f(x)\left[(x+a)/(x+b)\right]^p$" by Professor H. P. Thielman, College of St. Thomas.

2. "An extension of the definition of inversion" by W. R. McEwen, University of Minnesota.

3. "The report of a committee of the A.A.A.S. on mathematical instruction for purposes of general education" by Professor K. W. Wegner, College of St. Catherine.

4. "Fixed-point theorems with application to elementary mathematics" by Dr. H. A. Arnold, University of Minnesota.

5. "A simple mechanical application of elliptic integrals" by Dr. Isaac Opatowski, University of Minnesota.

6. "On the graph of a certain function" by Dr. M. G. Scherberg, University of Minnesota.

7. "Finite geometry and the frangent" by P. T. Gilbert, Jr., University of Minnesota, introduced by Professor Bussey.

8. "The foci of cubic curves" by M. C. Waddell, University of Minnesota, introduced by Professor Bussey.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Thielman showed that the successive terms of many well known series satisfy the functional equation of the title. The general solution of this equation was given, and the behavior of this solution at infinity was considered. On this basis conclusions were drawn in regard to the convergence and divergence of certain series.

2. The definition of an inverse curve was extended by Mr. McEwen to include the case in which the radius of the circle of inversion is a root of a pure imaginary number.

3. A preliminary report of the special committee of the American Association for the Advancement of Science on the improvement of science and mathematics teaching in colleges and universities was given by Professor Wegner. The committee undertook first to determine by way of a questionnaire the point of

view of science and mathematics teachers with respect to certain issues involved in the problem and also to discover the present practices in those courses designed primarily for purposes of general education. The returns of the questionnaire sent to mathematics teachers are tabulated in this report.

4. Dr. Arnold showed, principally by examples, how most of the important problems on the solution of equations, algebraic, differential or functional, may be formulated in fixed-point form. Various generalizations to functional equations of the Brouwer fixed-point theorem concerning continuous functions defined over an n-dimensional cube or an n-dimensional "tetrahedron" were stated and discussed.

5. It was shown by Dr. Opatowski that the forces acting in an ordinary automobile brake may be evaluated in a very simple way by means of Legendre's elliptic integrals.

6. It is readily shown that the graph of the multi-valued function $f(x)=0$ for x irrational and in $0 \leqq x \leqq 1$, $f(x)=1/s$ for $x=r/s$ a rational number in $0 \leqq x \leqq 1$, consists of the irrational point in $0 \leqq x \leqq 1$ and the intersections among the lines

$$y = x, \qquad y = x/2, \qquad y = x/3, \cdots$$

$$x = 0, \qquad y = 1, \qquad y = \tfrac{1}{2}, \qquad y = \tfrac{1}{3}, \qquad y = \tfrac{1}{4}, \cdots .$$

Dr. Scherberg then showed with a simple geometric argument that $f(x)$ is continuous at the irrational points.

7. Mr. Gilbert defined in a square field of points (a, b), a and b least non-negative residues of a modulus $m=\Pi_1^n p_i^{\alpha_i}$, a line to consist of all points such that for any two, (a, b), (a', b'), $a'-a \equiv qx$, $b'-b \equiv qy$, x and y being relatively prime L.N.R.'s whose ratio y/x is shown uniquely congruent to one of the $\psi(m)$ irreducible quotients, defined as fractions a/b of L.N.R.'s where (1) $a<b$ if $b>1$, (2) b and (3) a are, in order, the least possible. The chief law governing I.Q.'s is: $a/b \equiv a'/b' \cdot \sim \cdot ab' \equiv a'b$. The lines on a point are classified and counted, and the principal result is that the total = the frangent, $\psi(m)$, in number, and it is proved that $\psi(m)=\Pi_1^n p_i^{\alpha_i-1}(p_i+1)$. Horizontal (or vertical) lines, lacking a point in every row (or column) number $\psi(m)-m=\Pi_1^n p_i^{\alpha_i-1}\sum_0^{n-1} E_i$, E_i the ith elementary symmetric function of the p_i. Horizontal-vertical lines number $\psi(m)+\phi(m)-2m$. Rectangular lines (forming a rectangular pattern) total 2^n-2. The properties of $\psi(m)$ and its analogies with and the geometrical application of $\phi(m)$, as also those of the symmetric sequence $\{(I(d), m/d)\}$, d the divisors of m in order and $I(d)$ the least of equal intervals of symmetry in the list of totitives, were discussed.

8. Mr. Waddell gave a discussion of the Plücker definition of foci of plane algebraic curves, with attention to the number of foci which a curve will possess. In particular, the number and location of the foci of cubic curves was brought out, and several examples given. Mention was also made of the application of this definition to finite geometries of prime modulus.

A. L. UNDERHILL, *Secretary*

ON THE REDUCTION OF A MATRIX TO A CANONICAL FORM*

E. T. BROWNE, University of North Carolina

1. Introduction. If A is a square matrix of order n, and T is a non-singular matrix of the same order, the matrix

$$B = T^{-1}AT \tag{1}$$

is called the transform of A by T. Then A and B are called *similar*. If we consider A and B as the matrices of two homogeneous linear transformations, (*i.e.*, of projective transformations in $(n-1)$-space, or of affine transformations with fixed origin in n-space),

$$y = A(x), \qquad y' = B(x'),$$

it is well known that the transformations have identical geometric properties. Algebraically, a necessary and sufficient condition that A and B be similar is that their characteristic matrices $\lambda I - A$, $\lambda I - B$ have the same invariant factors, or if we prefer, the same elementary divisors.

An important problem in the theory is to select from the set of all matrices B which are similar to A one or more matrices which have the simplest form, or which are most suitable for studying the properties of the matrix. Such matrices are called *canonical forms* of A. Three of these so-called canonical forms are worthy of especial mention; (1) the *rational* or *natural* canonical form, which puts into evidence the (rational) invariant factors of the matrix, (2) the *classical* or Jordan canonical form which puts into evidence the Weierstrassian elementary divisors of the matrix, and (3) the Jacobson canonical form which puts into evidence the irreducible elementary divisors in a field $\mathfrak{F}$, and which includes the Jordan canonical form as a special case when $\mathfrak{F}$ is an algebraically closed field, for example, the field of all complex numbers.

Many writers obtain a canonical form in the following manner. They first establish the important theorem:

THEOREM I. *Two n-square matrices A and B are similar if, and only if, their characteristic matrices $\lambda I - A$, $\lambda I - B$ have the same elementary divisors, or, if we prefer, the same invariant factors.*

Having established this theorem they then write down a matrix B of simple form which has the same elementary divisors (or invariant factors) as the given matrix A and they take this as a canonical form to which, by Theorem I, A can be reduced by a transformation of the type (1). This is the point of view of Muth [1], Bôcher [2], Wedderburn [3], and others. On the contrary, other writers first obtain *ab initio* a canonical form B for a given matrix A, and from this they derive Theorem I as a consequence. The derivation of the canonical form becomes then the starting point of the theory of the similarity of two ma-

* Presented to the Southeastern Section of the Mathematical Association of America at Athens, Ga., March 30, 1940.

trices and of the equivalence of pairs of bilinear forms, *etc.* This latter point of view is adopted by Schreier and Sperner [**4**], van der Waerden [**5**], Kowalewski [**6**], Krull [**7**], Dickson [**8**], Turnbull and Aitken [**9**], and others.*

The methods employed by the last four writers depend on essentially the same idea, *viz.*, that of *chains of vectors*. To begin with, a chain of maximum length is selected. However, a criticism to be levelled at these methods is that no clue is given that will enable one to know when he has a chain of maximum length, except that in the contrary case there will appear in the course of the computation a chain of greater length [**10**].

The method of reduction given by Schreier and Sperner is not subject to this criticism. It is based essentially on the Fundamental Theorem of Abelian Groups (although it can be developed without explicit mention of that theory), and is very elegant. Indeed, one might use regarding it the words that Gundelfinger used regarding Lagrange's method for reducing a quadratic form to a sum of squares: "For elegance it leaves nothing to be desired." However, the point of view is slightly different from the classical point of view adopted by the above-mentioned four writers.

It is the purpose of this paper to give an *a priori* derivation of the three canonical forms. We also employ the notion of chains of vectors. No knowledge of invariant factors or of elementary divisors is presupposed. Indeed, we invoke only the Hamilton-Cayley theorem, *viz.*, that *every matrix A satisfies its own characteristic equation*, a fact that is easily established without any use of elementary divisors or of canonical forms [**11**]. Thus a method of reduction is given which in its practical application, particularly when n is small, has some advantages over those previously given.

2. The reduced characteristic function of A. Consider then a square matrix A of order n with elements in a field $\mathcal{J}$. Let A^α be the first matrix in the sequence

$$I, A, A^2, \cdots, A^{\alpha-1}, A^\alpha \tag{2}$$

such that the matrices in the sequence are linearly dependent. That is, let

$$A^\alpha = a_1A^{\alpha-1} + a_2A^{\alpha-2} + \cdots + a_{\alpha-1}A + a_\alpha.$$

If we write

$$\phi(\lambda) = \lambda^\alpha - a_1\lambda^{\alpha-1} - a_2\lambda^{\alpha-2} - \cdots - a_{\alpha-1}\lambda - a_\alpha, \tag{3}$$

then $\phi(A)=0$, while $\psi(A)\neq 0$ for any function $\psi(\lambda)$ of lower degree than α. We shall call $\phi(\lambda)$ the *reduced characteristic function* of A. Since the elements of A lie in a field $\mathcal{J}$, clearly the coefficients of $\phi(\lambda)$ also lie in $\mathcal{J}$, and, moreover, by the Hamilton-Cayley theorem, $\alpha \leqq n$.

This reduced characteristic function of A is unique and, moreover, if $\psi(\lambda)$ is any function such that $\psi(A)=0$, then $\phi(\lambda)$ is a factor of $\psi(\lambda)$ Furthermore, the

* For a complete bibliography on the subject up to 1933, see MacDuffee, The Theory of Matrices, Berlin, 1933, p. 72.

reduced characteristic function of any matrix B similar to A is identical with that of A.

3. Chains of vectors. Consider now a vector $x=(x_1, \cdots, x_n)$ over the field $\mathfrak{J}$. For simplicity of notation we shall look upon such a vector as a one column matrix. The vector Ax is then a one column matrix obtained by forming the product of A by x in the usual manner. We suppose then that x is not the zero vector $(0, \cdots, 0)$, and form the *chain* of vectors

$$x,\ Ax,\ A^2x,\ \cdots,\ A^{\mu-1}x,$$

with x as leader. Let us suppose that these vectors are linearly independent, but that the vector $A^\mu x$ depends linearly on them, *i.e.*, we suppose that

$$A^\mu x = b_1 A^{\mu-1}x + b_2 A^{\mu-2}x + \cdots + b_{\mu-1}Ax + b_\mu x,$$

where the b's belong to $\mathfrak{J}$. If we denote by $F(\lambda)$ the polynomial

$$F(\lambda) = \lambda^\mu - b_1\lambda^{\mu-1} - b_2\lambda^{\mu-2} - \cdots - b_{\mu-1}\lambda - b_\mu,$$

it is clear that $F(A)x=0$. The polynomial $F(\lambda)$ is therefore said to annihilate the vector x, and clearly no polynomial of lower degree possesses that property. We shall call $F(\lambda)$ the *reduced characteristic function* of x with respect to the matrix A, and shall say that x *belongs to the polynomial* $F(\lambda)$. In justification of this terminology, we quote without proof the following easily established theorem [**12**]:

THEOREM II. *The reduced characteristic function $F(\lambda)$ of a vector x (with respect to a matrix A) is unique, and moreover, if $\psi(\lambda)$ is any polynomial such that $\psi(A)x=0$, then $F(\lambda)$ is a factor of $\psi(\lambda)$.*

Since $\phi(A)=0$, every vector is annihilated by $\phi(\lambda)$, and we have the following:

COROLLARY. *The polynomial to which any vector belongs is a factor of $\phi(\lambda)$.*

Fundamental for our purpose is the following:

THEOREM III. *If $\phi(\lambda)$ is the reduced characteristic function of a matrix A, we can find vectors x, with elements in $\mathfrak{J}$, which belong to $\phi(\lambda)$.*

Proof. To prove this theorem, let $\phi(\lambda)$ when resolved into irreducible factors in $\mathfrak{J}$ be

$$\phi(\lambda) = p_1^{q_1}p_2^{q_2}\cdots p_s^{q_s}, \qquad (q_i \geqq 1;\ i = 1, \cdots, s),$$

where the p's are distinct and irreducible in $\mathfrak{J}$. Write

$$\phi(\lambda) = p_1^{q_1}P_1 = p_2^{q_2}P_2 = \cdots = p_s^{q_s}P_s.$$

The matrix $p_i^{q_i-1}P_i(A)$ is different from zero, since $\phi(\lambda)$ is the reduced characteristic function of A. If the kth column of this matrix does not consist entirely of

zeros, the unit vector e_k with 1 in the kth position

$$e_k = (0, \cdots, 0, 1, 0, \cdots, 0)$$

is such that

$$p_i^{q_i-1} P_i(A) e_k \neq 0.$$

If we write

$$z_i = P_i(A) e_k,$$

then $p_i^{q_i-1}(A) z_i \neq 0$, while $p_i^{q_i}(A) z_i = \phi(A) e_k = 0$. Hence z_i belongs to the polynomial $p_i^{q_i}$ and, moreover, it is a vector over the field $\mathcal{J}$.

The vector

$$x = z_1 + z_2 + \cdots + z_s$$

is not zero and is easily shown to belong to the polynomial $\phi(\lambda)$. For let x belong to $\psi(\lambda)$, so that

$$0 = \psi(A) x = \Sigma \psi(A) z_i.$$

Since for $i \neq j$, P_i is divisible by $p_j^{q_j}$, we have, on operating with $P_i(A)$,

$$0 = \psi(A) P_i(A) z_i, \qquad (i = 1, \cdots, s).$$

Hence ψP_i is divisible by $p_i^{q_i}$, and since the latter is relatively prime to P_i, it follows that ψ must be divisible by $p_i^{q_i}$, $(i = 1, \cdots, s)$, and therefore by their product $\phi(\lambda)$. That is, $\psi(\lambda) = \phi(\lambda)$.

4. The reduction of A to rational canonical form. We suppose that the elements of A lie in some field $\mathcal{J}$. Our problem is to find a non-singular matrix T with elements in $\mathcal{J}$ such that $B = T^{-1} A T$ assumes a canonical form.

We first establish the following:

LEMMA I. *If $v_1, v_2, \cdots, v_n$ denote the n linearly independent column vectors of the non-singular matrix T, and if $Av_k = b_{1k} v_1 + b_{2k} v_2 + \cdots + b_{nk} v_n$, then the elements $b_{1k}, b_{2k}, \cdots, b_{nk}$ are precisely the elements in the kth column of the matrix $B = T^{-1} A T$.*

Proof. First of all, we note that since the vectors $v_1, v_2, \cdots, v_n$ are by hypothesis linearly independent, the vector Av_k, which is the column vector in the kth column of the matrix AT, is always expressible in the form stated in the lemma. If now we denote by $V_1, V_2, \cdots, V_n$ the row vectors of the matrix T^{-1}, we have by well known properties of determinants,

$$V_i \cdot v_k = \delta_{ik},$$

where the notation $V_i \cdot v_k$ denotes the inner product, and δ_{ik} is the Kronecker symbol which equals 1 for $i = k$, 0 for $i \neq k$. It follows then that $V_i \cdot Av_k = b_{ik}$, and the lemma is established.

Now let $\phi(\lambda)$ in (3) be the reduced characteristic function of A, and let x_1 be a vector with elements in $\mathcal{J}$ which belongs to $\phi(\lambda)$. The vectors

$$x_1,\ Ax_1,\ A^2x_1,\ \cdots,\ A^{\alpha-1}x_1, \tag{4}$$

are linearly independent, while

$$A^{\alpha}x_1 = a_1A^{\alpha-1}x_1 + a_2A^{\alpha-2}x_1 + \cdots + a_{\alpha}x_1. \tag{5}$$

The vectors (4) may be thought of as constituting a basis of a linear vector space Γ_1 of dimension α, which is an invariant sub-space with respect to A. If $\alpha=n$, the vectors

$$v_1 = x_1,\ v_2 = Ax_1,\ \cdots,\ v_n = A^{n-1}x_1,$$

constitute a basis of the entire vector space. Since from (4) and (5),

$$\begin{aligned} Av_i &= v_{i+1}, \qquad\qquad (i = 1, \cdots, n-1),\\ Av_n &= a_nv_1 + a_{n-1}v_2 + \cdots + a_1v_n, \end{aligned}$$

it follows from the lemma that if these vectors be taken as columns of the matrix T, then

$$N_1 = T^{-1}AT = \begin{pmatrix} 0 & 0 \cdots 0 & a_n \\ 1 & 0 \cdots 0 & a_{n-1} \\ 0 & 1 \cdots 0 & a_{n-2} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 & 0 \cdots 0 & a_2 \\ 0 & 0 \cdots 1 & a_1 \end{pmatrix}. \tag{6}$$

This is the rational canonical form of a matrix A whose reduced characteristic function in (3) is of degree n.

If, however, $\alpha<n$, we can find a vector y, say one of the unit vectors, which is not in the space Γ_1, *i.e.*, which is linearly independent of the vectors (4). We now establish the following:

LEMMA II. *If x_1 is a vector belonging to the reduced characteristic function $\phi(\lambda)$ of A so that the linear vector space Γ_1 generated by x_1 is of dimension $\alpha<n$, and if y is any vector not in the space Γ_1, we can find a vector x_2 of the form $x_2=y-h(A)x_1$ such that x_2 is not in the space Γ_1 and such that the invariant sub-space Γ_2 generated by x_2 has no vector (except the zero vector) in common with Γ_1.*

Proof. To prove this lemma, we form the chain with y as leader,

$$y,\ Ay,\ A^2y,\ \cdots,\ A^{\beta-1}y,\ A^{\beta}y,$$

where $A^{\beta}y$ is the first vector in the sequence which depends linearly on the vectors (4) and on the preceding vectors in the chain. We then have a relation of the form

$$A^\beta y = b_1 A^{\beta-1} y + b_2 A^{\beta-2} y + \cdots + b_\beta y + G(A) x_1,$$

or, if we define

$$F(\lambda) = \lambda^\beta - b_1 \lambda^{\beta-1} - \cdots - b_{\beta-1}\lambda - b_\beta,$$

the relation can be written

$$F(A) y = G(A) x_1, \tag{7}$$

and $F(\lambda)$ is the polynomial of lowest degree such that a relation of this type holds.

Suppose now that y belongs to the polynomial $\psi(\lambda)$ and that on dividing $\psi(\lambda)$ by $F(\lambda)$ we have

$$\psi(\lambda) = Q(\lambda)F(\lambda) + R(\lambda),$$

where $R(\lambda)=0$ or is of lower degree than $F(\lambda)$. Then from (7) we have, on operating with $Q(A)$,

$$Q(A)F(A)y = \psi(A)y - R(A)y = Q(A)G(A)x_1. \tag{8}$$

Since $\psi(A)y=0$, if $R(\lambda)\neq 0$, this contradicts our supposition that $F(\lambda)$ is the *polynomial of lowest degree* such that a relation of the type (7) holds. Hence, $R(\lambda)=0$ so that from (8) we have $Q(A)G(A)x_1=0$. From this it follows by Theorem II that $Q(\lambda)G(\lambda)$ is divisible by $\phi(\lambda)$, and therefore by $\psi(\lambda)=Q(\lambda)F(\lambda)$, so that $G(\lambda)$ is divisible by $F(\lambda)$. If now we write $G(\lambda)=F(\lambda)h(\lambda)$, we have from (7),

$$F(A)[y - h(A)x_1] = 0.$$

The vector $x_2=y-h(A)x_1$ then belongs to the polynomial $F(\lambda)$. Moreover, x_2 is clearly linearly independent of the vectors (4) since y itself is, and finally, in view of our assumption as to the minimum degree of $F(\lambda)$, the space Γ_2 generated by x_2 has no vector in common with Γ_1. The lemma is, therefore, established.

If then $\alpha<n$, we find a vector x_2 satisfying the conditions in the lemma. If the sub-space Γ_2 generated by x_2 is of dimension β, the vectors (4) and the vectors

$$x_2,\ Ax_2,\ A^2x_2,\ \cdots,\ A^{\beta-1}x_2, \tag{9}$$

are linearly independent.

If $\alpha+\beta=n$, the vectors (4) and (9) will form a basis of the entire n-dimensional vector space. That is, they may be taken as columns of a non-singular matrix T, and just as above,

$$B = T^{-1}AT = \begin{pmatrix} N_1 & \\ & N_2 \end{pmatrix},$$

where N_1 is the matrix in (6) with $n=\alpha$, and N_2 is a square matrix of order β of the same type as N_1 but with $b_\beta, b_{\beta-1}, \cdots, b_1$ as elements in the last column.

In this case, the matrix B is in rational canonical form, and the reduction is complete.

If, however, $\alpha+\beta<n$, we can find a vector y, linearly independent of the vectors (4) and (9). By Lemma II, for a suitable choice of the polynomial $h(\lambda)$, we can determine a vector x_3 of the form

$$x_3 = y - h(A)x_1$$

such that the invariant sub-space Γ_3 generated by x_3 has no vector in common with Γ_1, although it may have vectors in common with Γ_2. This vector x_3 is linearly independent of the vectors (4) and (9). For if

$$x_3 = G_1(A)x_1 + G_2(A)x_2,$$

then

$$y = [h(A) + G_1(A)]x_1 + G_2(A)x_2,$$

and this contradicts our assumption as to the choice of y.

We then form the chain with x_3 as leader,

$$x_3,\ Ax_3,\ \cdots,\ A^{\gamma-1}x_3, \tag{10}$$

where we suppose that $A^{\gamma}x_3$ is the first vector in the chain that is linearly dependent on the vectors (4) and (9) and on the preceding vectors in the chain.

If $\alpha+\beta+\gamma=n$, the vectors (4), (9), and (10) form a basis of the n-dimensional vector space. In the contrary case, we can find a vector x_4 linearly independent of these vectors and such that the invariant sub-space Γ_4 generated by x_4 has no vector in common with Γ_1, although it may have vectors in common with Γ_2 and Γ_3. We then form the chain with x_4 as leader,

$$x_4,\ Ax_4,\ \cdots,\ A^{\delta-1}x_4, \tag{11}$$

where $A^{\delta}x_4$ is the first vector in the chain which depends linearly on the vectors (4), (9), and (10) and on the preceding vectors in the chain.

If $\alpha+\beta+\gamma+\delta<n$, the process may be continued, until a set of n linearly independent vectors is obtained. The procedure when such a set has been found will be illustrated for the case $\alpha+\beta+\gamma+\delta=n$. In this case we may take as basis of our vector space, *i.e.*, as columns of the matrix T, the vectors (4), (9), (10), and (11). If we write

$$\begin{aligned} v_i &= A^{i-1}x_1, && (i = 1, \cdots, \alpha),\\ v_{\alpha+i} &= A^{i-1}x_2, && (i = 1, \cdots, \beta),\\ v_{\alpha+\beta+i} &= A^{i-1}x_3, && (i = 1, \cdots, \gamma),\\ v_{\alpha+\beta+\gamma+i} &= A^{i-1}x_4, && (i = 1, \cdots, \delta), \end{aligned}$$

then

$$\begin{aligned} Av_i &= v_{i+1}, && (i = 1, \cdots, \alpha - 1),\\ Av_\alpha &= a_\alpha v_1 + a_{\alpha-1}v_2 + \cdots + a_1 v_\alpha, \end{aligned}$$

while for $j>\alpha$,

$$Av_j = [v_{\alpha+1}, \cdots, v_{\alpha+\beta+\gamma+\delta}],$$

where the brackets denote some linear combination of the vectors in the brackets. The matrix $B=T^{-1}AT$ will then be of the form

$$B = T^{-1}AT = \begin{pmatrix} N_1 & \\ & A_1 \end{pmatrix},$$

where N_1 is the matrix in (6) with $n=\alpha$.

Now let $\phi_1(\lambda)$ be the reduced characteristic function of A_1. Since $\phi(B)=0$, clearly $\phi(A_1)=0$, so that by the remark at the end of section 2, $\phi_1(\lambda)$ is a factor of $\phi(\lambda)$. We can now apply to A_1 the same process that we applied to A and obtain

$$\begin{pmatrix} N_1 & & \\ & N_2 & \\ & & A_2 \end{pmatrix},$$

where N_2 is of the same form as N_1 in (6) but with the elements in the last column the coefficients of $\phi_1(\lambda)$.

The process can be continued until finally A is reduced to the form

$$\begin{pmatrix} N_1 & & & \\ & N_2 & & \\ & & \ddots & \\ & & & N_s \end{pmatrix}.$$

This is the rational canonical form of the matrix A. Clearly the reduced characteristic function of each N_i is a divisor of the reduced characteristic function of N_{i-1}, $(i=2, \cdots, s)$. These functions will be found to be precisely the *invariant factors* of the matrix $A-\lambda I$.

Illustration 1. Let A be the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A + 2I,$$

so that the reduced characteristic function $\phi(\lambda)$ of A is $\lambda^2-\lambda-2$. By inspection the vector $x_1=(1, 0, 0)$ is seen to belong to the polynomial $\phi(\lambda)$. Forming the chain

$$x_1, \ Ax_1 = (0, 1, 1), \ A^2x_1 = (2, 1, 1) = (A + 2)x_1,$$

we find that x_1 generates an invariant sub-space Γ_1 of dimension 2. The vector $y = (0, 0, 1)$ is easily seen to be linearly independent of the two vectors x_1, Ax_1. On forming the chain with y as leader we find that

$$Ay = (1, 1, 0) = Ax_1 + x_1 - y,$$

i.e.,

$$(A + 1)y = (A + 1)x_1,$$

or

$$(A + 1)(y - x_1) = 0.$$

The vector $v_3 = y - x_1$ then belongs to the polynomial $\lambda + 1$, and generates an invariant sub-space Γ_2 of dimension 1, which has no vector in common with the sub-space Γ_1. If then we choose the 3 vectors

$$\begin{aligned} v_1 &= x_1 = (1, 0, 0), \\ v_2 &= Ax_1 = (0, 1, 1), \\ v_3 &= y - x_1 = (-1, 0, 1), \end{aligned}$$

as basis of the 3-dimensional vector space, or as columns of the non-singular matrix T, then $T^{-1}AT$ is easily seen to be of the form

$$\begin{pmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which is the natural or rational canonical form of A.

Illustration 2. Let A be the matrix

$$A = \begin{pmatrix} -2 & -7 & 22 & 33 \\ 1 & 3 & -11 & -22 \\ 0 & 0 & -2 & -7 \\ 0 & 0 & 1 & 3 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} -3 & -7 & 22 & 33 \\ 1 & 2 & -11 & -22 \\ 0 & 0 & -3 & -7 \\ 0 & 0 & 1 & 2 \end{pmatrix} = A - I,$$

so that the reduced characteristic function of A is $\phi(\lambda) = \lambda^2 - \lambda + 1$. If we choose the vector $x = (1, 0, 0, 0)$, then

$$\begin{aligned} Ax &= (-2, 1, 0, 0), \\ A^2x &= (-3, 1, 0, 0) = Ax - x, \end{aligned}$$

so that x belongs to the polynomial $\phi(\lambda)$. That is, x generates an invariant sub-space Γ_1 of dimension 2. The vector $y = (0, 0, 1, 0)$ is seen by inspection not to belong to the space Γ_1. Then $Ay = (22, -11, -2, 1)$, while $A^2y = (22, -11, -3, 1)$

$=Ay-y$. This vector y then generates an invariant sub-space Γ_2 of dimension 2 with no vector in common with Γ_1. If we choose the four vectors

$$v_1 = x, \qquad v_2 = Ax, \qquad v_3 = y, \qquad v_4 = Ay,$$

as the basis of our 4-dimensional vector space, or as columns of the non-singular matrix T, then since

$$Av_1 = v_2, \qquad Av_2 = - v_1 + v_2, \qquad Av_3 = v_4, \qquad Av_4 = - v_3 + v_4,$$

it is clear that the matrix $T^{-1}AT$ is of the form

$$T^{-1}AT = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

which is the rational canonical form.

5. The Jordan canonical form of A. We now proceed to the derivation of the Jordan canonical form of the matrix A. For this purpose we may suppose that A is non-derogatory, *i.e.*, that the reduced characteristic function of A is the same as the characteristic function. In the contrary case, we can first reduce A to the rational canonical form N in (11) in which each block N_i is non-derogatory. We need then only reduce each N_i separately.

We therefore restrict our attention to the case where the reduced characteristic function $\phi(\lambda)$ is of degree n. We suppose now that $\mathcal{J}$ is an algebraically closed field, for example, the field of all complex numbers, and that $\phi(\lambda)$, when resolved into powers of distinct linear factors, is as follows:

$$\begin{aligned} \phi(\lambda) &= (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_s)^{q_s} \\ &= (\lambda - \lambda_1)^{q_1}P_1(\lambda) = (\lambda - \lambda_2)^{q_2}P_2(\lambda) = \cdots = (\lambda - \lambda_s)^{q_s}P_s(\lambda). \end{aligned} \tag{12}$$

Just as in section 3, we can find a vector z_j in $\mathcal{J}$ belonging to the polynomial $(\lambda-\lambda_j)^{q_j}$. Let us then write

$$v_i^{(j)} = (A - \lambda_j)^{q_j-i} z_j, \qquad (i = 1, \cdots, q_j;\ j = 1, \cdots, s). \tag{13}$$

We have here $\Sigma q_j = n$ vectors. These vectors are linearly independent. For if they were linearly dependent, there would exist a relation of the form

$$\sum_{j=1}^{s} g_j(A)z_j = 0, \tag{14}$$

where $g_j(\lambda)$ is a polynomial of degree less than q_j. Since $P_i(\lambda)$ is divisible by $(\lambda-\lambda_j)^{q_j}$ for $i\neq j$, we have, on operating on (14) with $P_i(A)$,

$$P_i(A)g_i(A)z_i = 0, \qquad (i = 1, \cdots, s).$$

Hence, $P_i(\lambda)g_i(\lambda)$ is divisible by $(\lambda-\lambda_i)^{q_i}$. But this is impossible since the first factor is relatively prime to, and the second factor is of lower degree than, $(\lambda-\lambda_i)^{q_i}$. Since

$$Av_i^{(j)} = [(A-\lambda_j)+\lambda_j]v_i^{(j)} = (A-\lambda_j)^{q_j-i+1}z_j + \lambda_j v_i^{(j)},$$

we have

$$Av_i^{(j)} = v_{i-1}^{(j)} + \lambda_j v_i^{(j)}, \qquad (i = 2, \cdots, q_j;\ j = 1, 2, \cdots, s);$$

while

$$Av_1^{(j)} = (A-\lambda_j)^{q_j}z_j + \lambda_j v_1^{(j)} = \lambda_j v_1^{(j)}, \qquad (j = 1, \cdots, s).$$

Hence it is clear that if we take these n vectors as basis of our n-dimensional space, or if we take them as columns of the non-singular matrix T, the matrix $T^{-1}AT$ will assume the form

$$J = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J_s \end{pmatrix},$$

where J_i is a q_i-square matrix of the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 \cdots 0 & 0 \\ 0 & \lambda_i & 1 \cdots 0 & 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ 0 & 0 & 0 \cdots \lambda_i & 1 \\ 0 & 0 & 0 \cdots 0 & \lambda_i \end{pmatrix}.$$

This is the so-called Jordan or *classical* canonical form of A.

Illustration 3. Reduce the following matrix to Jordan canonical form:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{pmatrix}.$$

The reduced characteristic function of A is found to be $\lambda^3-3\lambda^2+3\lambda-1 = (\lambda-1)^3$, so that A is non-derogatory. It will therefore be unnecessary to reduce A first to rational canonical form. By inspection the vector $x=(1, 0, 0)$ is found to belong to the polynomial $(\lambda-1)^3$. By computation we find that

$$A - I = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{pmatrix}, \qquad (A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 3 & -3 & 3 \\ 3 & -3 & 3 \end{pmatrix}, \qquad (A-I)^3 = 0,$$

so that $(A-I)x=(1, 2, 1)$, $(A-I)^2x=(0, 3, 3)$. If we take the three linearly independent vectors $(0, 3, 3)$, $(1, 2, 1)$, $(1, 0, 0)$ as the first, second, and third columns of the matrix T, then $T^{-1}AT$ will be in Jordan canonical form. In fact,

$$T^{-1}AT = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

6. The Jacobson canonical form. The Jordan canonical form was derived on the supposition that $\mathcal{J}$ was an algebraically closed field. In case $\mathcal{J}$ is not algebraically closed, we cannot in general arrive at the Jordan canonical form through a transformation of the type $T^{-1}AT$, where the elements of T lie in $\mathcal{J}$. There is a canonical form, however, to which we can always arrive by transformations within the field $\mathcal{J}$. This is the Jacobson canonical form, due to Nathan Jacobson [**13**], which may be looked upon as a generalization of the Jordan form, and which reduces to the latter in case $\mathcal{J}$ is algebraically closed.

As in the preceding section, we start with a matrix which is non-derogatory, *i.e.*, a matrix whose reduced characteristic function is the same as its characteristic function. The matrix N in (6) is a matrix of that type. Let us suppose that $\phi(\lambda)$, when resolved into powers of irreducible factors in $\mathcal{J}$, is

$$\phi(\lambda) = p_1^{q_1} p_2^{q_2} \cdots p_s^{q_s}, \tag{15}$$

where p_i is of degree d_i. Here $\Sigma d_i q_i = n$. As in section 3, we can find vectors x_i with elements in $\mathcal{J}$ which belong to the polynomials $p_i^{q_i}$, $(i=1, \cdots, s)$. Each vector x_i then generates an invariant sub-space Γ_i with basis

$$x_i, Ax_i, \cdots, A^{d_iq_i-1}x_i,$$

where each Γ_i has dimension d_iq_i.

It is easy to show that no two of the spaces Γ_i, Γ_j, $(i, j=1, \cdots, s)$, have a vector in common, and that the vectors

$$x_1, Ax_1, \cdots, A^{d_1q_1-1}x_1; \cdots; x_s, Ax_s, \cdots, A^{d_sq_s-1}x_s,$$

are linearly independent. With these vectors as columns of the matrix T, it is clear that $T^{-1}AT$ assumes the form

$$T^{-1}AT = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_s \end{pmatrix},$$

where each M_i is a square matrix of order d_iq_i of the type (6), and where now the elements in the last column are the coefficients in the expansion of

$$[p_i(\lambda)]^{q_i} = \lambda^{d_iq_i} - k_1\lambda^{d_iq_i-1} - \cdots. \tag{16}$$

That is, each M_i is a non-derogatory matrix whose reduced characteristic function $[p_i(\lambda)]^{q_i}$ is a power of an irreducible polynomial.

We can now proceed at once to the derivation of the Jacobson canonical form. Indeed, we need only reduce each M_i to that form.

Consider then a non-derogatory square matrix M of order n whose reduced characteristic function is a power of an irreducible polynomial in $\mathfrak{J}$. That is, we suppose that $\phi(\lambda)=[p(\lambda)]^q$, where

$$p(\lambda) = \lambda^d - \pi_1\lambda^{d-1} - \pi_2\lambda^{d-2} - \cdots - \pi_d. \tag{17}$$

Let z be a vector in $\mathfrak{J}$ belonging to the polynomial $[p(\lambda)]^q$, and consider the chain of vectors

$$v_{kd-i} = [p(A)]^{q-k}A^{d-1-i}z, \qquad (k = 1, \cdots, q;\ i = 0, \cdots, d-1).$$

These vectors are clearly linearly independent. We now inquire as to the form of the matrix $T^{-1}MT$ if these vectors be taken as columns of T.

From the manner in which the vectors were derived, we note first of all that for i not divisible by d, we have

$$Av_i = v_{i+1}.$$

Further, since

$$v_{kd} = [p(A)]^{q-k}A^{d-1}z,$$

and since

$$A^dz = \sum_1^d \pi_iA^{d-i}z + p(A)z,$$

we have

$$\begin{aligned} Av_{kd} &= \sum \pi_i[p(A)]^{q-k}A^{d-i}z + [p(A)]^{q-k+1}z \\ &= \sum \pi_iv_{kd-i+1} + v_{(k-2)d+1}, \qquad (k>1); \end{aligned}$$

and

$$Av_d = \sum \pi_iv_{d-i+1}.$$

Hence, $T^{-1}MT$ is a matrix of the form

$$\begin{pmatrix} N & K & \cdot & \cdot & \cdot \\ \cdot & N & K & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & N & K \\ \cdot & \cdot & \cdot & \cdot & N \end{pmatrix}, \tag{18}$$

containing q N's in the main diagonal and K's in the super diagonal. Here N and K are d-square matrices of the form

$$N = \begin{pmatrix} 0 & 0 \cdots 0 & \pi_d \\ 1 & 0 \cdots 0 & \pi_{d-1} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 & 0 \cdots 1 & \pi_1 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 \cdots 0 & 1 \\ 0 & 0 \cdots 0 & 0 \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 & 0 \cdots 0 & 0 \end{pmatrix}.$$

That is, N is of the form in (6) with the coefficients of $p(\lambda)$ in the last column, while K consists entirely of zeros except for a 1 in the upper right-hand corner. The matrix in (18) is the Jacobson canonical form of a non-derogatory matrix whose reduced characteristic function is the power of an irreducible polynomial in $\mathcal{J}$.

In the particular case in which $\mathcal{J}$ is an algebraically closed field, the polynomial $\phi(\lambda)$ in (15) resolves completely into powers of linear factors so that each $d_i=1$. In this case, N and K reduce to the 1 by 1 matrices

$$N = (\pi_1), \qquad K = (1),$$

so that the Jacobson canonical form in (18) is simply the Jordan canonical form of a matrix with a single elementary divisor $(\lambda-\pi_1)^q$.

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ON THE TECHNIQUE OF GENERALIZATION*

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1. Introduction. Generalization is surely an important activity for mathematicians. Yet the ways and means of securing generalizations have never been systematically considered in mathematical literature. What accounts for this apparently strange lacuna? There are various reasons, among which, it seems, is the prevalence of a feeling, necessarily vague, that the analysis of the process of generalization, if undertaken at all, should be relegated outside the domain of professional mathematics—a feeling that such a study should belong, let us say, to *Psychomathematics*, coining a word whose meaning is readily inferred. By Psychomathematics, I understand, namely, that union of mathematics and psychology—using the latter term in a broad, not technical sense—whose function it is to explain how mathematical ideas arise, and to formulate heuristically helpful principles in mathematical exploration.

Now, to be sure, mathematicians indulge in Psychomathematics to a certain extent, just as artists must, to some extent, become philosophers and inquire what it is that the essence of beauty consists of—first in general, and then in their special province. But for fear of being misunderstood, I must say that the term philosophy, as used here, signifies not speculative or merely expository theory, but a system of principles fundamental for advanced strategy. In chess, for example, one such principle is that it is highly important to have a strong center; another principle, that, other things being equal, the move made should, as much as possible, enhance the mobility of one's forces, and correspondingly reduce the mobility of those of the opponent. From this latter principle, we have, for instance, the corollary that, except for particular objectives, pawns are not to be moved out of their original position. No chess player can attain high rank without conscious and intimate possession of such operative principles of strategy. A pianist, likewise, if of high ideals, would have operative principles with reference to distinctness, resonance, and beauty of tone, with reference to suppleness and power of musculature, with reference to organic unity and nobility of expression.

The technique of generalization belongs, as I said, to Psychomathematics, the field comprehensively concerned with the strategy of mathematical invention. And although inventive mathematicians are necessarily keen for new ideas, they virtually never pursue Psychomathematics systematically and with due appreciation of its potentialities. So that not infrequently, even among distinguished mathematicians, we may notice that a certain acquired technique or strategy is exercised just in its initial stages, without its being consciously and steadily improved. The technique of an eminent mathematician will, of course, be of high competitive order. But my interest here is to venture a glimpse into potentialities of principles for *improving* one's technique, be one's present powers high or low.

* Presented to the Mathematical Association of America at Columbus, Ohio, December 29, 1939.

Occasionally, of course, there will be a reference to Psychomathematics in the literature, as when E. H. Moore speaks at the beginning of his *General Analysis* of a heuristic principle of generalization; namely, the principle that the existence of analogies beckons generalization, points to the existence of a general unifying theory exhibiting these analogies as particular cases. And many a piece of research owes its origin to the operative stimulus of this heuristic, strategic principle.

Here is another instance. A distinguished mathematician, greatly interested in a certain problem,—which happened to concern equations having only real roots as related to certain continued fractions,—when he first saw a paper giving its solution, declared with a twinkle in his eye, "How remarkable it is that a difficult problem is rendered easy if you hit upon a good way of generalizing it!" Now the twinkle in his eye disclosed that the principle of generalization to which he was alluding was still new to him, very much as the thought of industrial democracy crossed the mind of many a democrat with a strange novelty in the days of World War I.

The remarks I have made were intended to indicate that only incidental attention has heretofore been given by mathematicians to such matters as the technique of generalization, the very terminology, perhaps, sounding strange to some ears. And I have also indicated a reason for this scantiness of attention. I may add, in this connection, that a vague feeling seems to pervade some minds that the creative process must not be analyzed or inspected for fear that the Muses will take a walk. But such a feeling would find it a task to justify itself. Indeed, it is a timid feeling, afraid not merely of what it professedly fears, but somewhat of itself, and it may be safely assumed that wherever such a feeling is entertained, there is misconception and contradiction. Witness the artist in one breath excoriating all who would dare discuss the nature of beauty, and in the next, unburdening himself of his own philosophy of beauty. Those who warn against inspection and analysis talk as if just around the corner, if we have only the temerity to look, we may peer into the very heart of the Absolute, the Soul of Beauty, the quintessence of creativeness. It seems to me that our reverence for the Absolute is greater if we are not afraid that the humble analysis which we poor mortals can bring forth at the present stage of civilization, with the heritage of racial limitation as it is, will get us dangerously close to the Heart of the Universe. In fact, the problem of securing useful principles in the strategy of generalization—or, indeed, in mathematical strategy in general—is no more profound *in genre* than many another problem naturally solved in the course of devoted industry.

Let me clarify my meaning through an illustration. A high school teacher of trigonometry may have solved numerous text-book trigonometric identities, all the time in fear that a new one might overtax his ingenuity. It may daunt him even to face the idea of searching for the secret, *i.e.*, the technique of solving all such identities. Nevertheless, as we know, this secret is near at hand; it is only his fears and self-imposed limitations which prevent him from discerning it.

After consideration, it must be said that essentially similar to this is the problem of finding the secrets of an advanced technique of generalization. Indeed, with the present-day equipment of the fraternity of mathematicians, similar too, is the problem of finding the secrets of an advanced technique of mathematical research in general, as research is ordinarily conceived, of which general technique, the technique of generalization is a part. And when I speak of an advanced technique, I mean one of manifold and decisive utility for student and professional mathematician. To forestall misunderstanding, I disclaim having myself reached a formulation of such an advanced technique. My purpose here is to call attention to the problem, and to indicate first steps. A more elaborate treatment will appear, among other things, in a forthcoming book.

Before I begin to discuss these first steps, I wish to add another remark. In the matter of improving our technique, it is important to bear in mind that the mere knowledge of a principle is of slight value; what counts is the degree to which the principle becomes consciously and intimately operative in our mind. A centrally operative principle is like another eye in our explorative faculties. Such eyes, no less real than physical eyes, give us our individual view of the world and determine our evaluations and the course of our action.

For a mathematician, such eyes define his major interests and the character of his operative strategy. And strategically speaking, the difference is vast between a more or less subconscious operation of a heuristic principle and a conscious and growing understanding of its nature and potentialities. The effectiveness of a living principle grows immeasurably through conscious possession of it, through progressively more refined discriminations bearing on its true significance, through zealous search for varied applications.

I turn now to the consideration of the technique of generalization in itself. We have to distinguish the generalization of concepts, of problems, of theorems, and of processes.

We take up first the question of generalizing concepts.

2. Illustration of the generalization of a concept. And I begin with an illustration, namely that of the concept of function. This notion gained clarity and refinement of meaning with advancing mathematical experience and reflection, and it was not until the latter part of the nineteenth century that the essence of functionality was recognized and the notion attained its present generality. The earlier idea of function was that of a relationship of one number to another or to a finite set of numbers; as, for example, $y=x^2$, $y=\log x$, $w=x^2-yz$. Moreover, earlier mathematicians seemed tacitly and commonly to suppose that expressibility is a necessary requisite for a functional relationship; that is to say, a functional relationship must be statable in terms of "accredited" mathematical entities and processes. Just what meaning should be attached to the term *accredited* seems to have been only vaguely grasped. Furthermore, whether additional attributes, such as continuity, were to be demanded, was not clearly settled. It was such vagueness which ranged celebrated mathematicians on op-

posite sides of disputed questions—for example, John Bernoulli and d'Alembert in the controversy on the problem of the vibrating string. And on account of this vagueness, both sides, though tenaciously and heatedly supported, were equally indefensible. Eventually the question was faced: What is the essence of functionality? And the answer was: Not expressibility in terms of certain mathematical entities by means of certain mathematical processes; not that the two mathematical entities involved in the functional relationship are the one a number and the other a number or a set of numbers; and not a complex of restrictive properties including continuity. The essential nature of functionality, as it emerged through experience rather than directed reflection, was *correspondence*. And if we abstract this item of *correspondence* from the compound of attributes which mathematicians of an earlier day vaguely attached to the notion of function, we secure the modern notion of function which formulates clearly the genuine essence of functionality. The conception of function thus arrived at is as follows:

A function is a mathematical system consisting of a set A (the range of the independent variable) and a set B (the range of the dependent variable) and a correspondence which mates with every element of A some element of B.

In this definition, we understand by *mathematical system* nothing other than a finite set of mathematical entities (sets, functions, relations, operations, and so on) having specified properties. Thus a group is a system which consists of a set and an operation (generalized multiplication) subject to certain properties, customarily termed *postulates*. Likewise, a field, a linear order, an abstract euclidean geometry, a topological space are systems. Utilizing this generalized notion of mathematical system, we see that function is something of the same sort as group, field, linear order, topological space, and so forth.

The definition of function just given, as our discussion has shown, formulates the essence of functionality in its pure nakedness; it thus achieves the highest generality of conception while preserving the heart of the idea of function. Since the definition does complete justice, as we may say, to the kernel of the intuitive idea of functionality, it would seem difficult, indeed, to extend it legitimately any further.

According to this modern definition of function, the weight of an elephant is an example of an honest-to-goodness function, the range A of the independent variable being the totality of elephants, and the range B of the dependent variable the set of real numbers, or positive real numbers, if you wish. Similarly, the area of a polygon is a function, the variable polygon the independent variable, the corresponding area the dependent variable. Functions of curves, usually called *functionals*, are functions. A sequence $\{a_n\}$, which mates a number a_n, real or complex, with every positive integer n, is a function. So is a point transformation, or a functional transformation. Relations and operations also are functions. Tensors are functions, and if those who are somewhat baffled by tensors will comprehend their functionality they will have a good basis for a full understanding.

3. Analysis of the technique of generalizing concepts. By examining our procedure for generalizing the notion of function, we may note various items in the technique of generalizing concepts. A generalization of a concept or property may be conceived as consisting of an *enlargement of the class* of objects to which the concept is to be applicable. Thus sequences, functionals, relations, transformations, are functions in our generalized, but not in the primitive sense. From another point of view, we may think of our generalization of function as consisting of discarding such properties as expressibility, or real number character, or continuity, and demanding nothing more than correspondence. In other words, from this second point of view, the generalization of function consists of an *abstraction*. If we take the term generalization in the narrower sense of *extension of class*—and on its face the term suggests this meaning—we may say that generalization and abstraction are substantially equivalent processes. Another equivalent way of conceiving of generalization, as the case of function showed, is as a *search for essence*. We formulate the observations just made in the following principle, which for convenience of reference, I shall call the *Principle of Equivalence*. It runs as follows:

The *generalization* (in the sense of extension of meaning to a larger class), *abstraction* (in the sense of passing from property to implied property), and *isolation of the essence* of a concept or property are substantially equivalent processes.

I will add some remarks with the purpose of elucidating this principle of equivalence.

First, it is to be understood that the essence of a property is not absolute, uniquely determined by the property, but is relative to the point of view. We shall illustrate this relativity of essence by considering the generalization of continuity.

Continuity, for a real one-valued function, may be defined in different ways, among which are the following three: $f(x)$ is continuous at ξ if (1) [*sequential* definition] for every sequence ξ_n converging to ξ, $f(\xi_n)$ converges; (2) [ϵ, δ definition] for every positive ϵ, there exists a positive δ such that $|x-\xi|<\delta$ implies that $|f(x)-f(\xi)|<\epsilon$; and (3) [*neighborhood* definition] for every positive ϵ, there exists a neighborhood N of ξ such that, for every ξ of N, $|f(x)-f(\xi)|<\epsilon$.

In the first or sequential definition, (apart from the given function $f(x)$) there are no concepts mentioned except that of *convergent sequence*. This definition, therefore, lends itself to extension to functions defined on any set A whatsoever, instead of the linear continuum, if only A is such that "convergent sequence of elements of A" has meaning. More explicitly: Let A be any set whatsoever (range of the independent variable, according to our modern conception of function); and $f(e)$, where e is a variable element of A, any real function on A, *i.e.*, any correspondence associating a real number $f(e)$ with every element of A. Furthermore, let the statement "e_n is convergent" have meaning if e_n is a sequence of elements of A. It is then apparent that the definition of continuity for real functions of a real variable extends directly to real functions defined on such a set A. And if, in addition, several simple properties are demanded (postu-

lated) for convergent sequences of A—for example, that a sub-sequence of a convergent sequence is convergent—we secure, among other conceptions, the conception of the so-called *L-space* of Fréchet, a mathematical system, as indicated, consisting of a set A for which *convergent sequence* has meaning and which complies with certain elementary demands (postulates), like the one just mentioned.

And just as the sequential definition leads naturally to the notion of Fréchet's L-space and the generalization of continuity to real functions defined on such a space, so does the ϵ, δ definition lead to so-called *metric space* and the generalization of continuity to such a space. Furthermore, the ϵ, δ definition leads to another generalization, obtained by writing the inequality $|f(x)-f(\xi)|<\epsilon$ as two inequalities, $-\epsilon<f(x)-f(\xi)<\epsilon$, and demanding just one of these instead of both. We thus gain the notion of semi-continuity—upper semi-continuity or lower semi-continuity according to whether we demand the inequality $f(x)-f(\xi)<\epsilon$, or the inequality $-\epsilon<f(x)-f(\xi)$.

The third or neighborhood definition is a very slight modification of the second, but suggests generalizations not as naturally suggested by the second. For, the third definition contains the phrase "for every x of N," and for one versed in mathematical strategy, such a phrase immediately suggests a generalization obtained by replacing the full demand "for every" by an approximate demand. And such an approximate demand may be formulated by allowing a set of exceptions, for example, a denumerable set, or an exhaustible set, or a set of measure 0, or a set of metric density 0. In this way, we gain, among others, the notions of d-continuity, e-continuity ("d" referring to denumerability, "e" to exhaustibility), and metric continuity (called "approximate continuity" by Denjoy). Again, in a way analogous to that for the sequential and the ϵ, δ definitions, the neighborhood definition leads to the generalization of continuity to real functions defined on a *topological* space.

These illustrations of different ways of generalizing continuity show, then, that the essence of a concept is relative, depending on the manner of its formulation.

The principle of equivalence states that generalization, abstraction, and the isolation of essence are substantially equivalent processes. But a conversion from one to another of these may offer heuristic advantage, just as integration by parts may offer heuristic advantage. Integration by parts changes the form of a problem, substitutes, for the old, a new problem which may be easier to handle. Likewise, to change our objective from the extension of class to the abstraction of a property or to the search for essence is a change of form. And a mere change of form, no matter how slight, may help, and frequently indicates the solution. We have seen in the example of continuity, how slight changes of form give new directions of generalization. If mathematicians had had something like the principle of equivalence as an intimate, operative tool—an explorative eye functioning organically in research—it is inconceivable that the modern conception of function would have been so long emerging. Such an ex-

plorative eye would at once have recognized that a concept so fundamental as function should not be a compound of expressibility, reality, continuity, and what not else, together with correspondence. Even ordinary language wants basal words to mean something simple, not compound. And this should be true *a fortiori* of basal mathematical terms. The explorative, generalizing eye would at once have been challenged by this vague, compound sense of function, and stimulated by such discontent, one assuredly comes to a solution. Expert mathematicians can greatly improve their strategy by organically acquiring such explorative eyes, just as expert pianists can improve their playing by learning new ways of incorporating power and flexibility in their muscles. Life is change—the depth and pace of life depend on the character of the change. A pianist may evince life by taking up a new sonata without noticeably improving his technique (which word I do not use as divorced from expression). Likewise, a mathematician may write another monograph out of a rather fixed technique, however competitively high it may be. But the acquisition of a new explorative eye is something of a different *genre*. It is like ascending to a new plane.

3.1. Change of form. From the example of the generalization of continuity, it appears that change of form is an important phase of the technique of generalization. Change of form is, indeed, of fundamental importance in mathematics—of an importance on a par with that of generalization itself. To be expert in generalization, one must be expert in the process of change of form.

A good speller is not one who needs to be born a good speller. If one learns when to go to the dictionary, one has the means of becoming a good speller. So it is with reference to becoming an expert in change of form. It does not require a new kind of capacity to be an expert form-changer. If we hold it important to have such capacity, we can readily find means of acquiring it. It would be sufficient, whether we are working in a narrow or wide field, to take the corresponding mathematical literature as raw material, note the occurrence of different types of change of form—and even if the field is large, most likely these types will be classifiable in small compass—and make these types intimately operative in our work. If we examine mathematical literature with reference to the strategy intimately exercised by mathematicians, we shall find that even eminent experts are not always intimate with simple changes of form in their own specialty, the printed analog of the twinkle of delight in the eye betraying that such changes are not infrequently at the frontier rather than the center of their technique—just as the playing of concert artists reveals that such fundamental items of technique as being deeply relaxed before each stroke, are not always guaranteed possessions of the performer.

I shall mention a number of important changes of form which, because of their basal character, may be useful in any branch of mathematics:

(a) The change *from the intuitive to the rigorous*, which constitutes the passage from vague knowledge, or inspired feeling, to formulated science—as when speed, curvature and other concepts, experimentally and vaguely felt as quantities, are formulated as derivatives; when linear order is formulated in terms of

rank and transitivity; cardinal number for finite or infinite sets by means of correspondence; measure in terms of outer and inner refined approximations.

(b) *Decomposition*, the breaking up of a property or condition α into two or more properties or conditions, which together are equivalent to α. This is exemplified in the generalization of continuity to semi-continuity, which was accomplished by breaking up $|f(x)-f(\xi)|<\epsilon$ into the two conditions $-\epsilon<f(x)-f(\xi)<\epsilon$. Decomposition, as a change of form, may be regarded as the process of dividing a set into sub-sets. It plays an important rôle in the theory of sets and functions. Such results as the theorem that the algebraic numbers are denumerable, that a set of isolated points is denumerable, that there are as many points in the straight line as in the plane, are proved basally by means of the technique of decomposition. And in many modern and apparently more technical considerations, as for example, those leading to the comprehensive theorems of Denjoy on derivatives and integrals, the technique of decomposition comes in again and again as an important part of the reasoning. It is apparent, too, to one who looks into the matter, that even experts manifest inadequate intimacy with the phases of decomposition in their own field.

(c) *From description to structure, representation*, as when we pass from the qualification of a number or function by means of an equation to an explicit formula. Other examples are the passage from the definition of a closed set, as one containing its limit points, to its representation as the complement of the sum of non-overlapping intervals. Similar passages are effected for a perfect set, a set measurable Jordan or Lebesgue, a normally ordered set, a complex analytic function, a group, an algebra. The virtue of this change of form lies in its substitution of pictures for logical conditions, and a picture offers integrated intimacy, and is invaluable in gaining new knowledge, in discerning the new in the intimate old. We hold effortlessly in our memory thousands of items after making a journey, but logical conditions are, and remain, foreign to the mind. If a property lacks picturesqueness it is comparatively meager in new suggestions, and it may be that only by dint of laborious logic can we derive implications from it—a laboriousness aggravated if the technique of weaving mathematical thoughts is not systematically explored and cultivated. Much of the apparent intricacy of mathematical literature can be explained as due to navigation *via* logical processes lacking systematic psychomathematical motivation, and concepts inadequately intimate, inadequately portrayed in pictures. To this strategy of passing from the descriptive to the structural belongs also the idea of positivization of definition, as when an infinite set is redefined as a set which may be biuniquely mated with a proper sub-set of it. Also here belongs the idea, which G. Hessenberg emphasized, of avoiding the *reductio ad absurdum* as much as possible, a minor principle of strategy frequently violated by mathematicians for no good reason.

I mention further, without exemplifying, such important changes of form of wide application as the passage from the infinite to the finite; from the absolute to the relative; and from the exact to the approximate (which was illus-

trated in one of the ways of generalizing continuity).

Most of the changes of form are of application also *vice versa.*

I have discussed in some detail generalizations of the notions of function and of continuity. On account of time limitation, I must now proceed more briefly.

Just as the case method in law derives general principles from the study of factual litigation, so can principles for the technique of generalization be derived from the study of particular generalizations. We have indicated such derivation of principles from the particular generalizations of function and continuity. A study of other important examples will reveal the operation of other principles. But, in fact, even the principle of equivalence alone, and the changes of form I mentioned, together with some others, constitute a nucleus for the technique of generalization more comprehensive in application than one might suppose before examination. This nucleus, if appropriately interpreted and refined, enables one to see the essential sameness of thought process in various mathematical achievements, which to the eye, not so equipped, would appear as discoveries of different motivation. With such a nucleus, the mere passage from positive integral exponent to more general exponents, if properly understood in the light of technique, removes the mystery of the passage from positive integers to integers, from integers to rational numbers, from rationals to irrationals—which passage no longer appears as just *sui generis*—to complex numbers, to quaternions and linear associative algebras. It makes it much less likely that students will find it difficult to be at home in postulate systems. It shows that if mathematicians had consciously possessed some such technique as intimate, consciously operative principle, they could not have felt mystified if asked to consider for the first time such problems as summing divergent series, or defining factorial $\frac{1}{2}$, or fractional differentiation. And today, too, with such technique, we should be much less mystified when confronted by various current problems.

By which, of course, I do not mean to imply that I advocate curtailment of originality and invention. Quite the contrary. A surer and more facile technique, if properly utilized, means not immersion in technique but expression of a higher order, a potential elevation of the plane of discovery.

I shall now say a few words about the generalization of problems.

4. Generalization of problems. Obviously, when a concept is generalized, a good many problems relating to the concept, in its primitive sense, carry over to the generalized concept. And new discriminations turn up—some staring one in the face, so to speak—which suggest additional problems.

I will add only this on the generalization of problems—a story of personal experience. About eight years ago, I gave a course entitled, "Problems and Methods in Point Sets and Real Functions." It was a course in the technique for making up problems and for composing proofs. As raw material, we used literature articles like those in the *Fundamenta Mathematica*, and it was our

purpose, through the study of particular cases, to acquire somewhat of a technique for asking worthwhile questions in the field of our study. Since people are so apt to conclude too readily, that they, too, could have initiated ideas proposed by others, it was emphasized that we would not agree that a literature problem had been shown to be derivable *via* our acquired technique unless the evidence for such a conclusion was indubitably convincing. For several weeks, during which time we were assembling our technique, no student reporting brought such evidence. But thereafter we began to get an occasional success, and after about six weeks, the reports were uniformly successful. The latter reports were made in some such way as the following. Without at first mentioning the literature problem, the student reporting would talk, maybe 10–20 minutes, about fundamental questions in the field of point sets and real functions, and show how, by means of our acquired principles of technique, other questions were progressively suggested, until a class of questions was obtained including the literature problem, together with allied problems perhaps no less interesting.

5. Generalization of theorems. As to the generalization of theorems, only brief indications can be made, since the mere statement of interesting theorems together with the appropriate elucidation of terms, would consume proportionately too much time. On the whole, it may be said, that the problem of formulating a comprehensive and effective technique for generalizing theorems offers essentially no more difficulty than that of generalizing concepts. In this connection, it is helpful to realize that the most important work of the best mathematicians centers around the clarification of concepts rather than the derivation of theorems involving concepts already clarified.

Frequently, numerous theorems of the same type appear in mathematical literature, such as theorems asserting the existence of a point in the continuum, or theorems that certain sets are closed, or exhaustible, or measurable. If the ideal of unification and generalization were as operatively active among mathematicians as it should be, there would at once be deliberate attempts in such cases, with great faith in success, to find a common genesis for such allied theorems. But even in the case of the theorems asserting the existence of a point, which come at the very beginning of analysis, such genetic unification is neglected in the literature—a unification that can readily be accomplished satisfactorily. Likewise, for the theorems on closed sets. Likewise, in a much more satisfactory manner than has been done, can a unification be made for the theorems on exhaustible sets and for those on measurable sets. Likewise, for theorems based on the continuum hypothesis. And such unifications, interesting in themselves, are just illustrations of the efficacy of even a moderately developed technique of generalization.

It may be good to repeat that though mathematicians go after unifications, they do not take as central objective in their work the acquisition of new explorative eyes, in the intimately operative sense alluded to before. On this ac-

count, there is a deficiency of faith in succeeding on the quest for unification, and a reconciliation with work on narrower problems.

6. Generalization of processes. In regard to the generalization of processes, I will just mention the following instance. To secure an example of an irrational number, one should first examine known properties of rational numbers, or change the form of known properties, or deduce new properties. One such property is that the decimal development of a rational number is periodic. A non-periodic decimal, like $.1010010001 \cdots$, is therefore irrational. The way to obtain an example of a transcendental number is essentially the same, and so is the way for defining an example of a non-measurable set.

7. Concluding remarks. I will conclude with a number of remarks.

1. The true nature of mathematics has been obscured on account of its great debt to the physical sciences. The fact that mathematics has constantly received manifold stimulus and inspiration from natural science does not imply that the essential nature of mathematics compels it to be in intimate *liaison* with applications—no more than it follows that music, which receives cues and inspiration from nature, is not entitled to a presiding Muse of its own. Physical laws, after all, are special. Without entering into judgment of relative values for life, it seems that it must be said that when mathematics is bound up with applications it withdraws from its high function of generalizing and seeking essence. But if we go after particularities, we begin to lose faith in the potency of general technique. Herein is one of the reasons why the importance of the search for technique has been inadequately realized.

2. The technique of generalization includes also the technique of specialization. By generalizing and then specializing, we may secure a generalization that is especially interesting, especially useful. An advanced technique of generalization means acquired insight and discrimination as to significance, significance with respect to what is uninterestingly obvious, with respect to the marriage of the intuitive and the formulated, with respect to psychomathematical effectiveness as a tool in strategy. Some generalizations are so broad that they lose strategic effectiveness.

3. To be a good teacher is to be a good learner. Mathematics has supreme values to offer to the human mind—thus to the evolution of humanity—such as discipline in discrimination and clarity, and symphonic intellectual patterns. Likewise, mathematics has things to learn from other fields. Especially desirable it is for mathematicians to become increasingly sensitive and friendly to suggestions for effecting fundamental changes in the character of one's exploring consciousness—whether such changes are wrought by intellectual, or by physical, psychical, or spiritual means. A person of high ideals will discern common elements in the learning or exploring process of diverse fields; in other words, the essence (generalization) of the learning process is of direct concern to mathematicians. I shall mention one such common element by way of illustration. I call it *differential perfection*, *differential* referring to the momentary state of the

mind and body, and *perfection* being construed in the *relative* sense. For a pianist to possess differential perfection when at play, would mean that his hands and arms—indeed, his entire body—are as supple and strong as his consciously available technique can make them; that his spirit is lifted to as high a sense of beauty and nobility as his technique can command; that he is prepared to enter, as deeply as his genuine mastery will permit, the soul of the music he is to produce; that he is filled with the friendliest feelings for his audience; and perhaps, if he is of the great, that he humbly feels himself to be a messenger of the Infinite, a priest of Beauty to his listeners. Even the mere idea of differential perfection, if seriously entertained, could raise concert performances to new levels. The analogous potentialities for mathematicians are evident upon reflection, though the analog in mathematics of fingering in music, of the brush stroke in painting, may not be apparent. Musicians and artists are perforce more naturally than mathematicians—though rarely systematically and comprehensively—drawn to the discipline in differential perfection. We may aptly here cite Plato (who formulates a technique of realizing Beauty on an ever grander scale), Mozart (who emphasized that the easy way to advanced technique is *via* perfection), Rubinstein (who schooled himself daily for hours in certain phases of differential perfection), Balzac, Gorki, Charlie Chaplin, among numerous others. It has not been realized by mathematicians that it is of first importance for them to incorporate in their available technique a constantly improving—*i.e.*, more embracing—differential perfection.

CYCLIC PROPERTIES OF MIQUEL POLYGONS

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In this paper it is shown that a certain simple, yet quite arbitrary construction, if completed n times on any n-polygon gives a new polygon directly similar to the original, despite the fact that the polygons constructed at intervening stages may be as diverse in shape as is imaginable.

Let there be given a plane polygon (O) with n vertices and n sides; a point P in the plane of the polygon, but not on a side; and an angle θ_1 with $0<\theta_1<\pi$. Assume that neither the sides nor the vertices of (O) are distinct—simply that there are n of each. Let the vertices be denoted by $A_1, A_2, \cdots, A_n$ and the sides by $A_1A_2, A_2A_3, \cdots, A_nA_1$, where if A_k and A_{k+1} are distinct, the side A_kA_{k+1} is the line joining the points; but if A_k and A_{k+1} coincide, there must be given a line through the double point to be designated A_kA_{k+1}. Note that $A_{n+1}=A_1$, and more generally $A_{k\pm n}=A_k$ for every k.

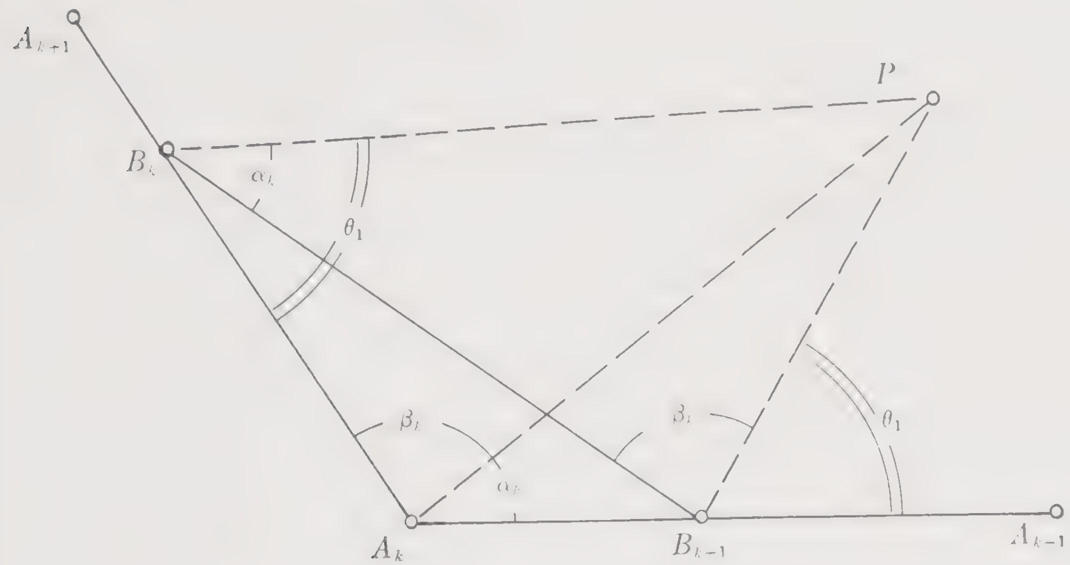
If every two consecutive vertices and every two consecutive sides are distinct, the polygon will be called *proper*; otherwise, *degenerate*. In the theorems that follow it is a notable feature that a single proof suffices for an enormous number of possible cases, this great simplification being accomplished by the notion of a directed angle, a notation thoroughly developed by R. A. Johnson (see his *Modern Geometry*).

Let the directed angle $\measuredangle ABC$ indicate the positive angle through which the line AB must be rotated about B to coincide with the line BC. Thus if $\measuredangle ABC = \measuredangle DEF$, the ordinary angles concerned are either equal or supplementary. Two implications of the directed angle notation are the following:

(I) P, A, Q, B are concyclic if and only if $\measuredangle PAQ = \measuredangle PBQ$.

(II) If $\measuredangle PAB = \measuredangle P'A'B'$ and $\measuredangle PBA = \measuredangle P'B'A'$, then the triangles PAB and $P'A'B'$ are directly similar.

Upon the polygon (O), the following constructions are to be made using the fixed point P and the fixed angle θ_1:



CONSTRUCTION 1

CONSTRUCTION 1. On the side A_kA_{k+1} determine a point B_k such that $\measuredangle A_kB_kP = \theta_1$. Complete for $k = 1, 2, \cdots, n$.

If B_{k-1} and B_k are coincident with A_k, define the line $B_{k-1}B_k$ to be the line $A_{k-1}A_k$; if B_{k-1} and B_k are coincident but distinct from A_k, perform also the following:

CONSTRUCTION 2. Through the point $B_{k-1} = B_k$ draw the tangent line to the circle $B_{k-1}A_kP$ and define this line to be $B_{k-1}B_k$.

There is determined in this way a polygon (1) with its n sides $B_{k-1}B_k$ as well as its n vertices B_k completely defined. The polygon (1) will be called *a Miquel polygon of P with respect to (O).*

Let $\alpha_k = \measuredangle A_{k-1}A_kP$ and $\beta_k = \measuredangle PA_kA_{k+1}$. Then by construction or by the relation (I) it follows that

$$\alpha_k = \measuredangle A_{k-1}A_kP = \measuredangle B_{k-1}B_kP, \tag{III}$$

$$\beta_k = \measuredangle PA_kA_{k+1} = \measuredangle PB_{k-1}B_k. \tag{IV}$$

But these properties are independent of the angle of incidence θ_1, so that by relation (II) the triangle $PB_{k-1}B_k$ is determined to within a direct similarity when various choices of θ_1 are made. Composition of the several triangles (some may be degenerate) as $k=1, 2, \cdots, n$ gives precisely the polygon (1). But two polygons composed of similar triangles similarly placed are similar. Hence we have the following:

THEOREM 1. *All the Miquel polygons of P with respect to (O) are directly similar, with P as a self-homologous point.*

Using any convenient angle of incidence θ_2 construct a polygon (2) as the Miquel polygon of P with respect to (1). If this process be repeated n times—(j) being the Miquel polygon of P with respect to $(j-1)$—we have the following:

THEOREM 2. *The polygon (n) is directly similar to the original polygon (O), with P as a self-homologous point.*

For, let the polygon (n) have the vertices D_k and consider the two chains of equalities derived from the relations (III) and (IV), respectively:

(V) $\quad \sphericalangle A_{k-1}A_kP = \sphericalangle B_{k-1}B_kP = \cdots = \sphericalangle D_{k-1}D_kP,$

(VI) $\quad \sphericalangle PA_{k-1}A_k = \sphericalangle PB_{k-2}B_{k-1} = \cdots = \sphericalangle PD_{k-n-1}D_{k-n} = \sphericalangle PD_{k-1}D_k.$

Then by relation (II) the triangles $PA_{k-1}A_k$ and $PD_{k-1}D_k$ are directly similar, for $k=1, 2, \cdots, n$. But two polygons composed of similar triangles similarly placed are similar. Hence (n) is directly similar to (O).

COROLLARY 1. *If the sequence of constructions be continued, (i) is directly similar to (j) if $i \equiv j$* mod n.

COROLLARY 2. *If $j-i$ is the smallest number $<n$ such that (j) is directly similar to (i), then $j-i$ divides n.*

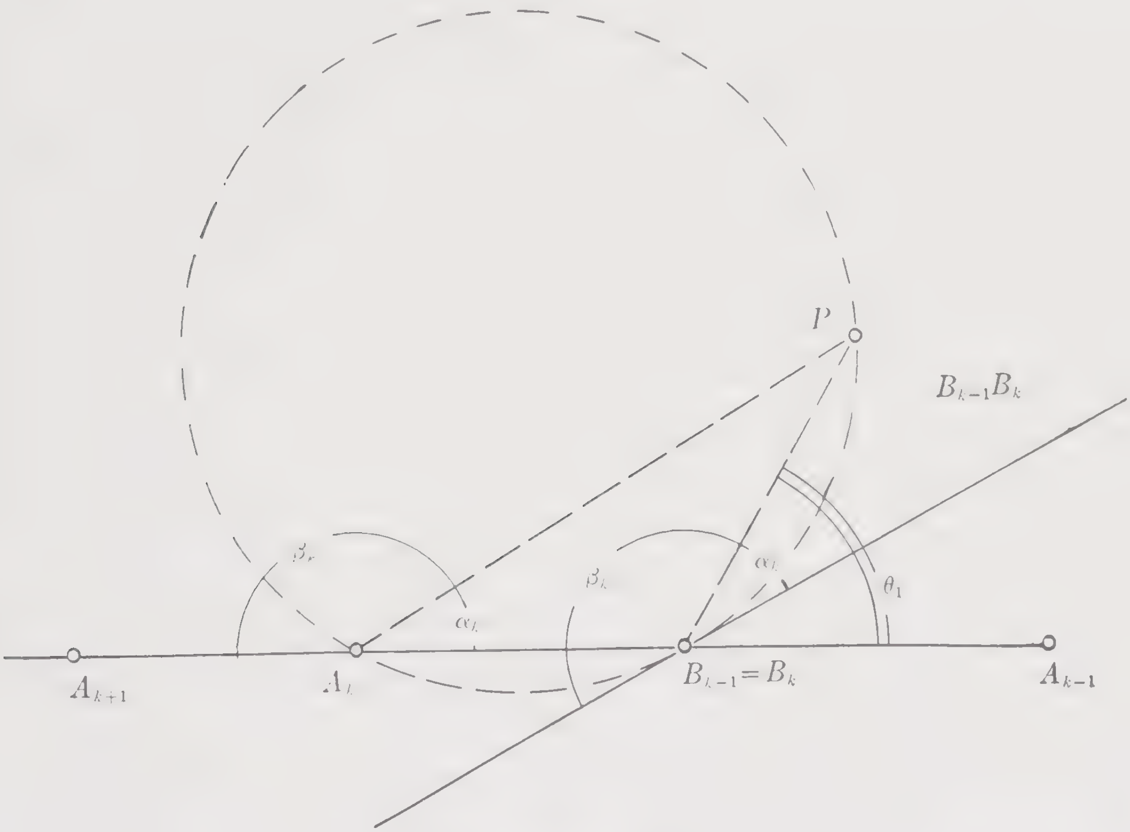
The situation in Theorem 2 may be clarified by using the notation defined in (III) and (IV), for it is the directed angle α_k which occurs in chain (V), and the directed angle β_{k-1} which appears in (VI). Then in the directed angle sense, the angles of the successive polygons are given by the table:

$$
\begin{array}{ll}
(0): & \alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_n+\beta_n \\
(1): & \alpha_1+\beta_2, \alpha_2+\beta_3, \cdots, \alpha_n+\beta_1 \\
(2): & \alpha_1+\beta_3, \alpha_2+\beta_4, \cdots, \alpha_n+\beta_2 \\
\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\
(n): & \alpha_1+\beta_1, \alpha_2+\beta_2, \cdots, \alpha_n+\beta_n
\end{array}
$$

This very pretty set-up reveals that Construction 2 will be essential only if P is so chosen that some $\alpha_i+\beta_k \equiv 0$ mod π. In particular, if (O) is assumed to be proper this relation cannot hold for $k=i$ nor for $k=i-1$; but for any other value of k it can hold, and can be interpreted to mean, by virtue of relation (I), that P

is on the circumscribed circle of the triangle determined by the sides $A_{i-1}A_i$ and A_kA_{k+1} together with the line joining A_i and A_k. When $n=3$ there is one such circle; when $n \geqq 4$ there are at most $n(n-2)$ of these circles; for all other choices of P each polygon of the sequence will be proper.

Illustrative figures are most readily constructed, but not always of most convenient dimensions, when the angles of incidence are all right angles—and the Miquel polygons, the familiar pedal polygons. It is also helpful to realize that by constructions which are more or less dual to those previously given, a polygon (-1) can be constructed such that (O) is a Miquel polygon of P with respect to (-1).



Construction 2

The directed angle γ between the side $D_{k-1}D_k$ of (n) and the side $A_{k-1}A_k$ of (O) is given by

$$\gamma = \sum_{k=1}^{n} \beta_k + \sum_{k=1}^{n} \theta_k. \tag{VII}$$

Among the interesting loci problems suggested by this formula is the following: Suppose that each $\theta_k = \pi/2$; how must P be chosen so that $\gamma \equiv 0 \bmod \pi$, *i.e.*, so that the sides of (n) are parallel to the corresponding sides of (O)? The desired

locus includes any axis of symmetry for the polygon (O), as well as the center of the corresponding circle, if it happens that (O) can be inscribed in or circumscribed about a circle.

The Miquel polygons as described in Theorem 1 are a mere generalization of a known theorem for a triangle which R. A. Johnson has noted to be associated with a theorem of A. Miquel (1838). There seems to be nothing like Theorem 2 in the literature, even for a triangle, except for a problem by V. Thébault* in which the theorem is stated for the special case of a point P *interior* to a *square* using a *pedal* construction, with the Pythagorean theorem used in the proof.

A THEOREM ON A CYCLIC POLYGON

R. GOORMAGHTIGH, Bruges, Belgium

1. A theorem regarding a triangle. It is well known that the envelope of the Simson lines of a triangle is a three-cusped hypocycloid having as inscribed circle the nine-point circle of the triangle.†

This theorem may be used to prove the following theorem, of which we will derive an extension to any polygon inscribed in a circle.

THEOREM 1. *The parallel drawn from a moving point of the circumscribed circle of a triangle to the Simson line of that point envelopes a three-cusped hypocycloid having as inscribed circle the circle circumscribed to the triangle.*

Proof. Let $A_1A_2A_3$ be the triangle, H the orthocenter, M a point on the circle $A_1A_2A_3$, $A_1'A_2'A_3'$ the triangle homothetic to $A_1A_2A_3$, the homothetic center being H and the ratio 2, *i.e.*,

$$\frac{HA_1'}{HA_1} = \frac{HB_1'}{HB_1} = \cdots = 2.$$

Further, let M' be the homothetic of M. As the Simson line of M in the triangle $A_1A_2A_3$ passes through the midpoint of HM, the parallel drawn through M to the Simson line of that point with respect to the triangle $A_1A_2A_3$ is the Simson line of M' with respect to the triangle $A_1'A_2'A_3'$, and this proves the considered proposition, as the circle $A_1A_2A_3$ is the nine-point circle of the triangle $A_1'A_2'A_3'$.

2. A generalization. Let now $A_1A_2 \cdots A_{n-1}A_n$ be an n-sided polygon inscribed in the circle Γ; the Simson line of a point M on Γ with respect to the polygon may then be defined from the following theorem,‡ which we will proceed to prove.

* V. Thébault, National Mathematics Magazine, vol. 13, 1939, p. 338.

† J. Steiner, Über eine besondere Curve dritter Klasse (und vierten Grades), Crelle, vol. 53, 1857, p. 231; Cremona, Sur l'hypocycloïde à trois rebroussements, Crelle, vol. 64, 1865, p. 101; Sister M. C. Karl, The projective theory of orthopoles, this MONTHLY, vol. 39, 1932, pp. 327–338.

‡ E. M. Langley, Educational Times, vol. 51; M. Kobayashi, Tôhoku Mathematical Journal, vol. 28, 1927, p. 46.

THEOREM 2. *In the quadrilateral* $A_1A_2A_3A_4$ *the projections of* M *on the Simson lines of that point with respect to the triangles* $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$ *lie on a straight line which we shall call the Simson line of* M *with respect to the quadrilateral. Similarly, the projections of* M *on the Simson lines of that point with respect to the four quadrilaterals which have as vertices four of the five points* A_1, A_2, A_3, A_4, A_5 *are on a straight line, the Simson line of* M *with respect to the pentagon* $A_1A_2A_3A_4A_5$; *and so on.*

The circle Γ, with center O, being taken as base circle, let Ω be the unit point and $t_1, t_2, \cdots, t_n, \tau$ the turns* corresponding to $A_1, A_2, \cdots, A_n, M$; the conjugate to the complex number x will be denoted by $\bar{x}$, and the symmetric functions of $t_1, t_2, \cdots, t_n$ by $\sigma_1, \sigma_2, \cdots, \sigma_n$, *i.e.*,

$$\sigma_1 = \Sigma t_1, \quad \sigma_2 = \Sigma t_1t_2, \quad \sigma_3 = \Sigma t_1t_2t_3, \cdots, \sigma_n = t_1t_2 \cdots t_n.$$

If

$$s_1 = t_1 + t_2 + t_3, \qquad s_2 = t_2t_3 + t_3t_1 + t_1t_2, \qquad s_3 = t_1t_2t_3,$$

then the Simson line of M with respect to the triangle $A_1A_2A_3$ will have

$$x\tau - \bar{x}s_3 = (\tau^3 + s_1\tau^2 - s_2\tau - s_3)/2\tau$$

as its equation. The projection of M on this line is

$$x = (3\tau + s_1 - s_2\tau^{-1} + s_3\tau^{-2})/4;$$

as

$$4x\tau^2 = 3\tau^3 + s_1\tau^2 - s_2\tau + s_3,$$

$$4\bar{x}s_3 = \tau^2 - s_1\tau + s_2 + 3s_3\tau^{-1},$$

we have

$$4(x\tau^2 + \bar{x}s_3t_4) = 3\tau^3 + (s_1 + t_4)\tau^2 - (s_2 + s_1t_4)\tau + s_3 + s_2t_4 + 3s_3t_4\tau^{-1},$$

or, p_1, p_2, p_3, p_4 being the symmetric functions of t_1, t_2, t_3, t_4,

$$x\tau^2 + \bar{x}p_4 = (3\tau^4 + p_1\tau^3 - p_2\tau^2 + p_3\tau + 3p_4)/4\tau.$$

Since this equation is symmetric in t_1, t_2, t_3, t_4 it represents the straight line joining the projections of the point M on its Simson lines in the triangles $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, $A_1A_2A_3$.

Extending the foregoing process we find the equation of the Simson line of M with respect to the polygon $A_1A_2 \cdots A_n$ to be

$$x\tau^{n-2} + (-1)^n\bar{x}\sigma_n = 2^{-(n-2)}\tau^{-1}\big[(2^{n-2} - 1)\tau^n + \sigma_1\tau^{n-1} - \sigma_2\tau^{n-2} + \cdots + (-1)^n\sigma_{n-1}\tau + (-1)^n(2^{n-2} - 1)\sigma_n\big].$$

The equation of the parallel drawn from M to the Simson line is

$$x\tau^{n-2} + (-1)^n\bar{x}\sigma_n = \tau^{n-1} + (-1)^n\sigma_n\tau^{-1},$$

* F. and F. V. Morley, Inversive Geometry, 1933, p. 15.

and this parallel meets the circle Γ again at the point Q represented by $(-1)^n\sigma_n\tau^{1-n}$.*

The nth root of σ_n has n values represented on Γ by the vertices $L_1, L_2, \cdots, L_n$ of an n-sided regular polygon. For a given polygon $A_1A_2 \cdots A_n$, the polygon $L_1L_2 \cdots L_n$ is also a fixed one when a new unit point is chosen on Γ; for, if in the new system, Ω is represented by λ, $A_1, A_2, \cdots, A_n$ will be represented by $\lambda t_1, \lambda t_2, \cdots, \lambda t_n$; the new value of σ_n will be $\lambda^n\sigma_n$ and the values of the considered nth roots will also be multiplied by λ.

But the equation

$$-\sigma_n^{1/n}:\left[(-1)^n\sigma_n\tau^{1-n}\right] = \tau^{n-1}:(-\sigma_n^{1/n})^{n-1}$$

shows that, if L_i' is the image in O of any of the points L, we have, between arcs in the same sense of rotation on Γ, the relation

$$L_i'Q = (n-1)ML_i'.$$

Hence, from a well known property,† we have the following:

THEOREM 3. *If an n-sided polygon is inscribed in a circle Γ, the parallel drawn from a moving point M on Γ to the Simson line of M with respect to the polygon envelopes an n-cusped hypocycloid having Γ as inscribed circle.*

BOUNDS FOR THE ROOTS OF A TRINOMIAL EQUATION

E. C. KENNEDY, Texas College of Arts and Industries

We shall prove that the complex roots of the trinomial equation

$$Z^n + aZ^k + b = 0, \qquad ab \neq 0, \tag{1}$$

have certain bounds to their absolute values, and thus lie in certain rings whose centers are at the origin of the complex Z-plane.

THEOREM I. *If X_1 is the positive real root of the equation*

$$X^n + |a|X^k - |b| = 0, \tag{2}$$

and X_2 the positive real root of

$$X^n - |a|X^k - |b| = 0, \tag{3}$$

then the roots of (1) *lie in the ring $X_1 \leq |Z| \leq X_2$.*

* This gives a geometric meaning to the symmetric function σ_n: the point σ_n is the second intersection of the circle with the parallel (when n is even) or the perpendicular (when n is odd) drawn from the unit point Ω to the Simson line of Ω with respect to the polygon $A_1A_2 \cdots A_n$. This is a generalization of the theorem we have given, for the case $n=3$, in Mathesis, 1939, p. 73.

† H. Wieleitner, Spezielle ebene Kurven, Leipzig, 1908, p. 207.

Proof. From (1) we have, for any root Z,

$$|Z|^n + |a||Z|^k \geqq |b|. \tag{4}$$

If we assume that $|Z| < X_1$, then we have from (2)

$$|Z|^n + |a||Z|^k - |b| < 0,$$

which contradicts (4). Hence $X_1 \leqq |Z|$.

From (1) we also have, for any root Z,

$$|Z|^n \leqq |a||Z|^k + |b|. \tag{5}$$

If we assume that $|Z| > X_2$, then we have from (3)

$$|Z|^n - |a||Z|^k - |b| > 0,$$

which contradicts (5). Hence $|Z| \leqq X_2$.

THEOREM II. *If X_3 and X_4 are the positive real roots of the equations*

$$X_3^{n-k} = |b|^{(n-k)/n} - |a|, \tag{6}$$

$$X_4^{n-k} = |b|^{(n-k)/n} + |a|, \tag{7}$$

then the roots of (1) *lie in the ring* $X_3 < |Z_1| < X_4$.

Proof. It will suffice to show that $X_3 < X_1$ and $X_4 > X_2$, where X_1 and X_2 are the positive real roots of equations (2) and (3), respectively. Calling the left member of (2) $L_2(X)$, we have

$$\begin{aligned} L_2(X_3) &= X_3^k[X_3^{n-k} + |a|] - |b| \\ &= X_3^k|b|^{(n-k)/n} - |b|, \quad \text{by (6)}, \\ &< |b|^{k/n}|b|^{(n-k)/n} - |b|, \end{aligned}$$

since by (6), $X_3 < |b|^{1/n}$. Hence $L_2(X_3) < 0$, and therefore $X_3 < X_1$. Similarly we may prove $X_4 > X_2$.

The reader may note that Theorem I gives a sharper limitation on the roots of (1) than does Theorem II, but the bounds X_3 and X_4 are more easily found than X_1 and X_2.

In the case of the general reduced cubic $X^3 - aX - b = 0$, $(a, b > 0)$, we have for the positive root R the relationship $b^{2/3} \leqq R^2 \leqq b^{2/3} + a$. If $a > 0$ and $b < 0$, we have $b^{2/3} - a \leqq R^2 \leqq b^{2/3}$.

Example 1. Isolate in a ring the roots of $Z^5 - (5+12i)Z + (70+24i) = 0$.

Here $n=5$, $k=1$, $|a| = 13$, $|b| = 74$ and we have

$$X_3^4 = 74^{4/5} - 13, \quad X_4^4 = 74^{4/5} + 13,$$

and hence

$$2.07 \leqq |Z| \leqq 2.58.$$

Example 2. Isolate in a ring the roots of $Z^{\sqrt{10}}-21Z^{\sqrt{2}}+50=0$.
Here we have

$$X_3^{\sqrt{10}-\sqrt{2}} = 50^{1-1/\sqrt{5}} - 2, \; X_4^{\sqrt{10}-\sqrt{2}} = 50^{1-1/\sqrt{5}} + 2,$$
$$3.56 \leqq |Z| \leqq 4.38.$$

Example 3. Isolate the roots of $Z^5-Z+32i=0$.

Here we have $15 \leqq |Z|^4 \leqq 17$ or $1.97 \leqq |Z| \leqq 2.03$. In this case we can obtain further information about the roots by setting $Z=\rho(\cos\theta+i\sin\theta)$, obtaining $\cos 5\theta/\cos\theta=1/\rho^4$. Using the expansion for $\cos 5\theta$ and taking $\rho^4=15$ we are led to the equation

$$16\sin^4\theta - 12\sin^2\theta + 1 = 1/15,$$

a quadratic equation in $\sin^2\theta$. Then using $\rho^4=17$ we get another quadratic equation. Solving these two trigonometric equations we isolate the five roots as follows:

$$\begin{aligned} 1.97 \leqq\ & \rho_i \leqq 2.03, \qquad (i = 1, 2, 3, 4, 5); \\ 54°23' \leqq\ & \theta_1 \leqq 54°26', \\ 125°34' \leqq\ & \theta_2 \leqq 125°37', \\ 197°16' \leqq\ & \theta_3 \leqq 197°22', \\ 242°38' \leqq\ & \theta_4 \leqq 242°44', \\ & \theta_5 = 90°. \end{aligned}$$

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

A PROGRAM IN FRESHMAN MATHEMATICS DESIGNED TO CARE FOR A WIDE VARIATION IN STUDENT ABILITY

E. A. CAMERON, University of North Carolina

While the teaching problem caused by extreme divergence in student aptitude and ability is not new, in many institutions—due to a variety of reasons—the problem has become increasingly acute within the past few years. For this reason it is thought that readers of the MONTHLY would be interested in a brief description of how the problem, as it arises in freshman mathematics instruction, is being met at the University of North Carolina.

At North Carolina virtually all freshmen are required to take a year course in mathematics. The number of students in the freshman class is about 850, drawn from a rather wide geographical area and representing a wide range of ability, preparation, and interests. Our present program in freshman mathematics, which was put into operation in the fall of 1938, was designed to take care of the extreme individual differences found in this group.

The first step in our procedure is to separate the students into three groups according to ability. This segregation is effected as follows. Before registration all freshmen are given a mathematics placement test. The American Council's Coöperative General Mathematics Test is used for this purpose. This test is an objective type, machine scored test, consisting of 60 questions on arithmetic, algebra, geometry, and trigonometry. In the fall of this year the scores ranged from 0 to 53. On the basis of the scores made on this test the entering freshmen are divided into three groups. The first group consists of the highest 10% of the freshman class, the second group of the next 65%, and the third group of the lowest 25%. These three groups are called the A-group, the B-group, and the C-group respectively. The students are then registered in sections according to their classification.

At the first meeting of each class a second standardized test is given, this test being on algebra only. A record card is made out for each student, upon which is entered the student's high school record, the scores he made on the two mathematics tests, and the score made on the American Council College Aptitude Test. The record card of each student is then studied, and on the basis of the complete picture portrayed on that card, the final classification is made. This means that some students have to be changed from one type of section to another, but the number of such changes that have to be made is less than 10% of the total registration.

Next I want to describe briefly the character of the courses given to the

three groups. For several years at North Carolina, students going into the School of Commerce have been given a somewhat different course in freshman mathematics from that given students who are going into the College of Arts and Sciences. Thus we have a two-way division of the students—longitudinally, one might say, into Commerce and Arts students, and latitudinally into the three groups on the basis of ability. Hence, there are six different freshman mathematics courses. The three courses given to the Commerce students differ from the ones given to the Arts students principally in the substitution of topics in the mathematics of economics and finance for a large part of the usual trigonometry. Therefore, for the purpose of this paper, it will be sufficient to confine the description to the courses given to the students in the Arts division.

The B-group students are given a course of the same character as is found frequently throughout the country—a unified course consisting of material drawn from college algebra, trigonometry, and the elements of analytic geometry and the calculus. The B-sections meet three hours a week throughout the year.

The C-group is made up largely of students frankly deficient in native ability, or preparation, or both. They undoubtedly constitute our most troublesome problem in freshman mathematics instruction. There are two directions in which we have attempted to adapt a course to the ability of this group. In the first place, more time is spent on a given amount of material than is the case in a B-section. This is accomplished by having the C-sections meet five hours a week throughout the year. However, the students receive only three hours credit. The other device goes in the direction of somewhat simpler subject-matter. Last year the C-group used the same text as the B-group. This year the first part of the year was devoted to the teaching of those minimum essentials of algebra without which any further work in mathematics is impossible. The rest of the year's work consisted of a simplified survey course, covering the most elementary parts of trigonometry, analytics, and the calculus. In this survey course an attempt is made to acquaint the students with the nature and uses of the branches of mathematics considered, without developing many more techniques than are necessary for this purpose. The limitations of the students in the C-sections are such as to render unlikely the need by this group of college mathematics as a tool, except in rare cases. Hence we do not consider this course as preparatory to further work in mathematics, but as a course whose cultural value does amply justify its inclusion in a liberal education.

The students in the A-group constitute our happiest subject. The students in this group come to us with a pretty good command of the techniques of elementary algebra, and most of them have had some trigonometry. For some of the students in this group, the freshman course is a terminal course, while for others it serves as an introduction for further work in mathematics. A complete description of the course given this group would take too much space, so I can only indicate briefly the nature of the course. The A-classes meet three hours a week throughout the year. We are using as a basic text a book somewhat more

mature and philosophical in its approach than most freshman texts, and are supplementing it freely with additional topics. We encourage the students to do some outside reading by having a shelf of parallel reading books on reserve, and requiring reports from time to time. Briefly, this course differs from the usual freshman course in the choice of material, the rapidity of covering the material, and in an unusually persistent attempt to give the students an adequate appreciation of the essential nature of mathematics, and its significance in the development of civilization and in contemporary life.

One definite measurable result of our program is that the percentage of students passing is somewhat higher than before the plan was put into operation. The greatest benefit that has come from the plan lies in the improved teaching conditions in the classroom. The very poor and the very good students are separated from the average; so that now we do not drown the one and bore the other while teaching at the middle. The segregation enables us to adapt the content of the course to the level of the student, which has increased interest and decreased discouragement on the part of the students.

The functioning of universal education—a practice recent in time and limited in geographical position—has swept into the colleges of this country hordes of students, including the brilliant and well-equipped, and likewise the ill-favored, the ill-prepared, and alas sometimes the illiterate. This situation constitutes a challenging problem for our colleges and universities, and one of tremendous importance. A program such as ours, which provides different treatment for different students, may superficially seem to violate democracy in education; but we are firmly convinced that it is helpful in the education of a democracy.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The Department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ANOTHER FORM OF THE RUSSELL PARADOX

L. S. JOHNSTON, University of Detroit

It is quite common in non-profit institutions such as colleges, churches, hospitals and the like, for the Secretary of the organization or of the Board of Trustees of the institution to be not a member of the organization served.

Several such secretaries decided to promote their common welfare by organizing Local No. 1, Amalgamated Secretaries of Organizations of Which They are Not Members, and wrote into its constitution the following provisions:

1. Any person who is secretary of an organization of which he is not himself a member may be a member of this Union.

2. No person who is a member of the organization he serves as secretary may be a member of this Union.

Then the Union tried to appoint a Secretary! Here is a statement of the Russell Paradox which is perhaps a little less artificial than the familiar "barber" formulation. In fact, it was suggested to the writer by one such secretary's remarking that he wished such a Union existed.

Note by the Editor. With the provisions given by Professor Johnston there is still the possibility of the Union appointing a secretary who is not one of its members. But if the word "may" in the first provision is replaced by "must," the Union must remain without a secretary. R. J. W.

SIMPLE HARMONIC MOTION WITH QUADRATIC DAMPING

C. R. WYLIE, JR., Ohio State University

Although text-books in mechanics conventionally treat simple harmonic motion subject to damping proportional to the square of the velocity in terms of an equivalent viscous resistance,* this device is unnecessary, for tables adequate for the solution of numerical problems of this type have been available for some years. These tables, together with the various infinite series from which they were obtained, are to be found in a paper by W. E. Milne.† For the sake of completeness it may be interesting to call attention here to another series associated with the problem which has the desirable property, not directly exhibited by the corresponding series of Milne, of yielding the finite form of the solution for the undamped case when the damping coefficient approaches zero.

* Timoshenko, Vibration Problems in Engineering, 1938, p. 32.

† W. E. Milne, Damped Vibrations, University of Oregon Publications, vol. 2, no. 2, 1923.

The differential equation to be solved is

$$(1) \qquad \frac{d^2x}{dt^2} \pm 2k\left(\frac{dx}{dt}\right)^2 + a^2x = 0.$$

We consider the motion from a left-hand extremum of $x_0<0$ to a right-hand extremum of $x_1>0$, and accordingly use the plus sign. For the first and second quarter-periods we have, respectively,

$$(2.1) \qquad t = \frac{2\sqrt{2}\,k}{a}\int_{x_0}^{\lambda x_0} \frac{dx}{\sqrt{(1-4kx)-(1-4kx_0)e^{4k(x_0-x)}}}, \quad 0 \leqq \lambda \leqq 1,$$

$$(2.2) \qquad t = \frac{2\sqrt{2}\,k}{a}\int_{0}^{\lambda x_1} \frac{dx}{\sqrt{(1-4kx)-(1-4kx_1)e^{4k(x_1-x)}}}, \quad 0 \leqq \lambda \leqq 1.$$

In (2.1) put $x=x_0(1-z)$ and $-4kx_0=\epsilon$. In (2.2) put $x=x_1(1-z)$ and $-4kx_1=\epsilon$. We then have, respectively,

$$(3.1) \qquad t = \frac{1}{\sqrt{2}\,a}\int_0^{1-\lambda} \frac{dz}{\sqrt{\dfrac{(1+\epsilon-\epsilon z)-(1+\epsilon)e^{-\epsilon z}}{\epsilon^2}}},$$

$$t = \frac{1}{\sqrt{2}\,a}\int_{1-\lambda}^{1} \frac{dz}{\sqrt{\dfrac{(1+\epsilon-\epsilon z)-(1+\epsilon)e^{-\epsilon z}}{\epsilon^2}}}$$

$$(3.2) \qquad = \frac{1}{\sqrt{2}\,a}\int_0^{1} \frac{dz}{\sqrt{\dfrac{(1+\epsilon-\epsilon z)-(1+\epsilon)e^{-\epsilon z}}{\epsilon^2}}} - \frac{1}{\sqrt{2}\,a}\int_0^{1-\lambda} \frac{dz}{\sqrt{\dfrac{(1+\epsilon-\epsilon z)-(1+\epsilon)e^{-\epsilon z}}{\epsilon^2}}}.$$

This last expression suggests the identity $\sin^{-1}\lambda=\pi/2-\cos^{-1}\lambda$.

Evidently (3.1) is the essential integral here. We rewrite it thus:

$$(4) \qquad t(\epsilon,\lambda) = \frac{1}{\sqrt{2}\,a}\int_0^{1-\lambda} \frac{dz}{\sqrt{z-(1+\epsilon)\left(\dfrac{z^2}{2!}-\dfrac{\epsilon z^3}{3!}+\dfrac{\epsilon^2 z^4}{4!}-\cdots\right)}},$$

and proceed to develop it in powers of ϵ. We find at once that $t(0,\lambda)=(1/a)\cos^{-1}\lambda$, which is precisely the usual solution for the undamped case. We find the various derivatives for $\epsilon=0$ by differentiating the integral with respect to ϵ, setting $\epsilon=0$, and integrating. In every case the integrals are elementary, and in fact

by obvious transformations the kth derivative can be made to depend upon the $k+1$ integrals

$$\int_{\cos^{-1}\mu}^{\pi/2} \frac{\cos^{2k}\theta\, d\theta}{\sin^{2(k-e)}\theta}, \qquad \mu = \sqrt{\frac{1-\lambda}{2}}, \quad (e = 0, 1, 2, \cdots, k).$$

The first few terms of the resulting expansion for the integral are

$$t = \frac{1}{a}\Bigg[\cos^{-1}\lambda + \left\{\sqrt{\frac{1-\lambda}{1+\lambda}}\,\frac{1-\lambda}{6}\right\}\epsilon + \left\{\frac{\cos^{-1}\lambda}{12} - \frac{1}{9}\sqrt{\frac{1-\lambda}{1+\lambda}}\,\frac{1+2\lambda}{1+\lambda}\right\}\frac{\epsilon^2}{2!}$$
$$+ \left\{\frac{\cos^{-1}\lambda}{6} + \frac{1}{1080}\sqrt{\frac{1-\lambda}{1+\lambda}}\,\frac{281 + 657\lambda + 493\lambda^2 + \lambda^3 + 6\lambda^4 + 2\lambda^5}{(1+\lambda)^2}\right\}\frac{\epsilon^3}{3!}$$
$$+ \left\{\frac{49}{96}\cos^{-1}\lambda - \frac{1}{4320}\sqrt{\frac{1-\lambda}{1+\lambda}}\left(\frac{3472 + 11649\lambda + 13588\lambda^2 + 6340\lambda^3}{(1+\lambda)^3}\right.\right.$$
$$\left.\left. + \frac{188\lambda^4 + 53\lambda^5 - 8\lambda^6 - 2\lambda^7}{(1+\lambda)^3}\right)\right\}\frac{\epsilon^4}{4!} + \cdots\Bigg].$$

The radius of convergence of this series depends on the particular value of λ under consideration. It is in fact the distance from the origin in the complex ϵ-plane to the nearest zero of

$$R(\epsilon) = \frac{(1 + \epsilon - \epsilon z) - (1 + \epsilon)e^{-\epsilon z}}{\epsilon^2} = 0 \quad \text{for} \quad z = 1 - \lambda.$$

If we put $\epsilon = u + iv$ we find this root to be the simultaneous solution of least absolute value (after $u = v = 0$ is rejected) of

$$(1 + u - uz)e^{uz} - (1 + u)\cos vz - v \sin vz = 0,$$
$$v(1 - z)e^{uz} - v\cos vz + (1 + u)\sin vz = 0.$$

To obtain an idea of how the radius of convergence r depends on λ, we develop this simultaneous solution in powers of $\lambda = 1 - z$. We find at once that

$$u = 2.089 - 47.47\lambda^2 + \cdots,$$
$$v = 7.461 + 41.76\lambda^2 + \cdots;$$

whence

$$r = \sqrt{u^2 + v^2} = 7.75 + 27.4\lambda^2 + \cdots.$$

The linear terms are missing because, curiously enough, R, regarded as a function of z, has a double root $z = 1$ for $\epsilon = 2.089 + i7.461$.

In conclusion we observe that the second extreme value x_1 required in the calculations may be found from the known initial value x_0 by solving the equation $(1 - 4kx_0)e^{4kx_0} = (1 - 4kx_1)e^{4kx_1}$ for the second root $x_1 \neq x_0$.*

* B. O. Peirce, The damping of the oscillations of swinging bodies by the resistance of the air, Proc. Amer. Acad., vol. 44, no. 2.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Bibliography of Mathematical Works Printed in America through 1850. By L. C. Karpinski. Ann Arbor, University of Michigan Press; London, Humphrey Milford and Oxford University Press, 1940. 26+697 pages. $6.00.

Die Arithmetik der Griechen. By K. Reidemeister. (Hamburger mathematische Einzelschriften, Heft 26.) Leipzig and Berlin, B. G. Teubner, 1940. 32 pages. RM 2.50.

Coleccion de Problemas de Algebra. By A. G. Ioachimescu. Traducida de la cuarta edicion rumana por B. I. Baidaff. Buenos Aires, Boletin Matematico, 1940. 16+173 pages.

Living Mathematics. A First Year College Course. By R. S. Underwood and F. W. Sparks. First edition. New York and London, McGraw-Hill Book Company, Inc., 1940. 9+365 pages. $3.00.

Pandiagonal Magic Squares of Prime Order. By A. L. Candy. Lincoln, Nebraska, A. L. Candy, 1940. 5+93 pages.

Geometria Descrittiva. Lezioni Redatte Per Uso Degli Studenti. By Enea Bortolotti. (R. Universita Degli Studi di Firenze.) Padova, Cedam, 1939. 715 pages. Lire 80.

Introduction to the Calculus. By Arnold Dresden. New York, Henry Holt and Company, 1940. 12+428 pages. $3.40.

Introductory Business Mathematics. By J. S. Georges and W. H. Conley. New York, Henry Holt and Company, 1940. 10+326 pages. $2.40.

Théorie Mathématique du Bridge à la Portée de Tous. By Emile Borel and A. Charon. (Monographies des Probabilités, fascicule 5.) Paris, Gauthier-Villars, 1940. 18+392 pages. Fr. 175.

Algebraic Theory of Numbers. By Hermann Weyl. (Annals of Mathematics Studies, Number 1.) Princeton, Princeton University Press; London, Humphrey Milford and Oxford University Press, 1940. 8+223 pages. $2.35.

Tables for Calculus. By C. K. Robbins and Neil Little. New York, The Macmillan Company, 1940. 22 pages. $0.15.

REVIEWS

Theory of Equations. By H. W. Turnbull. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 12+152 pages. 4/6s.

This book is an introduction to the elements of algebra. The contents of this small volume correspond roughly to the curriculum of the average college course. In some sections the author presupposes the knowledge of elementary facts on

determinants and trigonometry. The presentation of the material is clear. Reading for self-study should not be too hard for students. If additional exercises to illustrate the theory are supplied by the instructor, the volume is suitable for beginners.

O. F. G. SCHILLING

Integration. By R. P. Gillespie. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 8+126 pages. 4/6s.

This book covers briefly the topics usually included in the second half of our American calculus text-books, with some additional material. The chapter headings are: Introduction, Integration of Elementary Functions, Multiple Integrals, Curvilinear and Surface Integrals, The Riemann Integral, Infinite Integrals, The Riemann Double Integral. In the first four chapters, no attempt is made to be rigorous. The steps are justified by appeal to geometric intuition. In the chapter on infinite integrals, there is a good discussion of the Gamma and Beta functions, and of methods of evaluating various improper definite integrals. Five of the chapters have collections of problems, which are interesting and rather difficult.

The last three chapters are treated rigorously, and include the usual theory of Riemann integration. The rigor is not always iron-clad, however. In several places the author gives the impression that he believes that all finite functions are bounded, and that a function which fails to be bounded must assume the value ∞ at some definite point. On page 115 there is the following theorem:

If $\phi'(t)$, $\psi'(t)$ exist for the interval $\alpha \leqq t \leqq \beta$, the curve $x=\phi(t)$, $y=\psi(t)$ is rectifiable. If, further, $\phi'(t)$, $\psi'(t)$ are integrable over (α, β), the length of the curve is given by

$$\int_\alpha^\beta [\{\phi'(t)\}^2 + \{\psi'(t)\}^2]^{1/2}dt.$$

This is not true as it stands. In the proof it is assumed that $\phi'(t)$ and $\psi'(t)$ are bounded.

According to the definition of completeness of an orthogonal system given on page 103, no system would be complete. Instead of the condition that $\phi(x)$ is not identically zero, it should read that $\phi(x)$ is different from zero at a set of points of positive measure.

It is difficult to give a treatment of Riemann integration that is both rigorous and neat. It would be a great service if someone would develop for the use of advanced calculus students an elementary introduction to the theory of the Lebesgue integral, with a minimum of point set theory and the theory of measure.

ORRIN FRINK, JR.

Vector Methods Applied to Differential Geometry, Mechanics, and Potential Theory. By D. E. Rutherford. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 8+127 pages. 4/6s.

This interesting little book is one of a series of "University Mathematical Texts" published under the general editorship of Alexander Aitken and Daniel E. Rutherford. It is typically English and is designed to acquaint students who are reading for a degree in mathematics with vector methods as applied to differential geometry, mechanics, and potential theory. While the book may be used either as a text-book for a course in vectors, or as a source book for collateral reading, it is not intended to compete with the more advanced and specialized works.

The book begins with a brief and quite elementary treatment of vector algebra. Vector methods are then used in Chapter II to introduce the reader to differential geometry. Chapter III is devoted to showing how certain topics in mechanics may be treated naturally in the language of vectors. Next the vector operator " ∇" is discussed in some detail leading quite naturally to chapters on potential theory, Laplace's equation, and applications to the theory of gravitation and to hydrodynamics. A short chapter on four-dimensional vectors, several sets of exercises, a bibliography, and a good index are included.

The author has succeeded quite well in his effort to be as comprehensive as space would permit, but as he realizes, completeness is unattainable in so small a book.

The reviewer is of the opinion that a more rigorous and yet brief and elementary approach to the concept of vectors could have been made. He would have liked to see the traditional "consecutive points" approach to the development of the equation of the osculating plane replaced by something more tangible. A little more space would have permitted the inclusion of sections on geodesics and mapping. The proof given on page 43 in regard to the relation between work potential and the force components does not seem convincing. While it is common for writers on vector analysis to pass lightly over (or omit) the conditions necessary for the validity of such theorems as Green's and Gauss' theorems, the reviewer is of the opinion that an elementary yet more careful statement as to these conditions would have greatly improved the book.

In the light of its scope and purpose this book is well done and seems suited to the needs and traditions of the English school of mathematics. Many undergraduate students in American universities and colleges would find it difficult to use as a text-book, largely due to the fact that their background in mechanics and theoretical physics is insufficient for much of the material given in the chapters on mechanics, potential theory, and hydrodynamics; yet, the reviewer is thoroughly convinced that students of mathematics in this country should devote more attention to mechanics, physics, and related subjects, and accordingly recommends that this excellent little book be consulted by both teachers and students.

R. S. BURINGTON

Determinants and Matrices. By A. C. Aitken. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 7+135 pages. 4/6s.

The little book under review is an introduction to the theory of determinants and matrices and is very well done, considering the very definite limitations of space imposed on the author; the language is clear and concise and the material is handled with a minimum of unnecessary verbiage although the presence of such detail at the very beginnings of the subject is undoubtedly conducive to greater understanding on the part of the beginner.

The chapter headings are as follows: I. Definitions and fundamental operations of matrices; II. Definition and properties of determinants; III. Adjugate and reciprocal matrix, solution of simultaneous equations, rank and linear dependence; IV. Cauchy and Laplace expansions, multiplication theorems; V. Compound matrices and determinants, dual theorems; VI. Special determinants—alternant, persymmetric, bigradient, centrosymmetric, Jacobian, Hessian, Wronskian.

The first four chapters are very well done, the remaining two not so well, largely because the author attempts to crowd a mass of detailed material into a relatively small space; this fact is unfortunate because of the beautiful relations contained in much of this subject-matter.

The author inserts occasional exercises which serve both to illustrate facts already proved and to point out various simple theorems requiring proofs. In this way, for example, he connects the theory of matrices with the theory of the reduction of quadratic forms, but it seems as if insufficient emphasis is placed upon some topics due to this relegation to a list of exercises.

The reader may have already guessed that the author confines his attention to matrices over the field of ordinary complex numbers (although this restriction is nowhere explicitly stated). Much is lost by thus restricting the field; certainly attention should be called to the matric representation of quaternions and, even if the subject is not pursued further, to the concept of a linear associative algebra expressed as an algebra of matrices.

A final criticism is that quite frequently a general result is "proved" by carrying out the necessary details in a special case; this practice is resorted to too often in the book to inculcate rigorous habits in one attempting to pursue the subject further.

In summary, this reviewer feels that the book is easy to read, well organized, and of value as collateral reading by undergraduates but that the author has failed to effect any coördination of the material with modern algebra; such coordination could easily have been effected with no increase in the number of the pages by simply omitting Chapter IV. If Chapter V had been expanded somewhat, much improvement would have resulted; as matters stand now, the numerous topics treated and their cursory treatment give that chapter a very disjointed effect.

D. M. Dribin

Elementary College Mathematics. By E. L. Mackie and V. A. Hoyle. Boston, Ginn and Company, 1940. 424 pages. $2.80.

Elementary College Mathematics consists essentially of algebra and trigonometry, *treated separately*, with a slight introduction to analytics and the calculus.

The first chapter gives a long review of elementary algebra in order to help poorly-prepared students. The quadratic equation, determinants, and the theory of equations are adequately treated in other chapters. The material is frequently enriched by interesting applications. For example, the chapter on the quadratic equation includes an article on maxima and minima; and the graphical representation of functions is illustrated by areas, and average and instantaneous rates treated graphically. With regard to choice of material, the reviewer wishes the authors had substituted for the long discussion of determinants a more practical and cultural topic such as probability.

In analytics the book takes up only straight lines in connection with linear equations, and parabolas as an application of the quadratic equation. The chapter in the calculus deals with the simplest applications of differentiation and integration. The treatment is readable and enables the student to obtain fairly clear ideas of the methods of the calculus and their uses in modern life. This is mathematics for culture.

The last two chapters are devoted to a separate treatment of the usual course in plane trigonometry. One serious criticism of this treatment is that the applications of trigonometry are not sufficiently emphasized, and the number of statement problems is very small. An excellent set of five-place tables is included.

One of the best features of this book is the large number of well chosen illustrative examples throughout the text. The exercises consist of two parallel groups, one with answers at the end of the book, and one without answers to develop self-reliance.

On the whole, *Elementary College Mathematics* is interesting and teachable. It is suitable as a cultural course for those who plan to take mathematics for one year only, or as a preparatory course for those who intend to continue the study of mathematics. The typography is excellent. Teachers seeking a good text-book for a unified course will find this one well worth a careful examination.

J. Shibli

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.

At the present time there are over one hundred and sixty mathematics clubs listed in the files and records of this department. We hope that this year will be an active one for each of our large group of clubs and at the same time it will bring forth an increasing number of new clubs. May each organization continue to be of service to the work in mathematics carried on by the college and a real benefit to each of the individual members of the organization. As soon as new mathematics groups are organized, this department would appreciate hearing of the plans of the organization so as to announce these clubs in an early issue of this publication. Some new clubs reporting for the first time will be reviewed in this issue and reports from other new clubs will appear at a later time.

A number of universities and colleges report activities of mathematics clubs on their campus during some years and again at other times write that no active programs were held. For the benefit of those departments which do not wish to have a formal organization yet are interested in programs on popular subjects in mathematics for majors as well as other undergraduates, the suggestion of undergraduate mathematics lectures from Princeton University may prove helpful. We should welcome hearing from other departments which sponsor no mathematics clubs, regarding the activities planned each year for the benefit of their undergraduates and the general public.

UNDERGRADUATE MATHEMATICS LECTURES, PRINCETON UNIVERSITY

These lectures dealt with history of mathematics, applications of mathematics, and topics in pure mathematics and were usually given by members of the faculty. Many of the subjects treated were not new to most members of the audience, but the treatments did acknowledge the main logical difficulties encountered. A mimeographed outline was distributed before the start of each lecture, and this outline contained references to complete mathematical and historical treatments of the subject. The audience was also referred to A. N. Whitehead's *An Introduction to Mathematics* and E. T. Bell's *Men of Mathematics* as containing an adequate treatment of many of the subjects in the series. Provision was also made at each lecture for answering questions raised by members of the audience, and suggestions concerning these as well as topics for other lectures were welcomed by the committee in charge. Since the meetings were planned for the benefit of undergraduates only, faculty members and graduate students were not encouraged to attend.

Among the topics treated at the lectures during the past two years were the following:

Euclid's Elements, Book I
The arithmetic of infinite numbers
Geometry of four and more dimensions

The rôle of analogy in mathematics
Nuclear models
Mathematical induction
Groups
How to win at games of chance
What is an integer?
Non-euclidean geometry

CLUB TOPICS

For the benefit of those clubs which did not receive our spring letter, we should like to call attention to the following titles as timely for discussion at early fall meetings:

1. Apportionment of representatives
2. Proportional representation and preferential voting
3. Sample ballots and predicting election results
4. Forecasting the population of the United States
5. Mathematics and defense.

Professor Bushey of Hunter College of the City of New York sent us the references on the topics Apportionment of representatives, and Proportional representation and preferential voting which had been used as the basis for discussion at meetings of the *Hunter* chapter of *Pi Mu Epsilon.* Professor E. V. Huntington of Harvard University spoke on the topic of Congressional reapportionment at a recent meeting of the *Mathematics Club* of *Boston University*, and this suggests that other clubs may find this topic as worthwhile for discussion. The title "Your Chance to Win"* might seem appropriate for meetings prior to November 5th based on sample ballots and predictions of election results. At a meeting of the *Oregon Beta* chapter of *Pi Mu Epsilon* at *Oregon State College*, Dr. W. J. Kirkham discussed an aspect of this topic under the title "Statistical reasons for the apparent failure of the 1936 Literary Digest poll—how the figures should have been interpreted." Clubs or individuals finding suitable references on the topic of sample ballots should send these to this department as soon as possible, so that these too may be passed on to other interested clubs.† United States 1940 census figures will be available in a few months so the topic of population growth and the prediction of census figures may be another topic for study. We find an excellent article by David L. Kaplan in *Math X*, the magazine published by the *Mathematics Club* of *New York University*, *Washington Square*

* *Your Chance to Win*, by H. C. Levinson, Farrar and Rinehart, New York, 1939 deals with the laws of probability in various games of chance and is in itself suitable for use for program discussions.

† As we go to press we have the announcement of *The Pulse of Democracy: The Public Opinion Poll and How it Works*, by George Gallup and Saul F. Rae, Simon and Schuster, New York, 1940, \$2.50, which tells, with reservations, how the Gallup Poll and similar polls work. The method of representative sampling is discussed and the relative accuracy of the Literary Digest polls based on automobile owners and telephone subscribers in years previous to 1936 and the poll's failure in that year.

College, November 1939 issue, entitled "Forecasting the Population of the United States" and call to the attention of our readers this article as well as its bibliography appearing below.

Mathematics and defense is another subject which will be listed on many programs this year. A phase of it was discussed under the title "Mathematics and Military Applications" at a meeting of the *Alpha of Georgia* chapter of *Pi Mu Epsilon* at the *University of Georgia*. The references below, from earlier numbers of this MONTHLY, may prove helpful. Additional references will be welcomed by this department from those considering this and related topics in more detail.

51. *Apportionment of Representatives.*

Chafee, Z. Jr. Congressional reapportionment. Harvard Law Review, 42: 1015–47.

Huntington, E. V. Apportionment of representatives. Journal of the American Statistical Association, 17: 859–70.

Huntington, E. V. Apportionment of representatives in congress. Transactions of the American Mathematical Society, 30: 85–110.

Huntington, E. V. Methods of apportionment in congress. American Political Science Review, 25: 961–5.

Huntington, E. V. Apportionment of representatives in congress. Cowles Commission: 1937, 47–50.

Owens, F. W. On the apportionment of representatives. Journal of the American Statistical Association, 17: 958–68 and 1004–12.

52. *Proportional Representation and Preferential Voting.*

Hoag, C. G. and Hallett, G. Proportional Representation. Macmillan, New York, 1926.

Humphreys, John H. Proportional Representation.

Dines, L. L. Concerning preferential voting. This MONTHLY, 24: 321–5.

Lovitt, W. V. Preferential voting. This MONTHLY, 23: 363–6.

53. *Forecasting the Population of the United States.*

Burgess, R. W. Introduction to the Mathematics of Statistics. Houghton-Mifflin, 1927, 169–71.

Croxton, F. W. and Cowden, D. J. Practical Business Statistics. Prentice-Hall, 1934, 331–4.

Day, Edmund E. Statistical Analysis. Macmillan, 1930, 265, 266.

Mills, Frederick C. Statistical Methods. Henry Holt, 1938, 676.

Pearl, Raymond. The Biology of Population Growth. Knopf, 1925, 3, 4, 13, 14, 219.

Pearl, R. Introduction to Medical Biometry and Statistics. W. B. Saunders Co., 1930, chapter 17.

Standard Statistics Company. Standard Trade and Securities. Vol. 90, No. 24, Dec. 23, 1938, 71, 72.

World Almanac, 1939, 298, 325.

54. *Mathematics and Defense.*

Roever, W. H. Drawings and graphical solutions in navigation. This MONTHLY, 25: 415–27.

Whittemore, J. K. Firing data. This MONTHLY, 25: 360–70.

Richardson, R. G. D. Courses in college in preparation for the navy. This MONTHLY, 25: 1918, 321–5.

Mathematical instruction at naval training centers. This MONTHLY, 25: 326–8, 370.

Roman, I. A note on war savings stamps. This MONTHLY, 28: 307–8.

Laves, Kurt. How the map-problem was solved in the war. This MONTHLY, 26: 181–7.

Evans, G. W. Concerning haversines in plane trigonometry. This MONTHLY, 26: 69–72.

Ford, L. R. Elementary mathematics for field artillery. This MONTHLY, 26: 353–5.

Dadourian, H. M. Acoustic circles. This MONTHLY, 28: 111–4.

CLUB REPORTS, 1939–1940

NEW CLUBS

Mathematics Club, Campbell College

This club was organized in February, 1940 under the sponsorship of Miss Lucille Rorex. Two chapel programs were arranged during the remainder of the year at which the guest speakers were Professor H. A. Fisher of North Carolina State College and Professor W. W. Rankin of Duke University. Six semi-monthly meetings of the club were held after the constitution had been completed and ratified. Each meeting was characterized by a particular theme, upon which subject three or four student members spoke. These themes were: Development of numerals and signs, Men of mathematics, Mathematics for the million, Astronomy, Mathematical amusements. Officers for the first year were: Charles Scalzott, President; George Veitch, Vice-President; Peggy Lasater, Secretary-Treasurer; Miss Lucille Rorex, Adviser.

Mathematics Club, Pasadena Junior College

This club was organized on January 9, 1940 and has for its purpose to stimulate an interest in mathematics and to discuss topics not usually met in the classroom. Meetings were held every two weeks, with the programs planned and given by the students. The program chairman, Mr. H. Martens, gave mathematical problems and riddles at each meeting. Among the topics discussed were: The three famous problems, The history of the mathematical plus and minus sign, Constructions with ruler and compass, Fourth dimension, The history of the mathematical multiplication and division sign, A short biographical sketch of Archimedes, The history of the mathematical power and root sign, A short biographical sketch of Galileo. Two special lectures were given, by Dr. Saul Pollock who spoke on Space curves in the third dimension and Dr. William Pickering whose subject was The quest of the cosmic ray.

Kappa Mu Epsilon, Coker College

The *Mathematics Club* of *Coker College* became the *South Carolina Alpha* chapter of *Kappa Mu Epsilon* on April 5, 1940 when Miss Orpha Ann Culmer and four other members of the *Alabama Beta* chapter installed this chapter. Two meetings have been held each month and the programs were planned to show the close relationship of mathematics to other fields of learning. In addition there were lectures by professors in other departments, in which each stressed the need of a knowledge of mathematics for the work of his department. Two of these meetings were open meetings in an attempt to further develop the interest in and appreciation of mathematics not only for the members of the club but also for the students on the campus. Installed as the officers of this new chapter were: Frances Humphries, President Leibnitz; Edith Mitchell, Vice-President Pascal; Eunice Mitchell, Secretary Thales; Elsie Neighbors, Treasurer Gauss; Miss Caroline M. Reaves, Corresponding Secretary Descartes.

Archimedean Club, Winthrop College

This was another new club to report to this department. It devotes most of its programs to a discussion of problems arising in the teaching of mathematics. Three films—The isograph, Parabola, and Rectilinear coördinates—were shown at meetings in which members from other departments also participated. Officers were: James Small, President; Margaret Burgess, Vice-President; Annie Sarah Higgins, Corresponding Secretary; Margaret Messinger, Recording Secretary; Nellie Jackson, Treasurer; Claudine Derrick, Reporter.

Mathematics Club, McMaster University

Although this club has been in existence for eight years, the first report to this department was received recently from the President-elect Mr. Edward Munn. The club's program included talks on the following subjects: High school teaching in Ontario, Probability in modern physics,

Bessel functions. During the year the club visited the David Dunlap Observatory in Toronto. Plans for this year call for composite meetings including student talks, puzzles, plays, and quizzes at each meeting. Topics for these papers were assigned before the close of the school year in May, and include: The slide rule, Mathematical reasoning, The wheel and the cycloid, Magic squares, Inversion of plane analytic curves, Theorem of Fermat, Japanese mathematics, Pythagoras, History and theory of pi. Other members of the executive committee are: Wilda Morrow, Vice-President; Anne Bishop, Secretary-Treasurer; Glyn Reesor, Marion Brown, and Robert Graham as senior, junior, and sophomore representatives, respectively.

Rho Theta, St. Louis University

This honorary society has been meeting for the past four years and has held monthly discussion meetings on the following topics: Certain theorems of analytic geometry, A historical sketch of the theory of irrational numbers, The meaning and function of a picture. Other topics dealt with the use of mathematics in astronomy, photography, chemistry, physics, and geology. Officers were: Professor Francis Regan, Director; Richard Brooks, President; Alexander Yokubaitus, Vice-President; John McCann, Secretary-Treasurer.

Mathematics Club, Alfred University

Another new group reporting to this department enclosed its constitution which has been placed on file, and which could very well serve as a model for groups considering the organization of a mathematics club on their campus. The object of the organization is to promote a genuine interest in the study and the promotion of mathematics. Seniors and graduate students majoring in mathematics with a 2.00 average and juniors with a 2.20 average are eligible for membership in an honorary branch known as *Pi Delta Mu*. This name was chosen from the Greek translation of the motto, "Let us advance through mathematics." Particular honors will be given to the senior student at the end of each year who has been the most outstanding in the study of mathematics throughout the undergraduate years. The club presented the college library with a number of popular books on mathematics. Guest speakers included Professor Gehman of the University of Buffalo, Professor Aude of Colgate University, and Professor Walker of Cornell University who spoke on Averages, The use of matrices in elementary college mathematics, and Magic squares, respectively. Officers were: Esther Gent, President; Mildred Haerter, Secretary; Beth Olszowy, Treasurer.

Mathematics Club, Tennessee Polytechnic Institute

This club was organized October 10, 1933 under the direction of Dr. R. O. Hutchinson for the purpose of furthering mathematical endeavor and showing the progress mathematics has made since ancient times. In its first report to this department we learn that the loving cup given by the Christian Association for the best club stunt at the annual college night program was won by the *Mathematics Club* for its original stunt written by Dr. James A. Ward and entitled "A Wax Works Show." A copy of this is on file in our Stunt Library and is available for club use. Topics discussed at the regular meetings were: Matrices, Methods of solving quadratic equations, Trisection of angles, Pythagorean theorem, History and evolution of pi, The constant e, and "Flatland Fantasy." Officers were: Dr. James A. Ward, Director; Elizabeth Eastland, President; Millard De Berry, Vice-President; Margaret Plumlee, Secretary; Joseph Lane, Treasurer.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 430. *Proposed by Louis Bauer, Hofstra College, Hempstead, N. Y.*

Somebody received a check, calling for a certain amount of money in dollars and cents. When he went to cash the check, the teller made a mistake and paid him the amount which was written as cents, in dollars, and vice versa. Later, after spending \$3.50, he suddenly realized that he had twice the amount of money the check called for. What was the amount on the check?

E 431. *Proposed by V. Thébault, Le Mans, France.*

Find a number whose cube and fourth power together contain the ten digits, once each. (*Cf.* E 116.)

E 432. *Proposed by C. W. Trigg, Los Angeles City College.*

If a and b are the radii of two spheres, tangent to each other and to a plane, show that the radius x of the largest sphere which can pass between them is given by the formula

$$x^{-1/2} = a^{-1/2} + b^{-1/2}.$$

E 433. *Proposed by A. A. Bennett, Brown University.*

Two parallel vertical walls, separated by a distance of d feet, have level ground between them. Two ladders, of length a and b feet respectively ($a > b$), abut each against a foot of one of these walls and lean against the other wall, crossing each other at a height of c feet above the ground. Show that a solution in integers is given by

$$\begin{aligned} ka &= (su + tv)(s - t)(u + v), \\ kb &= (sv + tu)(s - t)(u + v), \\ kc &= (su - tv)(sv - tu), \\ kd &= 2(stuv)^{1/2}(s - t)(u + v), \end{aligned}$$

where s, t, u, v are any positive integers subject to the three conditions $u > v$, $sv > tu$, and $stuv$ is a perfect square, k being the greatest common divisor of the four right-hand members. What is the simplest particular solution in which a, b, c, d are all odd?

E 434. *Proposed by Daniel Arany, Budapest, Hungary.*

Let F_1 and F_2 be the foci of a variable ellipse, of major axis $2p$, inscribed in a given triangle whose orthocenter is H. Prove that $4p^2-HF_1^2-HF_2^2$ is constant.

E 435. *Proposed by David Segal, Kosow Huculski, Poland.*

Show that the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

is a necessary and sufficient condition for p to be an odd prime.

SOLUTIONS

E 363 [1939, 106]. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

Construct triangle ABC, given A, a, and (h_a+c-b).

Solution by V. W. Graham, Dublin, Ireland.

Let the bisector of the angle A meet the circumcircle of the triangle at D. Draw the diameter DF, perpendicularly bisecting BC at E. Let AG be the perpendicular from A to DF. From the right triangles DFC, DFA, we have

$$DC^2 = DF\cdot DE, \qquad FA^2 = FD\cdot FG.$$

Hence

$$DC^2 + FA^2 = DF(DF - EG) = 2R(2R - h_a). \tag{1}$$

Again, by Ptolemy's theorem,

$$DF\cdot CA \mp DC\cdot FA = DA\cdot FC = DA\cdot FB = DF\cdot AB \pm DB\cdot FA.$$

(The upper sign is to be taken if $b>c$, the lower if $b<c$.) Hence

$$\pm 2DC\cdot FA = DF(CA - AB) = 2R(b - c). \tag{2}$$

Adding (1) and (2),

$$(FA \pm DC)^2 = 2R\{2R - (h_a + c - b)\}.$$

Since a and A are given, DC and R are known. Hence FA may be found, and the vertex A located.

Also solved by V. Thébault and the proposer.

E 396 [1939, 652]. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

Given a triangle ABC, construct a point X such that the three lines drawn through X, each parallel to a side of the triangle and limited by the other two sides, are equal.

Solution by C. C. Oursler, Lancaster High School, Ill.

By drawing circles with centers A, B, C and respective radii BC, CA, AB, construct points A', B' which complete the parallelograms $CABA'$, $ABCB'$. Let L, M be the intersections of the bisectors of angles A, B with the respective opposite sides of the triangle. The intersection of the lines $A'L$, $B'M$ is the required point X.

The proof consists essentially in showing that $A'L$ is the locus of points from which lines drawn parallel to CA and AB intercept equal segments between the sides of the triangle. Through any point P on AL, draw lines parallel to CA and AB; let the former meet AB and BC in D and E, and let the latter meet BC and CA in F and G. Let X' be the intersection of lines drawn through E and F parallel to AB and CA, respectively. Since the segments of EX' and FX' cut off by the sides of the triangle are equal to PG and PD, which are sides of a rhombus, the point X' satisfies the conditions of the locus. Since parallelogram $EPFX'$ is homothetic to parallelogram $CABA'$, with L as center of similitude, the locus of X' (as P moves along the bisector of angle A) is a line. Hence the points A' and L, which lie on it, are sufficient to determine it. Similarly, there is a locus intersecting each of the sides, and the required point X is determined by any two of the three loci.

Also solved by W. E. Buker, Mannis Charosh, L. R. Chase, W. B. Clarke, A. A. Coffin, Wm. Douglas, V. W. Graham, L. M. Kelly, J. E. LaFon (with two extensions to three dimensions), W. R. McEwen, E. P. Starke, C. W. Trigg, and the proposer. Several of the solvers pointed out that the length of the three equal segments is $2/(a^{-1}+b^{-1}+c^{-1})$.

E 397 [1939, 652]. *Proposed by H. T. R. Aude, Colgate University.*

The symbols a, b, c, d, e, f are distinct digits in the denary scale. Find the three-figured number abc such that $(abc+1)^2=acdef$, and $(a+b+c)^{1/3}+1=(a+c+d+e+f)^{1/3}$.

I. *Solution by B. C. Zimmerman, Corozal, British Honduras.*

The perfect cubes in the second condition are clearly 8 and 27; therefore $a+b+c=8$. By the first condition,

$$a = 1 < b \leqq c/2.$$

Hence $b+c=7$, and the required number is 125.

II. *Solution by C. C. Sams, Mars Hill College, N. C.*

Since abc and $(abc+1)^2$ are to have the same first digit, we find $a=1$, and so $b>1$, $c>1$. We cannot have $b=0$ without $c=0$ or 1, save in the case $abc=109$, which makes $e=f$. Hence $acdef \leqq 19876$, and

$$120 \leqq abc \leqq 139.$$

Within this range there are six numbers N such that N and $(N+1)^2$ each have

distinct digits, as in the following table:

$N=abc$:	123,	125,	127,	132,	135,	136;
$(N+1)^2=acdef$:	15376,	15876,	16384,	17689,	18496,	18769.

Since the c's must agree, the only satisfactory value for N is 125.

The second condition is superfluous.

Also solved by W. E. Buker, Daniel Finkel, H. D. Larsen, H. R. Leifer, Hazel E. Schoonmaker, E. P. Starke, C. W. Trigg, Alan Wayne, G. A. Williams, and the proposer.

E 399 [1939, 653]. *Proposed by V. Thébault, Le Mans, France.*

Prove that the product of the first n positive integers $(1\cdot 2 \cdots n)$ is divisible by their sum $(1+2+\cdots+n)$ if and only if $n+1$ is not an odd prime.

Solution by Elmer Latshaw, Philadelphia.

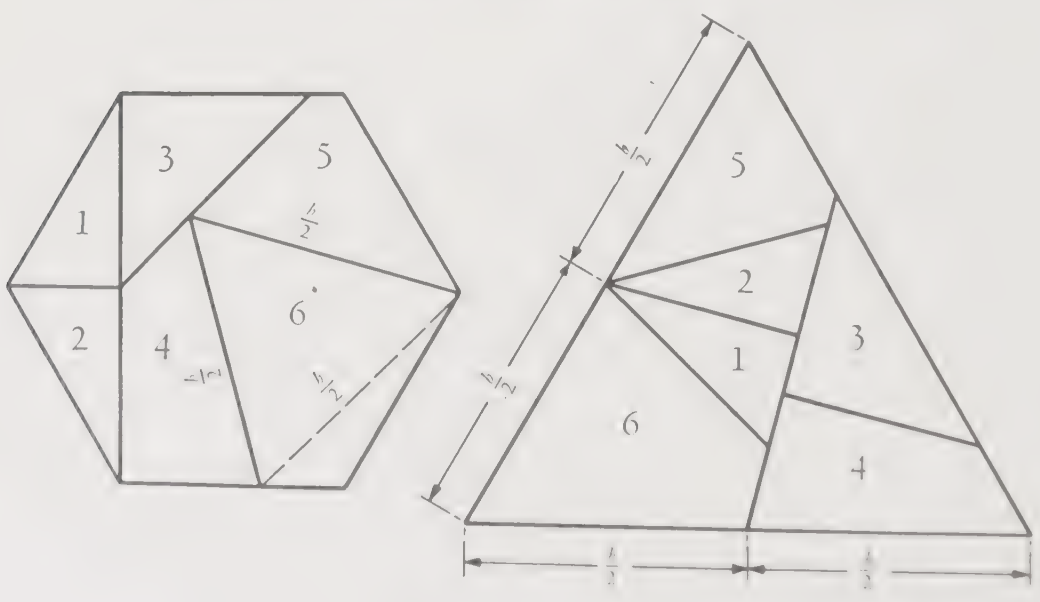
The sum divides the product if $n+1$ divides $2(n-1)!$. If $n+1$ is even, the factor $(n+1)/2$ occurs in $(n-1)!$. If $n+1$ is an odd square, $(n-1)!$ contains factors $(n+1)^{1/2}$ and $2(n+1)^{1/2}$. If $n+1$ is odd and composite, but not a square, it is the product of two unequal odd factors occurring in $(n-1)!$. Hence the only case when $n+1$ does not divide $2(n-1)!$ is when $n+1$ is an odd prime.

Also solved by W. E. Buker, Mannis Charosh, H. L. Lee, C. W. Moran, E. P. Starke, C. W. Trigg, Alan Wayne, G. W. Wishard, and the proposer.

E 400 [1939, 653]. *Proposed by H. S. M. Coxeter, University of Toronto.*

Show how to dissect a regular hexagon by straight cuts into the smallest possible number of pieces which can be reassembled to form an equilateral triangle (of the same area).

Solution by Michael Goldberg, Washington, D. C.



Also solved by Michael Goldberg (another way) and R. C. Yates (with the assistance of O. A. Nance and J. S. Smart), one of the pieces being in each case the trapezoid formed by one-half of the hexagon. Since there are at least three different six-piece solutions, one is tempted to investigate the possibility of a five-piece solution. Such a solution, or a proof of impossibility, will still be welcomed.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3965. *Proposed by H. S. Wall, Northwestern University.*

Show that for a real or complex x, $|x| \leqq 1$,

$$\frac{|x|}{1+|x|} \leqq |\log(1+x)| \leqq \frac{|x|(1+|x|)}{|1+x|}.$$

3966. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Consider the surface with the equation in rectangular coördinates $x^2+y^2+z^2=xyz+4$, which has the parametric representation $x=u+1/u$, $y=v+1/v$, $z=u/v+v/u$. Determine the rectilinear generators and the asymptotic lines.

3967. *Proposed by V. Thébault, Le Mans, France.*

For a given triangle ABC a second triangle $A'B'C'$ is formed where AA', BB', CC' are segments of altitudes and $AA'/BC=BB'/CA=CC'/AB=k$. (1) Show that the two triangles have the same centroid. (2) Examine the variation of the area of $A'B'C'$. (3) For what value of k do the two triangles have the same angle of Brocard? (4) If $k=\pm 1$, show that the centers of squares constructed exteriorly, or interiorly, on the sides of $A'B'C'$ are the vertices of ABC.

SOLUTIONS

3840 [1937, 395]. *Proposed by V. Thébault, Le Mans, France.*

Parallel planes, with arbitrary direction, drawn through the vertices A, B, C, D of a tetrahedron cut a given straight line Δ in $\alpha, \beta, \gamma, \delta$. The planes through the latter four points parallel, respectively, to the faces BCD, CDA, DAB, ABC determine by the intersections of sets of three a tetrahedron $A_1B_1C_1D_1$ symmetrically equal to the given tetrahedron $ABCD$ (J. Neuberg, *Mathesis*, 1891, p. 50). Find the loci of the vertices of the tetrahedron $A_1B_1C_1D_1$ and of the

center of symmetry of the two tetrahedrons when the direction of the parallel planes varies. This is an extension to space of 3817 [1937, 111].

Solution by R. Goormaghtigh, Bruges, Belgium.

Let DA, DB, DC be the axes of coördinates and $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ the coördinates of A, B, C. The coördinates of any point on Δ are

$$\xi + \lambda t, \qquad \eta + \mu t, \qquad \zeta + \nu t,$$

where ξ, η, ζ and λ, μ, ν are constants. The considered plane drawn through A has for its equation

$$l(x - a) + my + nz = 0,$$

and α is the point

$$\xi + \lambda(la - P)/Q, \cdots,$$

where

$$P = l\xi + m\eta + n\zeta, \qquad Q = l\lambda + m\mu + n\nu.$$

Hence, if the considered symmetry center exists, its coördinates are

$$u = \tfrac{1}{2}\xi + \lambda(la - P)/2Q, \cdots. \tag{1}$$

Indeed, the point δ has as coördinates

$$\xi - \lambda P/Q, \cdots,$$

and the plane equidistant from the face ABC and from the parallel plane drawn through δ is

$$\sum \frac{x}{a} - \frac{1}{2} \sum \frac{1}{a}\left(\xi - \lambda \frac{P}{Q}\right) - \frac{1}{2} = 0;$$

this equation is verified by the coördinates (1), which are therefore those of the considered center of symmetry.

Let now $(2u-\xi)/\lambda$, $(2v-\eta)/\mu$, $(2w-\zeta)/\nu$ be denoted by X, Y, Z; we have

$$l(\lambda X + \xi - a) + m(\mu X + \eta) + n(\nu X + \zeta) = 0,$$
$$l(\lambda Y + \xi) + m(\mu Y + \eta - b) + n(\nu Y + \zeta) = 0,$$
$$l(\lambda Z + \xi) + m(\mu Z + \eta) + n(\nu Z + \zeta - c) = 0.$$

The elimination of l, m, n leads easily to a linear equation in X, Y, Z; hence the loci of the symmetry center and the vertices of A_1, B_1, C_1, D_1 are five parallel planes.

Editorial Note. The second half of this argument is synthetic. From the above solution we see that the coördinates x, y, z of the vertex D_1 are fractions with numerators and denominators as linear homogeneous expressions in l, m, n and the denominators are the same, Q. Hence the locus of D_1 is a plane, straight

line, or a point; and in the latter two degenerate cases, Δ lies in a face of $ABCD$. This last statement may be proved analytically, but it is better to wait for the simpler synthetic proof below. We shall first assume that Δ does not lie in any face of $ABCD$; and then it follows that the loci of A_1, B_1, C_1, D_1 are planes, and Δ cuts the faces BCD, CDA, DAB, ABC in the definite points A', B', C', D'. Since Δ is a finite straight line we may suppose that D' is a finite point. The planes through the vertices parallel to ABC give $\alpha=\beta=\gamma=D'=D_{14}$, while A_{14}, B_{14}, C_{14} lie in a plane through D parallel to ABC. Hence this particular tetrahedron $A_{14}B_{14}C_{14}D_{14}$ is the symmetric of $ABCD$ with respect to the midpoint of DD'. Similarly, we have $A_{11}B_{11}C_{11}D_{11}$, $A_{12}B_{12}C_{12}D_{12}$, $A_{13}B_{13}C_{13}D_{13}$ with centers of symmetry with $ABCD$ at the midpoints of AA', BB', CC', respectively; and these three are simply translations of the first one by the vectors $A_{14}A'$, $B_{14}B'$, $C_{14}C'$. In order to prove that the above four plane loci are parallel we have merely to show that these three translator vectors are not all parallel. When this is proved we will have shown that the loci cannot be degenerate, and we shall have the theorem that the midpoints of AA', BB', CC', DD' lie in a plane, the locus of the centers of symmetry; the four plane loci of the vertices are parallel to this plane; and obviously the theorem of Neuberg cited in the problem is a corollary. If Δ passes through D, then $A'=B'=C'=D$, and it is obvious that not all three vectors of translation are parallel. If Δ does not pass through D, at least two points, say A', B' are distinct; and, if the three vectors were parallel, A', B', C', A_{14}, B_{14}, C_{14} would lie in a plane, and hence Δ lies in the plane of $A_{14}B_{14}C_{14}$ which is parallel to ABC. But then D' would have to be at infinity, contrary to our hypothesis; and thus the above theorem is proved.

We now consider the case where Δ lies in a face, say ABC, and does not contain two of its vertices; then A', B', C' are also in the plane of this face. Moreover, $A\alpha$, $B\beta$, $C\gamma$, are parallel straight lines; the plane through, say α, parallel to BCD cuts the plane of ABC in a line parallel to BC, while the plane through δ is the plane of ABC. Thus the face $A_1B_1C_1$ is in the plane of ABC, and by the above analytic proof and the synthetic considerations, the loci of A_1, B_1, C_1 are straight lines in this plane; obviously, $A_{11}B_{11}C_{11}$, $A_{12}B_{12}C_{12}$, $A_{13}B_{13}C_{13}$ are symmetrics of ABC; and the loci of A_1, B_1, C_1 are parallel straight lines in the plane of ABC, while the locus of the centers of symmetry is a parallel passing through the midpoints of AA', BB', CC'. Thus in this degenerate case the loci of the vertices of $A_1B_1C_1D_1$ are straight lines parallel to the locus of the centers of symmetry determined by the midpoints of AA', BB', CC'. If Δ passes through A but not through B or C, two of the loci coincide. If Δ passes through B and C, then $\beta=B, \gamma=C$ for all systems of parallel planes, and there is only one triangle $A_1B_1C_1$; and there is only one $A_1B_1C_1D_1$, the symmetric of $ABCD$ with respect to the midpoint of BC. This is the second degenerate case where all five loci are points. We have thus proved the theorem of 3817 [1937, 111] with its single degenerate case.

When we consider the extension of the theorem to n dimensions, the first part by analysis is obvious from the above, and there are $n-1$ possible degener-

ate cases. The second part of the argument offers only slight difficulty, and perhaps it may be convenient to use induction.

3885 [1938, 482]. *Proposed by V. Thébault, Le Mans, France.*

The product of n consecutive positive integers, n being odd, is divisible by their sum, except in the case where, n being prime, the arithmetic mean of the n integers is divisible by n. Examine the case where n is even.

Solution by E. P. Starke, Rutgers University.

The integer n being odd, let p be a prime divisor of n:

$$n = 2r + 1 = sp^i, \qquad i > 0, \qquad s \text{ not divisible by } p. \tag{1}$$

If m is the arithmetic mean, the n consecutive integers are

$$m - r,\ m - r + 1, \cdots, m + r. \tag{2}$$

The sum, Σ, of the numbers (2) equals mn. Their product, Π, is the product of m by the products of two sets of r consecutive integers,

$$(m - r)(m - r + 1) \cdots (m - 1) \quad \text{and} \quad (m + 1)(m + 2) \cdots (m + r).$$

Let $r/p = a + b/p$, $0 < b < p$. Then from $2r+1 = 2ap + (2b+1) = sp^i$, we have $2b+1 = p$ and $a = \frac{1}{2}(sp^{i-1} - 1)$. Thus $r/p > a = \frac{1}{2}(sp^{i-1} - 1) \geqq \frac{1}{2}(p^{i-1} - 1) \geqq \frac{1}{2}i$ for all $i \geqq 2$, $(p \geqq 3)$. Now each of the above sets of r consecutive integers contains a complete sets of p consecutive integers each. Since one integer out of each p consecutive integers is divisible by p, the product of the r integers is divisible by p^a. Hence Π/m is divisible by p^{2a} and thus by p^i. If now $i = 1$, s of the integers (2) are divisible by p; hence Π/m is divisible by p^{s-1}. Thus Π/m is divisible by p unless $s = 1$, $i = 1$, *i.e.*, $n = p$. In other words, if $n \neq p$, Π/m is divisible by n since it is divisible by every prime factor of n to an exponent at least as great as occurs in n; hence also, Π is divisible by $\Sigma = mn$. Finally if $n = p$, only one of the integers (2) is divisible by p. If that integer is not m, then Π/m is divisible by n and thus Π by Σ; if it is m, Π is not divisible by Σ.

If now n is even, then $n = 2^c \cdot d$, where d is odd and $c \geqq 1$. Let the n consecutive integers be $a - \frac{1}{2}n + 1$, $a - \frac{1}{2}n + 2, \cdots, a + \frac{1}{2}n$, or

$$\frac{(2a+1) - (n-1)}{2}, \frac{(2a+1) - (n-3)}{2}, \cdots,$$
$$\frac{(2a+1) - 1}{2}, \frac{(2a+1) + 1}{2}, \cdots, \frac{(2a+1) + (n-1)}{2}.$$

Then $\Sigma = \frac{1}{2}n(2a+1) = 2^{c-1}d(2a+1)$ and, if we combine pairs equidistant from the extremes, we have

$$\Pi = \{(2a+1)^2 - (n-1)^2\}\{(2a+1)^2 - (n-3)^2\} \cdots \{(2a+1)^2 - 1^2\}/2^n.$$

Evidently Π may be expressed as $\Pi = \{m(2a+1)^2 \pm k^2\}/2^n$, where m is some in-

teger and $k=1\cdot 3\cdot 5\cdot 7\cdot\cdot\cdot(n-1)$. Since the product of d consecutive integers is divisible by d, Π being the product of $n=2^c d$ integers is divisible by d^{2^c} and by $(2^c)^d$; hence certainly by d^2 and 2^{c-1}. Also, d being odd and less than n is a divisor of k. By the expression for Π in terms of k and m, d^2 divides $m(2a+1)^2$ and hence $d(2a+1)$ divides $m(2a+1)^2$. Thus $d(2a+1)$ divides Π if and only if it divides k^2. This is also the condition that $\Sigma=2^{c-1}d(2a+1)$ be a divisor of Π, since we have already seen that 2^{c-1} is a divisor of Π. Thus the necessary and sufficient condition that Π be divisible by Σ is that $2a+1$ be a divisor of k^2/d. Note that $2a+1$ must be greater than n if the integers are all positive, and that $2a+1$ is the sum of any two integers equidistant from the extremes.

For example, for $n=2$ there exist no two consecutive positive integers for which Σ divides Π. For $n=4$ we have $k=3$, and $2a+1$ must be a divisor (greater than 4) of 9; hence $a=4$, and the only set is 3, 4, 5, 6. For $n=6$ we have $k^2/d=75$; hence $2a+1=15$, 25, or 75 and there are three sets of six consecutive integers, *viz.*, those whose first integers are 5, 10, and 35.

Solved also by the proposer for n odd.

3889 [1938, 554]. *Proposed by V. Thébault, Le Mans, France.*

A triangle ABC is inscribed in a circle (O) with a fixed diameter Δ, and a transversal Δ', which turns about a fixed point, cuts BC, CA, AB in A_1, B_1, C_1. Let A_2 and A_3, B_2 and B_3, C_2 and C_3 be the orthogonal projections of A and A_1, B and B_1, C and C_1 on Δ. (1) Show that the circles with centers at the midpoints of AA_1, BB_1, CC_1 and passing through A_2 and A_3, and B_2 and B_3, C_2 and C_3 meet in a fixed point on the Euler circle of the triangle. (2) Find the locus of the second point of intersection of the three circles.

Solution by the Proposer.

The three circles of (1), (ω_a), (ω_b), (ω_c), whose centers ω_a, ω_b, ω_c are on the sides of the complementary triangle $A_mB_mC_m$, pass respectively through the symmetrics of A_2, B_2, C_2 with respect to its sides. Furthermore, these symmetrics coincide in the orthopole ϕ of the diameter Δ, which is a fixed point on the Euler circle of ABC. When Δ' turns about a fixed point P, the Newton line $\delta \equiv [\omega_a, \omega_b, \omega_c]$ envelopes a conic Σ inscribed in the triangle $A_mB_mC_m$. The circles (ω_a), (ω_b), (ω_c) intersect again in the symmetric ϕ' of ϕ with respect to δ. Hence the locus of ϕ' is the transformation by similitude with the ratio 2:1 of the pedal curve of ϕ with respect to Σ.

Editorial Note. The points A_m, B_m, C_m are the midpoints of the sides BC, CA, AB. The triangle ABC and Δ' form a complete quadrilateral whose diagonals are AA_1, BB_1, CC_1; and it is a familiar fact that the midpoints ω_a, ω_b, ω_c of these diagonals lie on a straight line δ, which is here termed the Newton line of the complete quadrilateral. The expression Euler circle of ABC means the circumcircle of the complementary triangle $A_mB_mC_m$, or in other words, the nine-point circle of ABC. A proof that the symmetric of A_2 with respect to B_mC_m, and the

other two similar symmetrics, coincide in a point ϕ of the nine-point circle is given in a note to the solutions of 3882, 3883 [1940, 403].

If P lies at a vertex, say A, then $B_1 \equiv C_1 \equiv A$, $\omega_c \equiv B_m$, $\omega_b \equiv C_m$, and δ is the fixed straight line B_mC_m. Hence ϕ' is a fixed point. If P lies on the straight line of a side, say AB, but not at a vertex, then $C_1 \equiv P$ and ω_c is a fixed point through which passes δ. The pedal curve of ϕ with respect to ω_c is a circle on $\phi\omega_c$ as a diameter. Hence the locus of ϕ' is a circle with center ω_c and radius $\phi\omega_c$. Conversely, if δ is fixed for two or more positions of Δ', P must be at a vertex. For, at least two of the points, say ω_a and ω_b must be fixed and distinct, and then A_1 and B_1 are fixed. But this is impossible for a variable Δ' unless $A_1 \equiv B_1 \equiv C$, and hence P must be at C. In a somewhat similar manner we can show that, if the variable lines δ pass through a fixed point, P must lie on a side of ABC not at a vertex.

Suppose now that P is a fixed point not on a side of ABC, then the lines Δ' through P cut projective ranges $\{A_1\}$, $\{B_1\}$, $\{C_1\}$ on the bases BC, CA, AB, and these ranges projected from the corresponding centers A, B, C upon the respective bases B_mC_m, C_mA_m, A_mB_m give the projective ranges $\{\omega_a\}$, $\{\omega_b\}$, $\{\omega_c\}$. Considering any pair of these last three, say the first two, the envelope of δ is a conic tangent to the two bases. When Δ' passes through C, $A_1 \equiv B_1 \equiv C$, $\omega_a \equiv B_m$, $\omega_b \equiv A_m$, and A_mB_m is a tangent element. Hence the conic is tangent to the three sides of $A_mB_mC_m$. Considering again the first two ranges, to C_m on the base B_mC_m, there corresponds on C_mA_m the intersection β of PB with this side, and thus β is the point of contact of the conic with this side. A similar construction gives the other two points of contact with the sides of $A_mB_mC_m$. Conversely, if a given conic is tangent to the sides of $A_mB_mC_m$, it is the envelope of lines δ for a suitable P. For the given conic is determined by the three tangent sides and the two points of tangency α, β of the sides B_mC_m, C_mA_m. Then the intersection of $A\alpha$ and $B\beta$ gives P, and it will follow that CP passes through γ, the point of contact with A_mB_m. If P is any finite point, the conic has a center which we now determine. Let Δ' be parallel to a side, say AB, then δ is also parallel to this side and equally distant from it and Δ'. Hence δ bisects PA and PB. Thus the midpoints A_m', B_m', C_m' of PA, PB, PC form a triangle symmetrically equal to $A_mB_mC_m$ and which also circumscribes the conic. The center of symmetry S is the center of the conic, and the points of contact of the sides of $A_m'B_m'C_m'$ are found by symmetry. We also find by symmetry with respect to S the triangle $A'B'C'$, the midpoints of whose sides are A_m', B_m', C_m', and the point P'. Thus the triangle $A'B'C'$ and P' determine the same conic. If P is at a vertex, say A, the conic degenerates into a straight line along B_mC_m with S at the midpoint of B_mC_m; while if P is on a side but not at a vertex, the conic degenerates into two points, one of which is determined as above from ABC and P and the other is determined similarly from $A'B'C'$ and P'. Thus, if P is on AB, P' is on $A'B'$ and the two points are respectively C_m' and C_m. If PB is parallel to CA, C_mA_m is an asymptote; and, if P is the symmetric of A with respect to A_m, the center is A_m and C_mA_m, A_mB_m are asymptotes. In order for the conic to be a parabola the

line at infinity must be a position of δ, and then the corresponding Δ' with P must be at infinity. Conversely, if P is at infinity not on a side of ABC, one position of Δ' is the line at infinity and then the corresponding δ is also at infinity. Hence the conic is a parabola. It will be seen that, if P is inside ABC, the conic is an ellipse; if P is a finite point outside and within an angle of ABC, the conic is a hyperbola; if outside at a finite point within a vertical angle, it is an ellipse.

3892 [1938, 631]. *Proposed by George Rutledge, Mass. Inst. of Tech.*

Show that, for all even values of n,

$$\int_0^1 \frac{1}{u}\left\{\left(\log u + \log\frac{1+u}{1-u}\right)^n - (\log u)^n\right\} du = 0.$$

I. *Solution by Fritz John, University of Kentucky.*

Set, for the positive integer n,

$$I = \int_0^1 \frac{1}{u}\left\{\left(\log u + \log\frac{1+u}{1-u}\right)^n - (\log u)^n\right\} du.$$

By the binomial theorem,

$$I = \sum_{\mu=1}^{n} I_\mu, \qquad I_\mu = \binom{n}{\mu}\int_0^1 \frac{1}{u}(\log u)^{n-\mu}\left(\log\frac{1+u}{1-u}\right)^\mu du.$$

The integrand of I_μ for the stated values of μ is easily seen to be continuous, or to become at most logarithmically infinite at $u=0$ and $u=1$. Integrating by parts, using $(\log u)^{n-\mu}/u = d/du(\log u)^{n-\mu+1}/(n-\mu+1)$, we get

$$I_\mu = -\binom{n}{\mu-1}\int_0^1 (\log u)^{n-\mu+1}\left(\log\frac{1+u}{1-u}\right)^{\mu-1}\frac{2du}{1-u^2}.$$

Setting $(1-u)/(1+u)=v$, we find

$$\begin{aligned} I_\mu &= -\binom{n}{\mu-1}\int_0^1\left(\log\frac{1-v}{1+v}\right)^{n-\mu+1}\left(\log\frac{1}{v}\right)^{\mu-1}\frac{dv}{v}, \\ &= -(-)^n I_{n-\mu+1} = -I_{n-\mu+1}. \end{aligned}$$

Therefore

$$I = \sum_{\mu=1}^{n} I_\mu = -\sum_{\mu=1}^{n} I_{n-\mu+1} = -\sum_{\mu=1}^{n} I_\mu = 0,$$

which is the desired result.

II. *Solution by W. R. Transue, Lehigh University.*

If, in the well known formula

$$\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-ux)^{-\beta}du = \frac{\Gamma(\alpha)\Gamma(\gamma-\alpha)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma,x),$$

where $F(\alpha, \beta, \gamma, x)$ is the hypergeometric function, we set $\alpha=x, \beta=-x, \gamma=1$, $x=-1$, we obtain

$$\int_0^1 u^{x-1}\left(\frac{1+u}{1-u}\right)^x du = \frac{\Gamma(x)\Gamma(1-x)}{\Gamma(1)} F(x, -x, 1, -1)$$
$$= \frac{\pi}{\sin \pi x}\left[1+\binom{x}{1}\binom{x}{1}+\binom{x+1}{2}\binom{x}{2}+\binom{x+2}{3}\binom{x}{3}+\cdots\right].$$

Since the first factor in the right member of the equality is an odd function of x and the second is even, the left member is an odd function of x.

Hence the Maclaurin expansion of

$$\phi(x) = \int_0^1 \left\{u^{x-1}\left(\frac{1+u}{1-u}\right)^x - u^{x-1}\right\} du$$

must contain only odd powers of x. Therefore $\phi^n(0)$, the nth derivative of $\phi(x)$ at $x=0$, must be 0 for all even values of n. Hence, for an even positive integer n,

$$\phi^n(0) = \int_0^1 \frac{1}{u}\left[\left(\log\frac{1+u}{1-u} + \log u\right)^n - (\log u)^n\right] du = 0.$$

Solved also by Harry Gershenson, Wang Chih Yi, and the proposer.

Editorial Note. The solution by Wang is similar to I, and the others somewhat similar to II using the function $Q(x)$ in the article by Rutledge and Douglass, *Table of definite integrals*, in this MONTHLY, 1938, p. 525, where other references are given.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Ill.

A conference in topology was held at the University of Michigan on June 24–July 6, 1940. The following lectures were given: Triangulated manifolds and differentiable manifolds, S. S. Cairns, Queens College; Topological functions, E. W. Chittenden, University of Iowa; Uniformity in topological space, L. W. Cohen, University of Kentucky; Extension and classification of continuous mappings, Samuel Eilenberg, University of Michigan; The rôle of local separating points in certain problems of continua structure, O. G. Harrold, University of Virginia; Abstract complexes, Solomon Lefschetz, Princeton University; Extensions of homeomorphisms in the plane, Saunders Mac Lane, Harvard University; Periodic and nearly periodic transformations, P. A. Smith, Barnard College; Regular cycles in compact spaces, N. E. Steenrod, University of Chicago; Topology of differentiable manifolds, Hassler Whitney, Harvard University; Uniform local connectedness, R. L. Wilder, University of Michigan; Topological transformation groups and foundations of geometry, Leo Zippin,

Queens College. It is planned to have the University of Michigan Press issue, during the coming year, a volume containing these lectures, as well as abstracts of a number of papers which were delivered during the Conference.

Professor J. L. Coolidge was awarded the honorary degree of LL.D. by Harvard University at Commencement 1940.

Professor Edward Kasner of Columbia University represented the Mathematical Association at the Conference on Science, Philosophy, and Religion in New York City, September 10 and 11.

Assistant Professor G. M. Ewing of the University of Missouri is on leave of absence for 1940–41 and will be at Princeton University for the year.

Dr. A. S. Gale of the University of Rochester has been made dean emeritus of the physical sciences.

Miss Harriet E. Glazier has retired at the University of California at Los Angeles with the title of assistant professor emeritus. She was in charge of the department of mathematics at Western College, Oxford, Ohio, from 1897 until 1920, at which time she took the position at U. C. L. A.

Dr. D. G. Fulton of Armour Institute of Technology has been appointed assistant professor at Ohio Northern University.

At Northwestern University Assistant Professor H. L. Garabedian has been promoted to an associate professorship, and Dr. M. E. Wescott has been promoted to an assistant professorship.

Assistant Professor S. H. Kimball of the University of Maine has been promoted to an associate professorship.

Professor L. C. Knight of the College of Wooster has retired after thirty-two years of teaching there.

Assistant Professor L. L. Lowenstein of Alfred University has been promoted to an associate professorship and made chairman of the department.

Dr. R. J. Michel of the University of Missouri has been appointed head of the department at Southeast Missouri State Teachers College.

P. E. Baur, associate professor of mathematics at Baldwin-Wallace College since 1920, died December 7, 1939.

Col. C. P. Echols of the U. S. Military Academy died on May 21, 1940. He was stationed at West Point for nearly forty years, the last twenty-seven of which he was professor and head of the department of mathematics. He retired in 1931. He was a charter member of the Mathematical Association.

C. R. Wilson, assistant professor of mathematics at Rutgers University, died on July 24 at the age of forty-four. He had been on the university staff for the past thirteen years.

Dr. F. G. Wren, Walker professor of mathematics at Tufts College, died on July 17, 1940. He had been a member of the faculty at Tufts College since 1894, and was dean of the faculty of arts and sciences for over thirty years.

REPORT OF THE WAR PREPAREDNESS COMMITTEE

The following report of the War Preparedness Committee of the American Mathematical Society and the Mathematical Association of America was presented at the Hanover meetings on September 9–10, 1940.

In fulfilling its mission to prepare the Society and Association to serve the country in time of war the Committee recognizes three objectives:

(1) The solution of mathematical problems essential for military or naval science, or rearmament.

(2) The preparation of mathematicians for research essential for objective (1).

(3) The strengthening of undergraduate mathematical education in our colleges to the point where it affords adequate preparation in mathematics for military and naval service of any nature. A study by a large group of mathematicians of the current routine military texts and sources wherever mathematics is involved in order that mathematicians may exert their proper influence on the teaching of military and naval science in time of war.

Organization. It is recommended that the present Subcommittee on Education be replaced by two new subcommittees corresponding to objectives (2) and (3) respectively and that the present Subcommittee on Research remain unchanged. So reorganized there would be three subcommittees designated as follows:

I. Committee on Research;
II. Committee on Preparation for Research;
III. Committee on Education for Service.

It is recognized that Committee III corresponds very closely to the special interests and aims of the Association. It is recommended that chairmen of Committees II and III be appointed at Hanover but that the completion of these committees be left until the new chairmen shall have had time to make recommendations as to the personnel of their committees.

Consultants. Having particular regard to the solution of problems, we recommend the appointment of Chief Consultants in each of the following fields: Ballistics, Aeronautics, Computation (numerical, mechanical, electrical), Cryptanalysis, Industry, Probability and Statistics. Other fields may well be added. The Chief Consultants would be responsible in research matters to the Research Committee without themselves taking on any of the executive duties of that committee. These consultants should also help the Committee on Preparation for Research when called upon. The three central committees should remain small so that they can act quickly. It is also recommended that there be *ordinary consultants* in each of the above fields. These ordinary consultants would be recommended for appointment so as to obtain some sort of correspondence with the subfields. Thus able young mathematicians so appointed might prepare themselves in advance to help in a given field. These consultants should also be appointed

so as to meet the needs of particular sections of the country for help in its industries. It is suggested that any mathematician who feels competent in a field might ask to be made a consultant, and might be accepted as such if that seemed desirable. A flexible group is indicated.

Duties of Committees. I. *Research.* Problems would be received and assigned to consultants when available, otherwise assigned to members of the Research Committee. Recommendations as to educational needs would be made to the Committee on Education for Research. Suitable contacts would be made by this committee. The proper assignment and orientation of young consultants should be the duty of this committee.

II. *Preparation for Research.* This committee would continue work already done on bibliographies, texts, seminars, publications, *etc.* Professor Tucker, representing the *Annals of Mathematics Studies*, has stated that there will be no difficulty in publishing suitable texts concerning applications of mathematics in war or industry. New texts in practically all of the above fields are needed and competent mathematicians have already been invited to write such texts. A small loan fund may be necessary to pay the initial cost of such Studies since these texts are published on a cash basis without profit.

III. *Education for Service.* The War Department intends in general to use sections of existing texts for instruction in the military schools. Those few texts which involve mathematics in an essential way should be carefully read by members of this committee, with two questions in mind: (1) *What elementary mathematics must be taught in order that these books be readily understood?* (2) *In what way could the material in these books of a mathematical nature be better written to save time and achieve the desired results?*

The following three resolutions are recommended for adoption by the Council of the A.M.S. and by the Board of Governors of the M.A.A.:

1. That all competent students in the secondary schools take the maximum amount of mathematics available in their institutions. In the case of many schools additions to the present curriculum will be necessary in order to furnish an adequate background for the military needs of the country.

2. That the colleges and universities at once make such revisions of their undergraduate courses in mathematics and add such courses to the curriculum as are necessary to prepare students in the elements of mechanics, probability, surveying, navigation and other essentials of military science.

3. That the graduate schools extend their courses in applied mathematics, such as dynamics, hydrodynamics, elasticity, aeronautics, ballistics, statistics, *etc.*, and that advanced students be urged to become highly qualified in one or more fields of applied mathematics.

These resolutions should be given immediate publicity in order that the changes recommended may be undertaken at once. In connection with resolutions 2 and 3, Committees II and III will be able to give a certain amount of advice at once and in a few months will undoubtedly be able to help considerably more.

We are appending two bibliographies corresponding to the needs, respectively, of those primarily interested in research and of those primarily interested in education for service. These bibliographies are selective rather than exhaustive because it is felt that such bibliographies will best serve immediate needs. More extensive bibliographies are available and will be mailed to those interested upon request. The report of the War Preparedness Committee *in extenso* containing letters from various authorities and

other important source material may be had upon writing to Secretary Richardson, as long as the supply lasts. Needless to say the Committee will help those who are working on war preparedness problems in any way in which it can.

Prepared for the Committee by
MARSTON MORSE, *General Chairman*

The above three resolutions were adopted by the Council of the American Mathematical Society and the Board of Governors of the Mathematical Association of America at a joint meeting at Hanover, N. H.

Professor M. H. Stone has been appointed Chairman of the Subcommittee on Preparation for Research and Professor W. L. Hart has been appointed Chairman of the Subcommittee on Education for Service. Chief Consultants have been appointed as follows:

Ballistics, Professor John von Neumann, Institute for Advanced Study; Aeronautics, Professor Harry Bateman, California Institute of Technology; Computation, Professor Norbert Wiener, Massachusetts Institute of Technology; Industry, Dr. T. C. Fry, Bell Telephone Laboratories; Probability and Statistics, Professor S. S. Wilks, Princeton University; and Cryptanalysis, Professor H. T. Engstrom, Yale University.

The bibliographies are published in the September number of the *Bulletin of the American Mathematical Society.*

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Morgantown, W. Va. April 20; Grove City, Pa., November 2.
ILLINOIS, Bloomington, May 10–11.
INDIANA, Richmond, May 3–4.
IOWA, Mt. Vernon, April 19–20.
KANSAS, Wichita, March 31.
KENTUCKY, Lexington, April 27.
LOUISIANA-MISSISSIPPI, Oxford, Miss., March 8–9.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Richmond, Va., May 11; Washington, D. C., December 7 or 14.
MICHIGAN, Ann Arbor, April 26–27.
MINNESOTA, Mankato, May 4.
MISSOURI, Warrensburg, April 19.
NEBRASKA, Omaha, May 9–11.
NORTHERN CALIFORNIA, Berkeley, January 27.
OHIO, Columbus, April 5.
OKLAHOMA, Oklahoma City, February 16.
PHILADELPHIA, November 23 or 30.
ROCKY MOUNTAIN, Fort Collins, Colo., April 19.
SOUTHEASTERN, Athens, Ga., March 29–30.
SOUTHERN CALIFORNIA, Compton, March 2.
SOUTHWESTERN, Tucson, Ariz., April 22–23.
TEXAS, Dallas, March 29–30.
UPPER NEW YORK STATE, Hamilton, May 11.
WISCONSIN, Milwaukee, May 4.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

ORGANIZATION MEETING OF THE UPPER NEW YORK STATE SECTION

A meeting to organize the Upper New York State Section of the Mathematical Association of America was held at Colgate University, Hamilton, New York, on Saturday, May 11, 1940. Sessions were held both in the morning and in the afternoon. Luncheon and dinner were served to members and guests in the James C. Colgate Student Union. Professor H. T. R. Aude of Colgate University presided at the morning session, Professor F. F. Decker of Syracuse University at the afternoon session, and Professor W. M. Carruth of Hamilton College at the dinner. The president of the Association, Professor W. B. Carver of Cornell University, presided at the business meeting which concluded the the afternoon session.

The attendance was ninety-six, including the following forty-three members of the Association: R. P. Agnew, E. B. Allen, H. T. R. Aude, A. B. Brown, H. S. Brown, F. J. H. Burkett, I. S. Carroll, W. M. Carruth, W. B. Carver, T. F. Cope, F. F. Decker, H. A. DoBell, W. H. Durfee, A. S. Gale, H. M. Gehman, H. N. Hubbs, W. A. Hurwitz, B. W. Jones, Caroline A. Lester, L. L. Lowenstein, Harriet F. Montague, D. S. Morse, C. W. Munshower, W. R. Murray, E. E. Nash, Abba V. Newton, E. R. Ott, B. C. Patterson, V. E. Pound, J. F. Randolph, C. E. Rhodes, M. A. Scheier, Joseph Seidlin, Virgil Snyder, Ellen C. Stokes, R. E. Street, Mary C. Suffa, R. J. Walker, A. K. Waltz, J. F. Wardwell, C. W. Watkeys, A. E. Whitford, W. D. Wray. Representatives were present from twenty colleges and universities.

A petition signed by forty-three members of the Association was addressed to the Board of Governors requesting the authority to organize a Section. A proposed set of by-laws had been prepared and was adopted, subject to the approval of the Board of Governors of the Association. The following officers were elected for the year 1940–1941: Chairman, H. M. Gehman, University of Buffalo; Vice-Chairman, A. D. Campbell, Syracuse University; Secretary, C. W. Munshower, Colgate University.

The following papers were read:

1. "A plan for freshman mathematics" by Professor T. F. Cope, Queens College.

2. "An application of analytic geometry to cryptography" by Dr. R. E. Huston, Rensselaer Polytechnic Institute, introduced by Professor Allen.

3. "Errors in American college mathematics texts" by Professor A. B. Brown, Queens College.

4. "Jacobian circles of the biquadratic" by Professor B. C. Patterson, Hamilton College.

5. "Complex roots of a polynomial equation" by Professor H. M. Gehman, University of Buffalo.

6. "Representation of a class of functions by Stieltjes integrals" by Dr. B. A. Lengyel, Rensselaer Polytechnic Institute, introduced by Professor Allen.

7. "A locus associated with the Pascal line" by Professor E. R. Ott, University of Buffalo.

8. "The convergence and summability of series" by Professor R. P. Agnew, Cornell University.

Abstracts of the papers follow, the numbers corresponding to the numbers above:

1. The plan described by Professor Cope has been tried for a year at Queens College. The three features of the plan are: (1) sectioning on the basis of a placement test into three groups; (2) in the case of the middle group a differentiation of content in the second semester according as the students are arts or science majors; (3) the encouragement of the able students of the superior group by putting them into the regular sophomore mathematics class in the second semester. The advantages claimed are: (1) that it greatly simplifies the teaching of freshman mathematics because of the homogeneity of the classes; (2) that the pace of the work is adapted to the ability of the students; and (3) that the content of the courses is adapted to the needs of the students.

2. A simple transposition code yields only a single representation for each letter; it is usually easy to decipher even a short message in such a code. Using no idea more complicated than translation of axes, Dr. Huston examined four codes in which each letter had more than a single representation. In the first of these, every letter was represented by every other letter with the representation in a particular sequence depending on the preceding letter. In the second, every sequence of two letters had 26^2 representations ranging from AA to ZZ. In the third, every sequence of three letters had 26^3 representations ranging from AAA to ZZZ, so that every three-letter word represented every other three-letter word. In the fourth code, every six-letter word had 26^6 legitimate spellings which ranged from $AAAAAA$ to $ZZZZZZ$.

3. The paper appeared in the June-July issue of the MONTHLY.

4. In the inversive plane the biquadratic curves include all curves which determine four points with a circle. It is known that such a curve is self-inversive (anallagmatic) with respect to four circles, its Jacobian circles. On the basis of the theory of bipolars, Professor Patterson presented a method for determining the Jacobian circles of a biquadratic and, as a consequence, the three polarities (homographies of period 2) under which the biquadratic is invariant.

5. Professor Gehman gave a graphic interpretation of a pair of complex roots of a polynomial equation. The theorem given was a generalization of a well known theorem for the case of the cubic equation.

6. A direct proof was given by Dr. Lengyel for the following theorem of Koopman and Doob: Let $w=u+iv$ be an analytic function in the upper half-plane, $y>0$, which satisfies the conditions $0<vy<K$, where K is a constant. Then $w(z)$ can be represented by a Stieltjes integral $\int_{-\infty}^{+\infty} d\psi(\lambda)/(\lambda-z)+a$, where a is a real constant, $\psi(\lambda)$ is a monotone increasing function; in fact,

$$\psi(\lambda) = \lim_{y\to+0} \frac{1}{\pi} \int_{-\infty}^{\lambda} v(x+iy)dx.$$

The proof was based on the Poisson integral theorem. First the existence of the limit defining $\psi(\lambda)$ was established, then the representation theorem was proved by a reasoning adapted from a paper of R. Nevanlinna.

7. When four fixed points and two variable points on a conic are considered as the vertices of an ordered hexagon, under certain restrictions the Pascal line of the hexagon envelops an algebraic curve. Professor Ott obtained parametric equations for certain of these curves, located their singularities, and obtained their Plücker characteristics.

8. Professor Agnew gave a general discussion of convergence and other methods of summability of series. Emphasis was placed upon relations between different methods of summability and upon Tauberian theorems.

C. W. MUNSHOWER, *Secretary pro tempore*

THE TWENTY-FOURTH ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-fourth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Colorado State College of Agriculture and Mechanic Arts, Fort Collins, Colorado, April 19 and 20, 1940. There were three sessions. Professor D. F. Gunder presided at each. The Saturday morning session was a joint meeting with the mathematics section of the Eastern Division of the Colorado Educational Association.

The attendance was seventy-six, including the following twenty members of the Association: C. F. Barr, J. R. Britton, I. M. DeLong, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, I. L. Hebel, C. A. Hutchinson, A. J. Kempner, Claribel Kendall, W. V. Lovitt, S. L. Macdonald, A. E. Mallory, W. K. Nelson, Greta Neubauer, M. G. Pawley, G. B. Price, O. H. Rechard, A. W. Recht, C. H. Sisam.

At the business meeting the following officers were elected for next year: Chairman, W. V. Lovitt, Colorado College; Vice-Chairman, J. C. Fitterer, Colorado School of Mines.

The following papers were presented:

1. "The line integral of curvature as a measure of its associated central angle" by Professor C. F. Barr, University of Wyoming.

2. "Determination of the differential equation and the equation of the orbit of a central force when the law of the force is known" by Professor J. R. Everett, Colorado School of Mines.

3. "Expansions of determinants of order four and five" by Professor W. V. Lovitt, Colorado College.

4. "The Gaussian solution of the trinomial equation" by Professor A. J. Lewis, University of Denver, read by A. M. Kahan.

5. "Projective representation of an affinely connected space" by T. C. Doyle, University of Wyoming, introduced by Professor Rechard.

6. "Some famous problems of modern mathematics" by Professor G. B. Price, University of Kansas.

7. "Mathematical Reviews" by Professor G. B. Price, University of Kansas.

8. "Report of the college-secondary mathematics coördinating committee" by Professor D. F. Gunder, Colorado State College.

9. "Discussion of trends in the teaching of mathematics in the junior high school" by Dr. G. S. Willey, Director of Instruction, Denver Public Schools; Professor L. Edwards, Colorado State College of Education; Superintendent M. W. Jessup, Bennett, Colorado, by invitation of the program committee.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The condition that a line integral of curvature along a polar curve be a constant multiple of its associated central angle is expressed by an ordinary differential equation of the second order, which is completely solvable. Embedded in this family of solutions are some of the best known angle-measurement theorems of elementary geometry. Professor Barr advanced the suggestion that the line integral under consideration may be used to unify and greatly extend these theorems. (A preliminary report.)

3. Professor Lovitt obtained the expansion of a determinant of order four by means of a schematic diagram comparable to that used for a determinant of order three.

4. Professor Lewis outlined Gauss's method of finding the real roots of trinomial equations by the use of addition and subtraction logarithms. He showed how the method could be simplified by modern computing machines.

5. Given an affinely connected space of N dimensions with connection components $\Gamma^i_{jk}(x)$ and curvature tensor $\Gamma^i_{jkl}(x)$, the partial differential equations

$$\frac{\partial^2 y}{\partial x^j \partial x^k} = \Gamma^r_{jk}\frac{\partial y}{\partial x^r} - \frac{1}{N-1}\Gamma^r_{jkr}y \tag{1}$$

will admit a system of $N+1$ fundamental solutions $y^\alpha(x)$ determined to within a projective transformation with constant coefficients providing the integrability conditions are satisfied. These solutions $y^\alpha(x)$ serve to define homogeneous coordinates of the point (x^i) and the geometry of the resulting projective representation will find its analytical expression in the invariant theory of (1) under the combined transformations $\bar{x}^i = \bar{x}^i(x)$ and $\bar{y} = \phi(x)y$, where ϕ is an arbitrary non-vanishing factor.

6. Professor Price gave the history and present status of Waring's Problem, the Four-Color Problem, the Jordan Curve Theorem, and the Problem of Plateau. The talk was illustrated with various models and demonstrations, including paper and rubber models of surfaces, a map on a wooden torus which required seven colors for its coloring, and soap film models for the Problem of Plateau. These problems were used: (1) to emphasize the great progress that has taken place in mathematics in recent times; (2) to illustrate the nature and source of problems in mathematics; (3) to point out the difference between a proof in mathematics and a proof in physics. The oldest of the four problems was first studied in 1636; although great progress has been made in recent years,

all four of them are still the subject of research. Problems in mathematics arise from (a) conundrums dealing with the positive integers, (b) a study of the physical world, and (c) from generalizations of simpler problems. Proof in mathematics consists of logical deduction; proof in physics consists of an induction from a large number of experiments.

8. The coördinating committee found that there was little discrepancy between the material presented in high school courses in mathematics and that required or expected of entering college freshmen. As a consequence, the present lack of preparation of college freshmen was attributed to lack of retention. Six suggestions for improving the general situation were offered by the committee. These were:

(1) Teach with emphasis on understanding rather than mechanical manipulation.

(2) Teach the correct terminology to further promote understanding.

(3) Teach the material in larger units with frequent repetition to prevent loss of sight of the subject as a unified whole.

(4) Induce other departments to use mathematics understandingly.

(5) Supplement or replace formal college entrance requirements by placement examinations.

(6) Improve the quality of teachers by requiring of them more and broader education in both mathematics and other fields.

9. The main purpose of education in the junior high school is the furthering of wholesome growth and development of the whole child through broad, meaningful experiences. Mathematics teachers, concerned with the total development of boys and girls, must consider how their subject will contribute to the concerns and problems of youth. The trend for teaching mathematics in the junior high school was summarized by this discussion as follows:

(1) Teaching all children only that which we know all children will use. Children interested in vocational phases may pursue mathematics materials in those fields.

(2) Choosing units of subject-matter through pupil-teacher planning.

(3) Permitting the "natural" method of learning, which is the only good teaching technique.

(4) Not carrying drill beyond the limits of its use by average adults in the community.

(5) Giving ample opportunity for experiences employing the four fundamental processes, simple fractions, percentage, and interest, stressing accuracy in each case.

(6) Permitting many informational problem-solving experiences that are meaningful to the pupils.

(7) Providing a program of diagnostic checking and remedial teaching.

(8) The placing of general, or social, mathematics through the ninth grade, with opportunity for electing algebra in the ninth grade for the present at least.

A. J. LEWIS, *Secretary*

THE TWENTY-SIXTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-sixth annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Wichita on Saturday, March 30, 1940. In the morning there was a joint session with the Kansas Association of Teachers of Mathematics. A social hour and luncheon at noon was followed by the showing of a sound film, "Rectangular Coördinates." After this the two organizations met for separate programs. Professor C. B. Read, chairman of the Section, presided at the morning session as well as at the Section meeting.

The attendance was one hundred and twenty-three, including the following thirty-five members of the Association: Sister Mary N. Arnoldy, R. W. Babcock, Wealthy Babcock, E. A. Beito, Lois E. Bell, Talmon Bell, C. V. Bertsch, Florence L. Black, E. E. Colyer, R. D. Daugherty, Lucy T. Dougherty, W. H. Garrett, W. A. Harshbarger, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, Anna Marm, U. G. Mitchell, O. J. Peterson, G. B. Price, C. B. Read, C. A. Reagan, B. L. Remick, D. H. Richert, J. A. G. Shirk, D. T. Sigley, G. W. Smith, R. G. Smith, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, J. J. Wheeler, A. E. White.

The officers elected for the coming year are: Chairman, G. B. Price, University of Kansas; Vice-Chairman, C. V. Bertsch, Southwestern College; Secretary, Lucy T. Dougherty, Junior College, Kansas City. The time and place of the next meeting were left to the executive committee.

The major portion of the program was given to the reports of the committee appointed at the 1939 meeting to arrange for a uniform test in mathematics. The test was prepared by the committee, and was given in September 1939 to entering freshmen in most of the colleges and universities in the state. Several of the institutions repeated the test at the end of the first semester. The committee through its different members presented the results of the test by summary, analysis, and interpretation. These reports are included in the abstracts below. In the free and informal discussion that followed, the problems developed were so many that the committee was continued with instructions to go on with the study another year.

The following eight papers and reports were presented:

1. "The Peano-Baker method and mean coefficients" by Professor W. C. Janes, Kansas State College.

2. "Mean deviation of ungrouped variates" by Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg.

3. "Mathematics journals" by Professor D. T. Sigley, Kansas State College.

4. "Results of Kansas mathematical tests as they apply to college courses"—reports and round table discussion.

   a. Professor U. G. Mitchell, University of Kansas.

   b. Professor W. H. Garrett, Baker University.

c. Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg.
d. Professor W. T. Stratton, Kansas State College.
e. Professor C. V. Bertsch, Southwestern College.

Abstracts of the papers and reports follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Janes gave a brief résumé of the Peano-Baker method of solving differential equations, together with a discussion of the procedure for obtaining approximate solutions by replacing the matrix of coefficients by a matrix of constants.

2. Professor Shirk showed that quite frequently the median is a more significant average than the arithmetic mean as a measure of the central tendency when the number of variates is rather small. Mean deviation is defined as the arithmetic mean of the absolute values of the deviations of the variates from any central tendency. It may therefore be taken from the median as well as from the arithmetic mean. He then developed a simple formula for finding mean deviation by the use of an addition-subtraction machine. The process is much simpler than the calculation of standard deviation. He also gave an easy proof that the mean deviation from the median is less than from any other point.

3. Professor Sigley, after a short philosophical discussion on mathematics journals, spoke particularly on *Mathematical Reviews*. The accomplishments and policies, as interpreted from the first three numbers of the publication, were discussed. Statistics covering number of subscriptions, papers reviewed, and so on, were presented.

4. a. Professor Mitchell presented the results of Kansas Mathematics Test, Number One, given to 4351 students entering Kansas colleges and universities in the fall of 1939. The test was prepared by a committee of the Kansas Section of the Mathematical Association of America and consisted of twenty questions in arithmetic and thirty-five questions in elementary algebra. Tabulations and analyses are not yet complete, but distributions of scores for 2903 students having one year or less, and for 1448 students having more than one year of high school algebra were presented. Also the total scores of a combined group of 4045 students, and a diagnostic analysis of answers to each of the fifty-five questions asked, for 2542 students. It is expected that the complete results will be published in the October 1940 issue of the *Bulletin* of the Kansas Association of Teachers of Mathematics.

b. Professor Garrett presented the results of a study made by the committee, showing the correlation of the grades made in the freshman classes in mathematics in twenty colleges and universities of Kansas the first semester of 1939–1940, and the scores made in the Kansas Test given at the beginning of the semester. Several charts were presented showing the relation of the class grades to the test scores, the distribution of grades in the various classes, the relation of the maxima and minima scores, and so forth. The data included the class records of fifty-one sections in five-hour college algebra, totaling 1150 students; seventeen sections of trigonometry, 358 students; thirty-eight sections of

three-hour college algebra, 784 students; and three sections of general mathematics, 80 students.

c. Professor Shirk expressed the opinion that the principal value of this test has been to reveal the very meager mathematical skills in the possession of the average college freshman. From a study of the first semester grades received in college mathematics in Kansas State Teachers College of Pittsburg, it was found that the correlation of the grades with the scores of the mathematics test was too low to be of the significance desired, and therefore this test would hardly serve as a means of dividing freshman students into sections. The correlation with the grades in college algebra was .411; the results of this test might be used with some value as additional information to the high school record, which was found to give a correlation with college mathematics of about .65. The arithmetical portion of the test showed great deficiencies in ordinary computations, and it was suggested that in forming a test for 1940, business men be asked to aid in devising a test which would be in line with what they felt their employees should know. This would also enlist the support of patrons of the public system in insisting upon better accomplishment in preparatory mathematics.

d. Professor Stratton said that at Kansas State College the test was given at the close of the semester, as well as at the beginning, and Professor A. E. White had worked out a number of correlations between the term grades and the test, for sections of three-hour and five-hour college algebra. The correlations were high. Professor Stratton also gave the average scores made by students from a number of the larger high schools and suggested the possibility of ranking the high schools as to the quality of their work done in mathematics.

e. Professor Bertsch said that the low scores in the arithmetic part of the test, and the general correlation between the arithmetic and the algebraic parts, led him to believe that the cause of the low results may be partially inherent in the work preceding the high school. He mentioned the trend in grade school children of excellent abilities to profess a dislike of arithmetic, while seemingly quite enthusiastic over other subjects. The early development of such an attitude cannot but have a bad influence on a student's success in mathematics. He then suggested some changes in the tests for the coming year, that may make them more valuable for diagnostic purposes.

LUCY T. DOUGHERTY, *Secretary*

THE SEVENTEENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The seventeenth annual meeting of the Nebraska Section of the Mathematical Association of America was held at Creighton University, Omaha, on Saturday, May 11, 1940. Professor A. K. Bettinger of Creighton University was chairman.

The attendance was thirty-six, including the following twelve members of

the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. L. Candy, H. M. Cox, J. M. Earl, J. D. Fitzpatrick, M. G. Gaba, A. L. Hill, F. E. Marrin, Lulu L. Runge.

Officers elected for the coming year were: Chairman, A. L. Candy, University of Nebraska; Secretary-Treasurer, Lulu L. Runge, University of Nebraska; Member of Executive Committee, A. K. Bettinger, Creighton University. The next meeting will be held at the University of Nebraska in Lincoln, May 1941.

After a one o'clock luncheon at the Omaha Athletic Club, Professor Brenke presided at a round table discussion on (1) unified mathematics examinations; and (2) classification tests in mathematics. There was also a discussion of Professor Weil's paper.

The following program was presented:

1. "On an identity by Bailey" by J. A. Daum, University of Nebraska.

2. "How can experimental psychological facts be utilized for teaching mathematics in secondary schools?" by Professor Hermann Weil, Nebraska Central College, introduced by Professor Bettinger.

3. "Algebraic proofs of Dwyer's identities" by T. E. Oberbeck, University of Nebraska, introduced by the Secretary.

4. "A certain one-to-one correspondence" by E. P. Coleman, University of Omaha, introduced by Professor Bettinger.

5. "What does a mathematics examination examine?" by H. M. Cox, University of Nebraska.

6. "Rectification of the ellipse" by Professor M. G. Gaba, University of Nebraska.

7. "Grades in freshman algebra as indicative of later success in engineering courses" by Professor C. C. Camp, University of Nebraska.

8. "Certain pseudo-periodic functions" by Professor W. A. Dwyer, Creighton University, introduced by Professor Bettinger.

9. "Note on perfect numbers" by Professor T. A. Pierce, University of Nebraska, by title.

Abstracts of the papers follow, numbered in accordance with their numbers listed above:

1. The identity

$$\theta_3(\alpha)\theta_3(\beta+\gamma)\theta_3(\beta+\delta)\theta_3(\alpha+\gamma+\delta) - \theta_3(\beta)\theta_3(\alpha+\gamma)\theta_3(\alpha+\delta)\theta_3(\beta+\gamma+\delta)$$
$$= \theta_1(\gamma)\theta_1(\delta)\theta_1(\alpha-\beta)\theta_1(\alpha+\beta+\gamma+\delta),$$

(W. N. Bailey, *Quarterly Journal of Mathematics*, 1936) was discussed by Mr. Daum. It was shown that this identity is equivalent to Jacobi's fundamental formula involving products of four theta functions. Relations involving certain of the functions

$$\phi_{\alpha\beta\gamma}(x, y) \equiv \frac{\theta_1'\theta_\alpha(x+y)}{\theta_\beta(x)\theta_\gamma(y)}$$

were also obtained.

2. Psychologists of the University of Marburg, Germany, have discovered that there are some children who have the ability to see very vividly objects which are not present to their sight at the moment. Such a child, after having been asked to look attentively at an object, is able, with eyes open or closed to *see* this object again. This is possible either immediately or after a certain lapse of time, even after the passage of several years. Although the stimulus object may be reproduced with almost photographic fidelity, eidetic images, as these images are called, differ from the original stimulus object as to color, form, and a number of details. They differ from hallucinations as well as from after-images. They differ from memory-images in that the phenomena are really seen. Dr. Weil's paper dealt with eidetic images as observed in plane and solid geometry.

3. Mr. Oberbeck used the arithmetic method of Uspensky to prove four of the arithmetic identities of W. A. Dwyer which appear in vol. 45 of the *Bulletin of the American Mathematical Society*. He also showed by simple arithmetic considerations that these four identities form a fundamental set of a certain type of identity.

4. Mr. Coleman considered the relation $t^2+yt+x=0$, where x and y refer to the rectangular coördinates of a point in the plane and t refers to a point on a line. By this relation an eminent contact between certain notions of the one-dimensional geometry and the corresponding notions of the two-dimensional geometry is shown. For an assigned point in the plane the relation becomes a quadratic in t which yields two points on the line. When a point is chosen in the plane so that the discriminant, y^2-4x, is equal to zero, the quadratic equation leads to one real distinct point on the line. A one-to-one correspondence is shown to exist between the points of the parabola, $y^2-4x=0$, and the points on the t-line.

5. Mr. Cox illustrated the concepts of examination reliability, examination validity, question difficulty, and question validity by data obtained from the use of the University of Nebraska mathematics classification examination in the Nebraska high schools. He considered the meaning of the "single score" as obtained from an achievement examination.

6. By elementary geometric methods Professor Gaba showed that the length of an ellipse of semi-major axis a and eccentricity e is

$$\lim_{k\to 0} (1+2k)\sum_{n=0}^{n=k-1} \frac{\sqrt{[(2k+1)^2+(2n+1)^2]^2-4e^2[(2n+1)^2-(2k+1)^2]^2}}{(k^2+k+n)(k^2+k+(n+1)^2)}.$$

7. This is a statistical study of prerequisite courses. Should the prerequisite course be given special consideration? Should credit be given on the same basis as for other courses? With the passing mark 60, in view of a rule that requires 4/5 of the grades for graduation to be 70 or higher, certain courses may need to be repeated. Would the mortality in future work be appreciably lessened if the credit in prerequisite courses was withheld for any mark under 70? Professor Camp made frequency tables and calculated correlation coefficients for grades

in algebra with subsequent courses. The records of 1010 students over a period of 6 years were tabulated by a student assistant. Among other things the ratios of students receiving grades less than 70 in one or more subsequent courses for the algebra grade groups 60–69 and 70–79 were calculated. These ratios did not differ sufficiently to justify withholding credit in prerequisite courses for students in the 60–69 group.

8. Professor Dwyer showed that a certain pair of identities, given by Uspensky (*Bulletin of the American Mathematical Society*, vol. 36, identities I, II), involving incomplete numerical functions in three variables result from the paraphrase of a certain theta function identity. The method of establishing the theta identity is that used by Basoco (*American Journal of Mathematics*, vol. 54).

9. By use of Gauss's theorem that the sum of the totients of the divisors of a number is equal to the number, Professor Pierce proved that a number is perfect if, and only if, the sum of the totients of the proper divisors of the number is equal to the totient of the number.

LULU L. RUNGE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The eighteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the University of Georgia, Athens, Ga., on Friday and Saturday, March 29–30, 1940.

There were in attendance about three hundred persons from fifty-three institutions, including the following eighty-one members of the Association: Martha E. Allen, D. H. Ballou, T. A. Bancroft, G. F. Barnes, D. F. Barrow, W. S. Beckwith, R. C. Blackwell, R. V. Blair, J. W. Blincoe, E. T. Browne, Iris Callaway, E. A. Cameron, C. L. Carroll, Jr., T. C. Carson, J. B. Coleman, R. W. Cowan, Orpha A. Culmer, Forrest Cumming, R. T. Donnell, B. F. Dostal, L. A. Dye, E. D. Eaves, W. W. Elliott, Floyd Field, Tomlinson Fort, W. A. Gager, L. L. Garner, Leslie J. Gaylord, J. J. Gergen, M. E. Gillis, L. J. Green, J. T. Hains, C. L. Hair, R. A. Hefner, Archibald Henderson, Ruby U. Hightower, P. R. Hill, C. W. Hook, J. V. Howell, H. M. Hughes, R. O. Hutchinson, J. A. Hyden, J. B. Jackson, E. S. Kennedy, F. W. Kokomoor, G. B. Lang, J. W. Lasley, Jr., F. A. Lewis, J. B. Linker, T. G. Loudermilk, E. L. Mackie, S. W. McInnis, W. J. Mays, J. F. Messick, E. R. C. Miles, Verdie F. Miller, W. B. Moye, W. P. Ott, C. R. Phelps, C. G. Phipps, W. W. Rankin, M. C. Rhodes, H. A. Robinson, C. C. Sams, C. L. Seebeck, Jr., C. Eucebia Shuler, T. M. Simpson, Augustus Sisk, A. R. Sloan, H. E. Spencer, F. H. Steen, Ruth W. Stokes, J. M. Thomas, Louise Thompson, T. L. Wade, Jr., J. A. Ward, D. L. Webb, C. W. Williams, W. L. Williams, F. L. Wren, R. C. Yates.

Sessions were held Friday afternoon and evening, and Saturday morning. Professor C. L. Hair, chairman of the Section, presided, except Friday evening, and part of Saturday morning when the Section was divided into sub-groups

according to the nature of the papers presented. Sub-groups were presided over by Professors F. L. Wren, J. B. Jackson, Forrest Cumming, F. A. Lewis, Archibald Henderson, and D. H. Ballou. On Friday evening a dinner was given in honor of the visiting speaker, Dean Tomlinson Fort of Lehigh University. At this time Professor Forrest Cumming presided.

At the business session on Saturday the following officers were chosen for 1940–41: Chairman, Forrest Cumming, University of Georgia; Vice-Chairman, J. W. Lasley, Jr., University of North Carolina; Secretary-Treasurer, H. A. Robinson, Agnes Scott College; Members of Executive Committee: D. H. Ballou, Georgia School of Technology, T. M. Simpson, University of Florida, F. L. Wren, George Peabody College for Teachers; Governor, Region No. 5, J. M. Thomas, Duke University. The next meeting was scheduled for March 1941, at the University of North Carolina.

The following forty-three papers were presented:

1. "The marked ruler" by Dr. R. C. Yates, Louisiana State University.

2. "Four models on conic sections for use in projective geometry" by Professor Ruth W. Stokes, Winthrop College.

3. "Freshman preparation" by Professor C. L. Hair, The Citadel.

4. "A problem in baseball" by Professor W. S. Beckwith, University of Georgia.

5. "Analysis of an ancient Hindu game" by Dr. L. D. Rodabaugh, University of Alabama, introduced by the Secretary.

6. "Southeastern mathematics text-book authors" by Professor H. A. Robinson, Agnes Scott College.

7. "*Mathematical Reviews* for the mathematician in a small college" by Professor F. W. Kokomoor, University of Florida.

8. "The story of the parallelogram" by Dr. R. C. Yates, Louisiana State University.

9. "The study of geometry in high school" by Professor W. W. Rankin, Duke University.

10. "The Lehmus-Steiner problem" by Professor Archibald Henderson, University of North Carolina.

11. "Mathematics and the sciences" by Dean Tomlinson Fort, Lehigh University.

12. "A note on the average deviation" by Dr. H. H. Germond, University of Florida, introduced by the Secretary.

13. "The Plate of Zones of Ghiyathu'd Din al-Kashi" by Dr. E. S. Kennedy, University of Alabama.

14. "The resolvents of a polynomial" by Professor J. M. Thomas, Duke University.

15. "A program in freshman mathematics designed to care for a wide variation in student ability" by Professor E. A. Cameron, University of North Carolina.

16. "A new course for commerce students" by Professor J. W. Lasley, Jr., University of North Carolina.

17. "*Mathematical Reviews* for the research mathematician" by Professor J. J. Gergen, Duke University.

18. "Abel's lemma and applications in infinite series" by Dean Tomlinson Fort, Lehigh University.

19. "A mathematical program for junior college students who do not plan to continue work in mathematics" by Professor W. A. Gager, St. Petersburg Junior College.

20. "Problems of adapting instruction in mathematics to the students entering the junior college" by Professor C. Eucebia Shuler, Georgia Southwestern College.

21. "Aids to motivation in junior college mathematics" by Professor Ruth W. Stokes, Winthrop College.

22. "Mathematical inadequacies of students entering the senior college" by Professor W. A. Beckwith, University of Georgia.

23. "The status of mathematics in a liberal curriculum" by Miss Curtis Ledford, Griffin, Georgia, High School, introduced by the Secretary.

24. "Models for illustrating special products and factoring" by T. G. Loudermilk, Decatur, Georgia, Boys High School.

25. "Correlating mathematics in the junior high school" by Martha E. Allen, Atlanta Junior High School.

26. "A diagonal property of determinants" by Professor E. R. C. Miles, Duke University.

27. "On the characteristic roots of a circulant" by Professor H. S. Thurston, University of Alabama, introduced by the Secretary.

28. "Some generalizations of the fundamental theorem of algebra" by Dr. Witold Hurewicz, University of North Carolina, introduced by the Secretary.

29. "Solution of a certain linear difference equation of the second order with polynomial coefficients" by Dr. R. W. Cowan, University of Alabama.

30. "Automorphisms of the Lie algebra of order 28" by C. L. Carroll, Jr., Georgia School of Technology.

31. "The number and computation of idempotents for any modulus" by H. M. Hughes, University of Tennessee.

32. "Line configurations associated with a group of order 48 and degree 6" by Dr. W. G. Warnock, University of Alabama, by title.

33. "An involutorial Cremona transformation associated with a pencil of quartic surfaces" by Dr. L. A. Dye, The Citadel.

34. "On geometries and invariants" by Dr. T. L. Wade, Jr., University of Alabama.

35. "Loci associated with certain circles in the plane" by Professor L. L. Garner, University of North Carolina.

36. "Twisted cubics associated with a space curve" by Dr. L. J. Green, Georgia School of Technology.

37. "Line coördinates" by R. T. Donnell, Castles Heights Military Academy, Tennessee.

38. "On the reduction of a matrix to rational canonical form" by Professor E. T. Browne, University of North Carolina.

39. "A note on the quartic and its Hessian" by Dr. J. A. Ward, Tennessee Polytechnic Institute.

40. "Formulas for coefficients in equations whose roots form certain sequences" by Dr. G. B. Lang, Emory University.

41. "An invariant of the set of affine connections defining vectors normal to every arbitrary curve of a Riemann space" by Dr. C. L. Seebeck, Jr., University of Alabama.

42. "A note on cube roots of complex numbers" by Dr. J. N. Mallory, Union University, introduced by the Secretary.

43. "A theory of runs of luck" by W. J. Mays, Masseyville School, Tennessee.

Abstracts of some of the papers follow, numbered in accordance with their listing above:

1. Dr. Yates gave a discussion of the constructional possibilities of the double-edged marked ruler as a geometrical tool. By inserting a marked portion between two given lines, similar to the trammel of Archimedes and the conchoid trisection trammel mentioned by Pappus, Dr. Yates showed that all problems leading to equations of degree not higher than four whose coefficients represent possessed lengths are solvable by this tool alone.

2. Professor Stokes displayed four projective geometry models and demonstrated their construction. She also presented a sound film on the parabola.

3. In his retiring address, Chairman Hair discussed the problem of freshman mathematics and the preparation of the pupil, asserting that the right kind of pre-college direction has been lacking.

4. Professor Beckwith discussed a problem of "minimum time" for throwing a ball from an outfielder to a catcher. He showed that when a ball is caught on the fall before a bounce, the time is longer than if caught on the rebound.

5. In this paper, the solution of a generalized game of "Kim" was presented from a matric view-point. Dr. Rodabaugh's technique was applied to a broad class of games.

6. Some seventy mathematics books published by authors living in the Southeastern Section were listed by Professor Robinson. Thirty books published recently were displayed in an exhibit.

7. Professor Kokomoor gave the history of "*Mathematical Reviews*," and showed the need for the publication by mathematicians located in institutions isolated from the research centers.

8. Dr. Yates gave an account of the dominant rôle the parallelogram has played in the theory of dissection, in restricted euclidean construction, and in the theory of linkages and mechanical motions.

9. Professor Rankin reviewed the status of the study of geometry in the high schools and pointed out some of the more recent efforts for improving the work in geometry. He invited college professors to give serious thought as to

ways and means of giving better preparation for the teachers of geometry.

10. Professor Henderson presented his subject in the form of a centennial anniversary paper, as it was in 1840 that Lehmus gave to Steiner the problem of equal internal base-angle bisectors of a triangle. Dr. Henderson sketched the history of the problem, and presented a number of solutions believed to be new, based on a uniform techníque.

11. This paper will appear in this MONTHLY.

12. Dr. Germond illustrated certain common errors found in statistics textbooks. He showed that unless a modification is made in the manner of computing the average deviation, it is not necessarily the least when computed about the median.

13. Dr. Kennedy gave a preliminary report on a study being made of an anonymous 15th century Persian manuscript which describes the construction and operation of an astronomical instrument invented by al-Kashi.

14. This paper will appear in this MONTHLY in November.

15. This paper appeared in the August-September, 1940, issue of this MONTHLY.

16. Professor Lasley gave an account of a new course in mathematics for commerce students which was developed at his university by the combined efforts of both mathematics and commerce departments.

18. In this address, Dean Fort was concerned with the formula for summation by parts and applications in the theory of infinite series. Applications were first pointed out in the study of the convergence of Dirichlet series, factorial series, Lambert series, and other related types. The second portion of the address was concerned with generalizations of the formula for summation by parts which are applicable in the theory of summability. Particular reference was made to recent research of the lecturer.

19. Professor Gager showed how he believed a mathematics program may be integrated in order to give the junior college terminal student the minimum essential tool mathematics for effective citizenship.

20. Professor Schuler's test showed that preparation of incoming students indicates a decrease in quality of present day training. She urged that pupils be taught to think analytically, to read reflectively, and to reason effectively in order that they might arrive at true conclusions with controlled emotions.

21. Professor Stokes discussed visual aids to motivation in junior college mathematics. She showed certain lantern slides, and a sound film on "Rectilinear Coördinates." Finally, she called attention to an electrically lighted stereographic instrument which may be used to illustrate propositions in solid geometry.

22. Professor Beckwith listed certain mathematical inadequacies found in students entering the senior college. He suggested that more emphasis should be placed on the fundamental operations of algebra and arithmetic.

23. Miss Ledford showed that the demands of our technical civilization make a study of mathematics a vital necessity to everyone.

25. Miss Allen gave a brief analysis of the type of mathematics a progressive

junior high school should carry on in order to have an integrated curriculum.

26. In a determinant of order n, the number of terms in the expansion which are free of elements of a particular diagonal is $\phi_1(n)=n!\Sigma(-1)^k/k!$. Professor Miles showed that if $\phi_k(n)$ denotes the number of terms free of elements of k parallel diagonals, then $\phi_k(n)$ is determined as the solution of a linear difference equation of order $k+1$ with polynomial coefficients.

29. Dr. Cowan assumed his solution to be in the form of a series in ascending powers of a parameter with undetermined coefficients. When the series was inserted in the difference equation, a set of non-homogeneous difference equations with constant coefficients were obtained which were solved by using the operator E. The resulting solution was simplified by means of Zeta-functions.

31. Mr. Hughes gave a simple formula for the number of idempotents for a given modulus and demonstrated some devices for simplifying their computations.

32. The operators of the imprimitive group of order 48 and degree 6 were applied to the fundamental identity of line geometry in S_3. The resulting identities were interpreted as lines.

34. In this paper, Dr. Wade pointed out certain advantages, particularly from the invariant view-point, of considering sub-geometries of projective geometry from the theories of tensors, as opposed to the view-point of regarding such geometries as the theories of sub-groups of the general linear projective transformation group.

35. Dr. Garner discussed a sequence of mutually equal circles which were tangent to each other in consecutive pairs, each circle being also tangent to a fixed circle. The point of contact of a typical pair of circles generated a continuous curve. Certain interesting properties of this locus were mentioned.

36. A canonical tetrahedron of reference can be found which yields a simple pair of power series expansions for a space curve in the neighborhood of an ordinary point. Dr. Green showed that geometical characterizations of them can be obtained from a study of the two-parameter family of five-point osculating twisted cubics.

37. Mr. Donnell gave the results of a study of the class of certain cubic curves by means of line coördinates.

38. Let A be an n-square matrix with elements in a field F. By invoking a knowledge merely of the theory of linear dependence and of the Hamilton-Cayley theorem, and without presupposing any knowledge of the notion of invariant factors, Professor Browne showed how to obtain a non-singular matrix T, with elements in F, such that $T^{-1}AT$ is in rational canonical form.

39. Dr. Ward derived a formula that expressed the roots of a quartic as functions of the roots of its Hessian in the special case where the Hessian is a cubic. Since all quartics may be so reduced, it is a general solution of the quartic. Similarities were pointed out between this method and a similar formula for the solution of a cubic.

41. Dr. Seebeck gave a necessary and sufficient condition for a symmetric affine connection of the vector associated with every arbitrary curve of a Rie-

mann space by means of covariant differentiation with respect to this affine connection to be normal to its associated curve.

42. Professor Mallory's work led to a method of finding certain solutions of a cubic equation. By making certain simplifications of the "irreducible case," a formula for the solution of the cubic by Tartaglia's method was derived.

43. The theory of runs of luck was developed from the solution of a certain difference equation. Mr. Mays developed an expression which gives approximately the number of repeated trials of an event of doubtful outcome necessary to obtain a given probability for a given run of luck.

H. A. ROBINSON, *Secretary*

BISECTING CIRCLES

P. H. DAUS, University of California at Los Angeles

1. Introduction. If a circle x (center X) cuts the circle a (center A) in the ends of a diameter, the circle x is said to bisect the circle a. In the geometry of inversion, orthogonality and bisection play analogous rôles. If A is a given point and x a given circle, and if the power p of A with respect to x is positive, then A is the center of a circle a orthogonal to x, and its radius is $t=\sqrt{p}$, where t is the length of the tangent from A to the circle x. An inversion with respect to the circle a leaves the circle x invariant. If, however, the power p of A with respect to x is negative, then A is the center of a circle a bisected by x, and its radius is $c=\sqrt{-p}$, where c is the minimal half-chord through A. An inversion with respect to the imaginary circle, center A, radius $=\sqrt{-c^2}$, leaves the circle x invariant. We call this imaginary circle the conjugate of a, and represent it by the symbol $\bar{a}$. The inversion with respect to $\bar{a}$ may be interpreted as a real inversion with respect to the circle a, followed by a reflection on the point A. For if any chord through A cuts x in P_1 and P_2, we have

$$AP_1\cdot AP_2 = -c^2.$$

If the inverse of P_1 with respect to a is P_1', and the reflection of P_1' on A is P_1'', then the equations

$$AP_1\cdot AP_1' = +c^2, \qquad AP_1\cdot AP_1'' = -c^2,$$

show that P_1'' and P_2 are identical, so that the circle x remains invariant.

It is the purpose of this paper to study circles with respect to this bisection property, and as illustrations to discuss the ruler and compass construction of circles determined by three conditions, when some of these conditions are this bisection property. The other conditions will be orthogonality and tangency,* represented by the symbols O and T, respectively.

* For circles involving these properties only, see a paper by the author, this MONTHLY, vol. 34, 1927, pp. 357–359.

All of the results of this article are not new; for example, a number of them are given or suggested at various places in Altshiller-Court, College Geometry (referred to hereafter as A.-C.), and in Coolidge, Treatise on the Geometry of the Circle and the Sphere (referred to hereafter as C.). Some specific references will be given later.

The analogy between orthogonality and bisection has its limitations. The property of orthogonality (or tangency) of two circles is a mutual property, but that of bisection is not. For that reason we must distinguish carefully between a circle *bisected by* a given a circle (represented hereafter by B) and a circle *which bisects* a given circle (represented by W). The symbol W_1B_2 stands for a circle (or construct a circle) which *bisects* one given circle and is *bisected by* two other given circles. When it is desirable to designate explicitly the given circles, we use such symbols as W_a, B_b, O_c, T_d, so that $x = W_aB_bO_c$ may be read: x is a circle which bisects the circle a, is bisected by the circle b, and is orthogonal to the circle c. Similar notations are self-explanatory.

2. The one-parameter family B_2. Focal points. If a circle x is to be bisected by two given circles a_1 and a_2, its center X must have equal powers with respect to both circles and hence must lie on their radical axis (Fig. 1). If the required

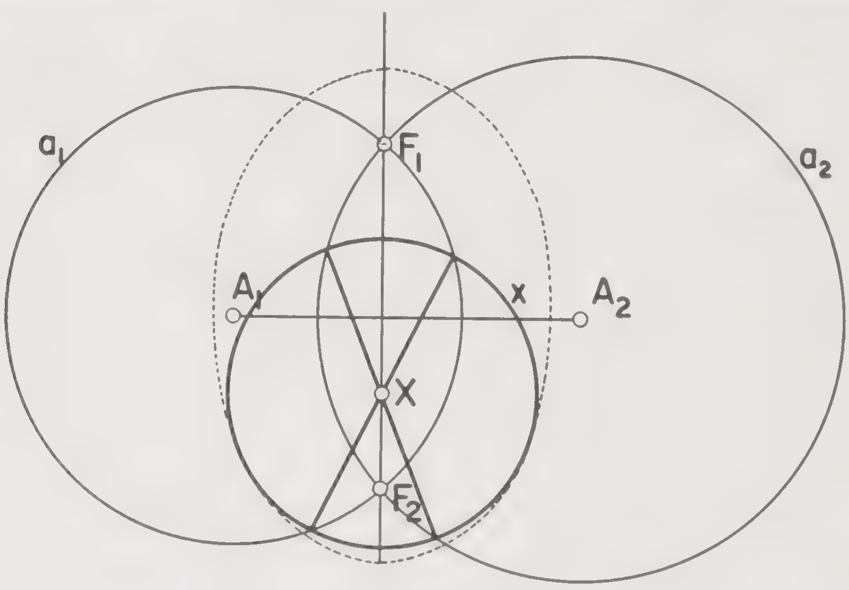


FIG. 1

circle is to be real, a_1 and a_2 must intersect, and any point between their intersections is the center of a unique circle which is bisected not only by a_1 and a_2, but by every circle coaxal with them. It can be shown analytically that the envelope of this family is an ellipse whose axes are the line of centers and the radical axis, with lengths of semi-axes as f and $\sqrt{2}f$, where f is half the length between the points of intersection F_1, F_2. For if we take the axes as indicated, the equation of the circle with center $X(0, t)$ is readily shown to be $x^2+y^2-2yt+2t^2-f^2=0$, and its envelope $2x^2+y^2=2f^2$. All circles of the family do not have real contact with this enveloping ellipse, especially when X is near F_1 or F_2. In particular, the point circles F_1 and F_2 are circles of this family; we call them the *focal points* of the elliptical family B_2, since they are the foci of the enveloping ellipse.

3. The family W_2. The bisectral axes. We have just seen that a coaxal system of circles of the intersecting type determines the one-parameter family of bisecting circles. Let us now discuss the converse property W_2, that is, the family* *which bisects* two given circles c_1 and c_2. Let X be the center of any circle of the system (Fig. 2). Then

$$r^2 = XC_1^2 + r_1^2 = XC_2^2 + r_2^2,$$

where r, r_1, r_2 are the radii of circles x, c_1, and c_2. Let us take the point C_1 as

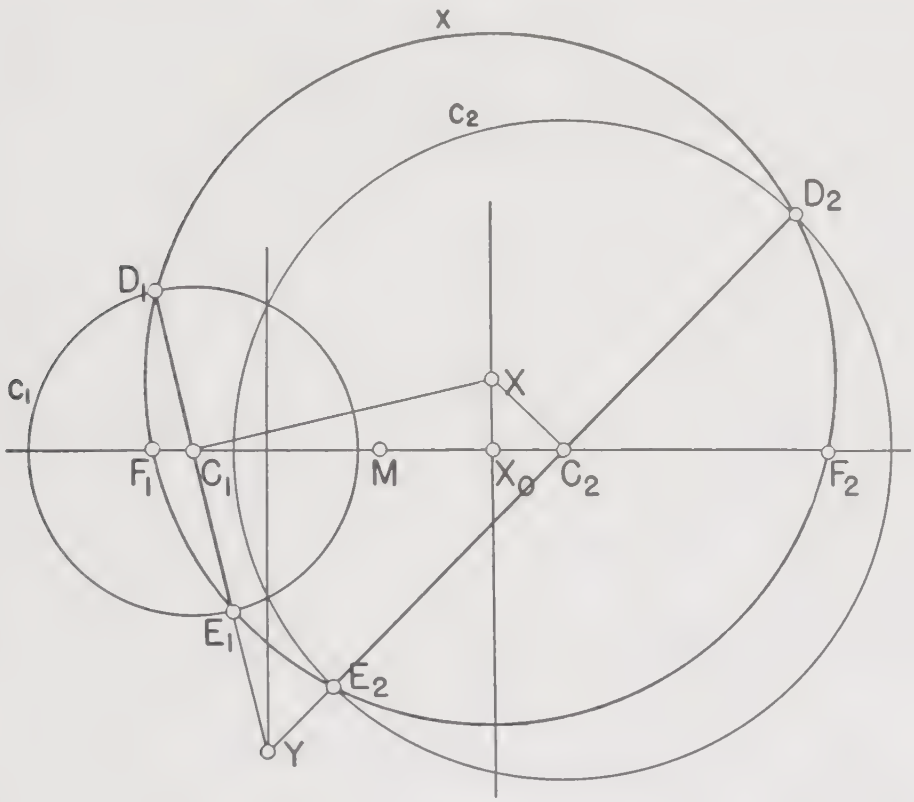


FIG. 2

origin and let the coördinates of X be (x, z) and $C_1C_2 = d$. Then

$$x^2 + z^2 + r_1^2 = (d - x)^2 + z^2 + r_2^2.$$

If we solve for x, we readily find

$$x = \frac{d^2 - r_1^2 + r_2^2}{2d} = \text{const.}$$

If $Y(y, w)$ is any point on the radical axis, a similar analysis gives

* See A.-C., pp. 168–171, §§334, 335, and 339; or C., p. 108, Theorems 207, 208.

$$p = YC_1^2 - r_1^2 = YC_2^2 - r_2^2,$$

from which we find

$$y = \frac{d^2 + r_1^2 - r_2^2}{2d}.$$

Hence $x+y=d$. We may state the results in the following manner:

THEOREM. *The locus of the centers of circles which bisect two given circles is a straight line, called their bisectral axis; it is the reflection of the radical axis of the given circles on the midpoint of their line of centers.*

If x_1 and x_2 are any two circles of the system W_2, then c_1 and c_2 are bisected by x_1 and x_2, so that all the x circles (including the line C_1C_2) are coaxal, passing through two points F_1, F_2 on C_1C_2, which are the focal points of the system B_2 determined by any two x circles. The points F_1 and F_2 may be determined by any x circle; a convenient one to use is that one whose center is X_0, the intersection of the bisectral axis and the line C_1C_2.

Suppose that x (center X) is any circle of the W_2 system, and that it determines the respective diameters D_1E_1 and D_2E_2 of c_1 and c_2. Let these diameters meet in the point Y. Then, from circle x, we see Y has equal powers with respect to c_1 and c_2, and so lies on their radical axis. Conversely, if Y is any point on the radical axis, we may determine the circle x and its center X, the points X, Y, C_1, and C_2 being concyclic with XY a diameter of this circle (Fig. 2). These results afford an immediate geometric proof of the properties of the bisectral line. Since XY is a diameter of the circle XYC_1C_2, X is symmetric to Y with respect to a point on the perpendicular bisector of C_1C_2, and as Y describes the radical axis parallel to this bisector, X describes a line symmetric to the radical axis with respect to this bisector, or symmetric to it with respect to the midpoint of C_1C_2.

4. Other one-parameter families. *The family* W_1O_1. Suppose x, with center X, is any circle which bisects a and is orthogonal to b. If we invert with respect to the circle $\bar{a}$ (center A, radius ia), that is, invert with respect to a and then reflect on A, the circle x remains invariant, while the circle b inverts into a circle b' orthogonal to x. Therefore the locus* of X is the radical axis of b and b', with x being a circle of the coaxal system orthogonal to that of b and b'.

The family B_1O_1. Suppose that x, with center X and radius r, is a circle bisected by a and orthogonal to b (Fig. 3). Then p_1, the power of X with respect to a is $-r^2$, and p_2, the power of X with respect to b is $+r^2$, so that $p_1/p_2=-1$. It follows from a well known theorem† that the locus of X is a circle m coaxal with a and b, whose center M divides AB in the ratio $AM/BM=p_1/p_2$; in this

* A.-C., p. 169, proves the locus is a line without using inversion.

† See Johnson, Modern Geometry, p. 87.

case M is the midpoint of AB. This circle is called the radical* circle of a and b. The center and then the radius of any circle of the family may thus be determined.

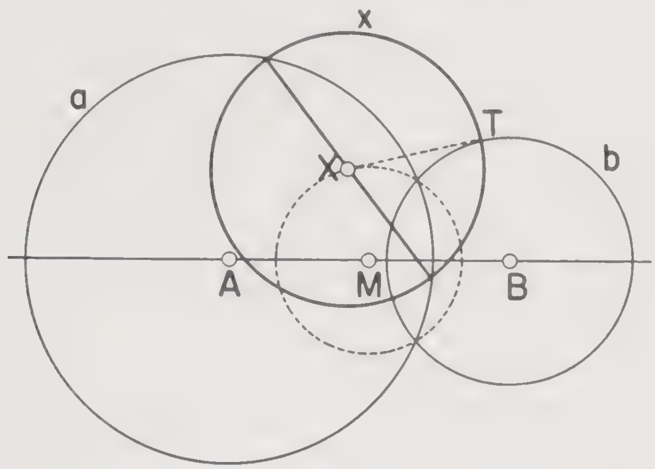


FIG. 3

The family W_1B_1. Suppose that x (Fig. 4) is any circle which bisects a and is bisected by b. Then

$$r^2 = r_1^2 + AX^2 = r_2^2 - BX^2,$$

where r, r_1, r_2 are the radii of circles x, a, b. Hence

$$AX^2 + BX^2 = r_2^2 - r_1^2 = \text{const.}$$

It follows from a well known simple theorem† that X describes a circle whose center is the midpoint of AB and whose radius depends upon r_1, r_2 and the distance $d = AB$.

We may construct this circle geometrically, knowing its center, by finding one point on it as follows. Consider any point circle E, taken for convenience on circle a. Let X be one of the two intersections of the diameter perpendicular to AE and the radical circle n of b and the point circle E. Then the circle center M and radius MX is the required circle. For

$$AX^2 = r^2 - r_1^2, \quad \text{and} \quad r^2 = -(BX^2 - r_2^2),$$

so that

$$AX^2 + BX^2 = r_2^2 - r_1^2,$$

* C., p. 106, Theorems 199–201. The term radical circle of two circles should not be confused with the radical circle of three circles, a term sometimes used for the circle orthogonal to three circles.

† Apply the cosine law to triangles AXM and BXM, and add. This shows that MX is a constant.

which shows that X is on the required locus. If the points of intersection of the circle n and this diameter are not real, we merely note that the required circle has its center at M and is coaxal with the diameter and the radical circle n. Variations of this construction may be made by taking E on b, or if a and b intersect, at their intersection. We shall refer to this circle as a *mid-center circle* of a and b.

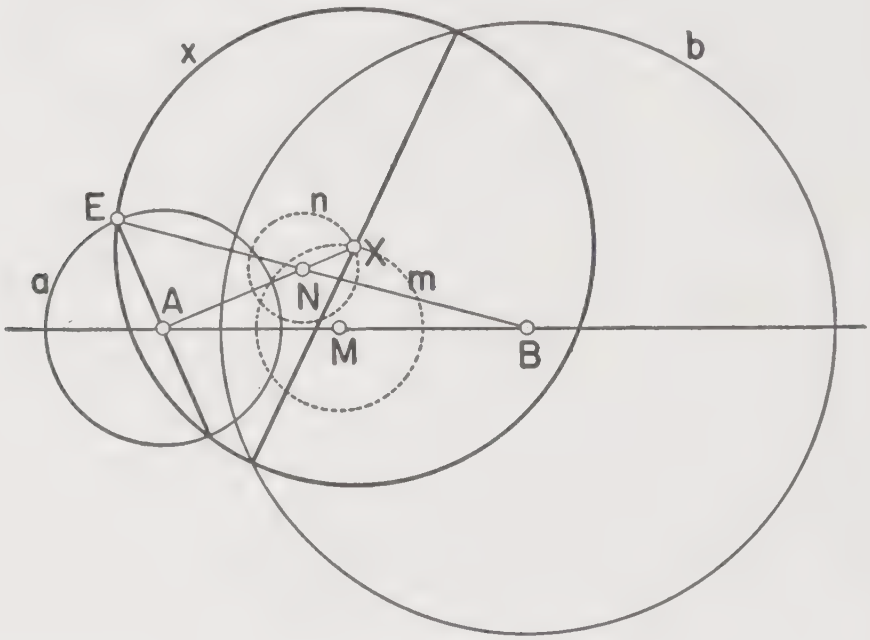


FIG. 4

Other one-parameter families such as OT_1, B_1T_1, W_1T_1 do not lead to circle loci, and so we shall not discuss them here, except to remark that the resulting loci involve general conics.

5. Linear construction problems. We shall now consider the problems of constructing a circle determined by three conditions, one of which is B or W, and the others selected from B, W, O, and T. First we consider those that are linear, namely, (1) B_3, (2) W_3, (3) W_1O_2, (4) W_2O_1. (1) For* B_3, the required center is obviously the radical center of the three given circles. This must be inside all three circles for a real solution. (2) For† W_3, the required center is the bisectral center (the intersection of the bisectral axes). Except in trivial cases, the solution is always real. (3) For the solution‡ of $W_aO_bO_c$, we invert with respect to $\bar{a}$, so that x is the circle orthogonal to b, c, and b'. (4) The solution of W_2O_1 becomes the well known problem of drawing a circle through the focal points F_1 and F_2 orthogonal to a circle. It may be remarked that an inversion with respect to $\bar{a}$ reduces the problem (4) $W_aW_bO_c$ to problem (3).

* C., p. 100, Theorem 182.

† A.-C., p. 172, §342.

‡ A.-C., p. 193, §385.

6. Quadratic construction problems. The problems are symbolized by (5) O_2B_1, (6) B_2O_1, (7) $W_1O_1B_1$, (8) W_2B_1, (9) B_2W_1, (10) W_2T_1, (11) $W_1O_1T_1$. We consider the problems (5), $O_aO_bB_c$, and (6), $B_aB_bO_c$, simultaneously. The center X of a required circle lies on the radical axis of a and b, and on the radical circle of b and c. There are two solutions X_1 and X_2. If the power of X with respect to b is positive, we obtain a real solution of (5), if negative, a real solution of (6). Figure 5 shows a case when we obtain one solution of each type $x_1 = O_aO_bB_c$, and $x_2 = B_aB_bO_c$.

We may solve (7), $W_aO_bB_c$, in several ways. The best perhaps is to recognize that an inversion with respect to $\bar{a}$ reduces it to (5); the center lies on the radical axis of b and b', where b' is the inverse of b, and on the radical circle of b and c.

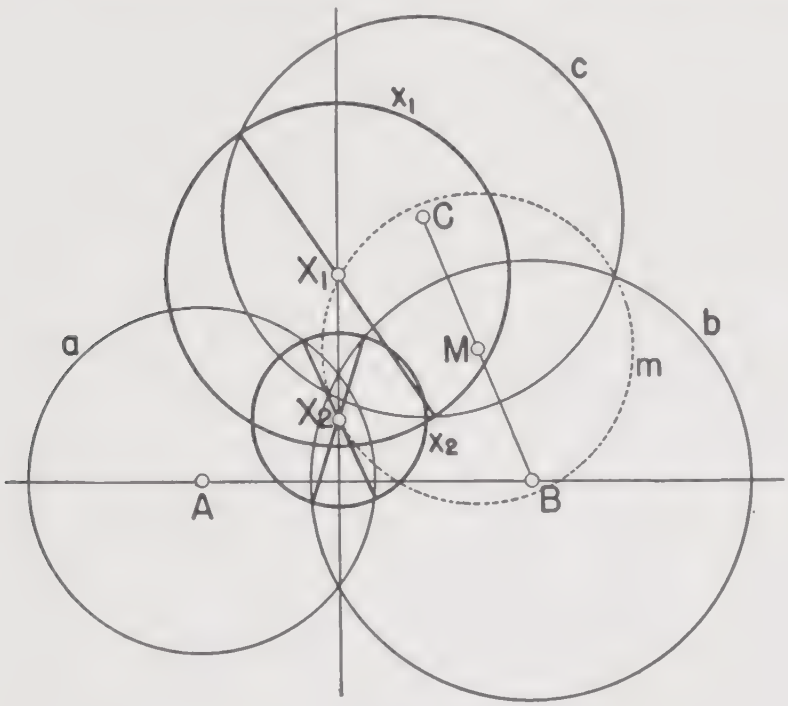


FIG. 5

Problem (8), $W_aW_bB_c$, becomes that of drawing a circle through the focal points F_1 and F_2 determined by the system W_aW_b and bisected by the circle c. The center lies on the bisectral axis of a and b, and on the radical circle of c and the point circle F_1.

For (9), $B_aB_bW_c$, we could locate X on the radical axis r of a and b, and on a mid-center circle of b and c. If we recognize that we may replace condition B_a by O_r, we see that we have problem (7), so that if we invert with respect to $\bar{c}$, with r' the inverse of r, X is on the radical circle of r' and b.

Problem (10), W_2T_1, is equivalent to the well known problem of drawing a circle through the two focal points tangent to a circle, the detailed construction being most readily done by Gergonne's method.

For (11), $x = W_aO_bT_c$, we invert with respect to $\bar{a}$. We have $x = O_{b'}O_bT_c$ so

that (11) reduces to the well known problem of drawing a circle orthogonal to two circles and tangent to a third circle, a construction for which is shown (Fig. 6).

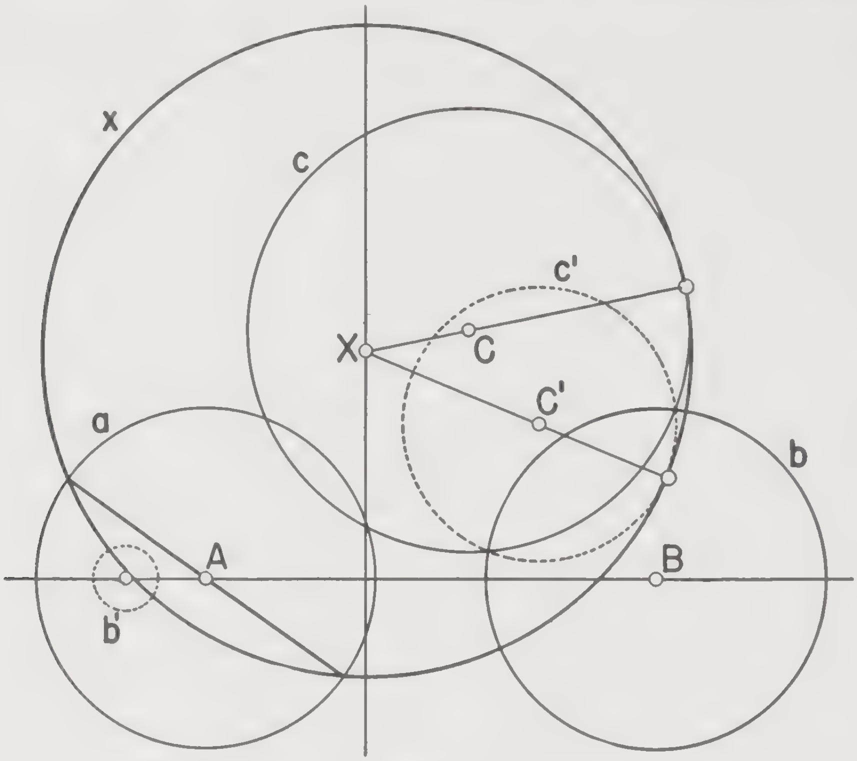


FIG. 6

7. Problems of degree higher than two. These problems divide into two categories, those that are constructible by ruler and compass, and those that are not. In the first category we have (12) W_1T_2. For if $x=W_aT_bT_c$, and if we invert with respect to $\bar{a}$, we have x tangent to the three circles b', b, c, and so x may be constructed. Of the eight solutions of this problem of Apollonius, only four are solutions of (12), for it must be noted that if x is to have a specified type of contact with b, it has the same type or opposite type of contact with b' according as A is inside of b or outside of b, since A is always inside of x.

Problems (11) and (12) are closely related; indeed, (11) is transformed into (12) by inversion. For if (11) is $x=W_aO_bT_c$, and we invert with respect to b, then $x=W_aT_cT_{c'}$, where the types of contact of x with c and its inverse c' are restricted as above. Figure 6 shows a required circle x orthogonal to b and b', its inverse with respect to $\bar{a}$, and tangent externally to both c and c', since B is outside c and x.

Conversely, the solution of (12) may be made to depend upon (11). For if x is tangent to b and c, it is orthogonal to the circles h and k, with respect to which b and c are mutually inverse. At least one of h and k is real. If they both are real, the four solutions of (12) come from the two solutions of each of $W_aO_hT_b$ and $W_aO_kT_b$. If the circle k is imaginary, its conjugate $\bar{k}$ is real, and the second of the above problems is replaced by a case of problem (10), $W_aW_{\bar{k}}T_b$.

The remaining cases (13) B_2T_1, (14) $B_1O_1T_1$, (15) $B_1W_1T_1$, (16) B_1T_2 lead to quartic problems which are irreducible in the realm $R^{1/2}$, that is, are not constructible by ruler and compass. Let us next consider problem (13). We may take, without loss of generality, as the first two circles the radical axis and circle a with center A on the radical axis. The third circle c will be arbitrary. If the required circle has center $X(0, y)$ and radius r, and circle c has center $C(u, v)$ and radius s, then

$$r^2 = 1 - y^2; \qquad (r \pm s)^2 = u^2 + (y - v)^2.$$

If we eliminate r and rationalize, we find

$$4y^4 - 8vy^3 + (4u^2 + 8v^2 - 4)y^2 - 4v(u^2 + v^2 - s^2 - 1)y + \left[(u^2 + v^2 - s^2 - 1)^2 - 4s^2\right] = 0.$$

This quartic is in general irreducible in $R^{1/2}$, for the reducibility would imply restrictions on the coefficients. We may best illustrate this irreducibility by a properly selected example, where we make one root rational and show the residual cubic is irreducible. Such an illustration is $u = 3/5$, $v = -1/5$, $s = 1/5$, so that the above quartic becomes, after the roots are multiplied by 5, with $z \equiv 5y$,

$$z^4 + 2z^3 - 14z^2 - 16z + 39 = 0.$$

It is readily verified that this has the solution $z = 3$ and that the residual cubic $z^3 + 5z^2 + z - 13 = 0$ is irreducible in the realm $R^{1/2}$, since it has no rational root.

We now show that the remaining three problems are fundamentally equivalent. We may replace (16) $B_aT_bT_c$, by the two problems $B_aO_hT_c$ and $B_aO_kT_c$ (type 14), or if k is not real, the latter one by $B_aW_{\bar{k}}T_c$ (type 15), where h and k are the circles with respect to which b and c are mutually inverse. Each of these problems, we shall point out, has four solutions, so (16) has eight solutions.

For either (14) or (15), the center lies on a circle and a conic, accounting for the four solutions. To illustrate the irreducibility of (14) in the realm $R^{1/2}$, we consider the special case of $B_aO_rT_c$, where the circle r is a line (Fig. 7). This limitation is equivalent to the problem $B_aT_bT_c$, when the circles b and c are equal and we require like contacts with them. Since we may now interpret x as bisected by both a and r, it is bisected by every circle coaxal with them, and hence without further loss of generality, we may take A on the line r. Suppose that r is the y-axis and a has center $A(0, u)$, radius s; c has center $C(v, 0)$, radius t; and the required circle x has center $X(0, y)$, radius r. Then the conditions which determine X are

$$(r \pm t)^2 = y^2 + v^2; \qquad r^2 = s^2 - (u - y)^2.$$

The elimination of r gives the quartic

$$4t^2(y^2 + v^2) = \left[2y^2 - 2uy + (u^2 + v^2 + t^2 - s^2)\right]^2,$$

which in general is irreducible in $R^{1/2}$. As before we illustrate this by a special example made up to have one rational root. Take $t=3$, $v^2=3$; $u=2$, $s^2=34$. Then the above equation becomes

$$y^4 - 4y^3 - 23y^2 + 36y + 54 = 0,$$

which has the rational root $y=-1$, and the residual cubic $y^3-5y^2-18y+54=0$

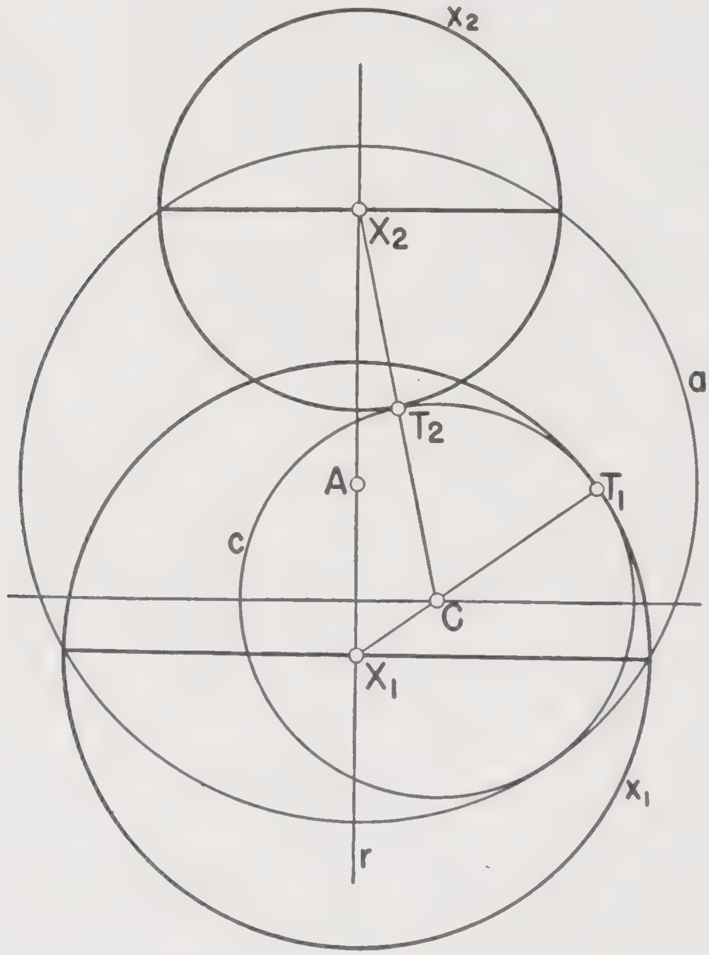


Fig. 7

has no rational root, so it is irreducible in the realm $R^{1/2}$. The roots of this cubic are roughly 6.5, 2.2, and -3.7. Figure 7 shows the circle corresponding to the root -1, having internal contact with c, and the circle corresponding approximately to the root 6.5, having external contact with c; the other two circles are real but are not shown in the diagram.

8. Special cases. Some of the circles may become points or lines, leading to special cases of the above problems. If a circle becomes a point P, the symbol B_P is meaningless, while the conditions $W_P=T_P=O_P$ are all equivalent and when interpreted as O_P the above constructions need no essential modifications.

Indeed the construction used to discuss the family W_1B_1 in §4 was a solution of the problem $W_aO_EB_b$, where E was a point circle.

If the circle becomes a line l, the symbol W_l is meaningless, while B_l is equivalent to O_l. Most of the constructions are made as before, but the number of solutions is often reduced. There are no essential changes involved except in problems containing the condition T. For example, the problem (13.1) B_2T_1, is now a quadratic one and hence can be solved by ruler and compass. We may take without loss of generality for the two circles (Fig. 8) the radical axis r and the circle a of the given coaxal system with center on the given line t. Let its radius be s. Let the center of the required circle x be $X(x, y)$ with the coördinate axes taken as t and the line through A perpendicular to t. Then

$$r^2 = y^2 = s^2 - AX^2 = s^2 - (x^2 + y^2),$$

so that $x^2+2y^2=s^2$. This shows that X lies on the radical axis and on an ellipse, whose major circle is the circle a and whose concentric minor circle has radius $s/\sqrt{2}$. Now we may find as many points as necessary on the ellipse and hence use the Steiner construction to find the points X.

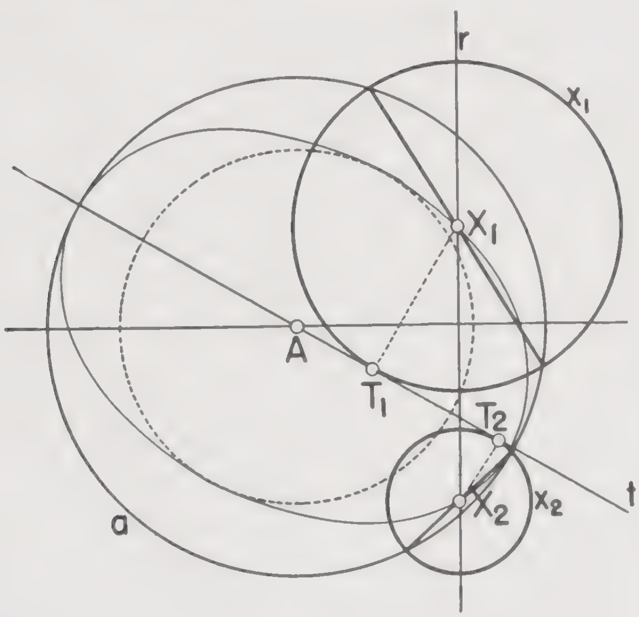


FIG. 8

As a final example, consider problem (16.1) when the two circles of problem (16) are lines. Now there are only four solutions and it is solvable by ruler and compass. For a required circle is defined by $x=B_aT_{l_1}T_{l_2}=B_aT_{l_1}O_m$, where m is one of the bisectors of the angle between the lines. This is the problem just considered and hence using each bisector and considering (13.1) twice, we obtain the four solutions.

CONES AND VECTOR SPACES

F. A. FICKEN, Cornell University

It has recently become possible to view the traditional "geometry of union and intersection" as a special case of the general theory of partially ordered sets.* The new theory has so adapted the terms "union" and "intersection" to its own purposes that they may be used in the older theory more sharply and picturesquely than ever. This remark is here applied specifically to a real vector or centered affine n-space A, in which C is a nondegenerate cone with its vertex at A's center. A vector sub-space V is passed through C's vertex, and their intersection is presented in detail as the union of suitably constructed spaces with that sub-space of V which lies on C. Since most of the results are believed to be known,† in one form or another, proofs are omitted.

Although our discussion applies to any real vector n-space, we think in concrete terms of an ordinary real euclidean n-space, discarding its metric properties. We introduce a particular cartesian coördinate system, and henceforth admit only any non-singular linear homogeneous transformation of coördinates. Then the initial origin always has coördinates $(0, 0, \cdots, 0)$, and will be called the center of the space, or the 0-vector. A directed segment issuing from the center will be called a vector. In a given coördinate system, the end-point of a vector has certain coördinates $(x^1, \cdots, x^n)$, which will be called the components of the vector x in that coördinate system. If $x=(x^i)$, $(i=1, \cdots, n)$, and $y=(y^i)$, then $x+y$ is defined to be the vector with components (x^i+y^i), and, if t is a real number, tx is defined to be the vector with components (tx^i). It is easily seen that these constructions lead to unique results, independent of the coördinate system with whose components the construction is actually performed. The following discussion is conducted in the space A whose elements are the vectors just defined, admitting these two operations—addition and multiplication by a real number.

If $x_1, \cdots, x_k$ are vectors, and $t^1, \cdots, t^k$ are real numbers, then $x=t^1x_1+\cdots+t^kx_k$ is thus a well-defined vector. As $t^1, \cdots, t^k$ take on all real values, the set of vectors x so obtained can be shown to be a vector space in the same sense that A is. "Space" shall henceforth mean A or one of its vector sub-spaces. Every space U contains at least one set of vectors $x_1, \cdots, x_r$ with the property that, for each vector x of U, there is one and only one set of real numbers $t^1, \cdots, t^r$ for which $x=t^1x_1+\cdots+t^rx_r$. The integer r is U's dimension, and the vectors $x_1, \cdots, x_r$ are said to be a basis for U.

The join $U\oplus V$ of spaces U and V is defined to be the space W of all vectors $W=u+v$, where u is in U and v in V; if U has dimension d and V has dimen-

* See, for example, the series of expository notes in the Bulletin of the American Mathematical Society, vol. 44, no. 12, Dec. 1938, pp. 793–827; particularly, Lattices and their applications, by Garrett Birkhoff, p. 800.

† *E.g.*, E. Bertini, Einführung in die Projective Geometrie Mehrdimensionaler Räume, chap. VI.

sion δ, then $U\oplus V$ is a space whose dimension is not less than the greater of d and δ. The intersection UV is the space W of vectors common to U and V. The space V is an inverse of U in W if $U\oplus V=W$ and UV contains only the 0-vector. We write $U\leqq V$ to denote that U is a sub-space of V.*

The definitions may be illustrated in a 3-space. Let e_1 be the vector $(1, 0, 0)$, $e_2=(0, 1, 0)$, $e_3=(0, 0, 1)$, and let E_i, $(i=1, 2, 3)$, be the 1-space t^ie_i, $-\infty<t^i<+\infty$. Then $E_1\oplus E_2$ is the xy-plane, $E_1\oplus E_2\oplus E_3$ is the whole space, $E_1\leqq E_1\oplus E_2$, $E_1E_2=0$ (the center), and E_1 is an inverse of E_2 in $E_1\oplus E_2$. Any 2-space (plane through the center) either coincides with $E_1\oplus E_2$ or intersects it in a 1-space (line through the center).

Let C be the cone $x^2+y^2-z^2=0$. A 1-space is outside, inside, or on C. A 2-space V intersects C either in the center only, or in a 1-space which may be called V's contact space, or in two 1-spaces which together form a nondegenerate cone in V. We turn now to the development of these relations in our space A.

Let $x\cdot y$ denote $\sum_{i,j=1}^{n}a_{ij}x^iy^j$, and let $x\cdot x$ be a non-singular indefinite quadratic form. A coördinate system exists in which

$$x\cdot x\equiv x^{1^2}+x^{2^2}+\cdots+x^{n_1^2}-x^{(n_1+1)^2}-\cdots-x^{n^2}.$$

In this coördinate system, the coördinate axes are a set of principal axes of the cone C whose equation is $x\cdot x=0$. Let $n_2=n-n_1$; for simplicity, it is assumed that $n_2\leqq n_1$. A vector x is called positive, negative, or isotropic according as $x\cdot x$ is positive, negative, or zero. If every vector in a space is positive, negative, or isotropic, the space is called positive, negative, or isotropic. If V is isotropic, we write V^0; if V has dimension k, we write V_k.

The positive and negative spaces are separated from each other by C, on which each isotropic space lies. If V is an arbitrary space, the essential point in this discussion is that V's behavior toward C can be exhibited very simply by expressing V as the join of an isotropic space and properly chosen positive and negative spaces. The isotropic space is V's contact-space, which will now be defined.

Vectors x and y are said to be polar (conjugate, with respect to C) if and only if $x\cdot y=0$. A vector is self-polar if and only if it is isotropic. The space V' of vectors polar to every vector of a space V is called the polar of V; evidently $(V')'=V$. For any V, the space VV' is on C and each of its vectors is polar to every vector of V; $V^0=VV'$ is called V's contact-space (with C).

If $VV'=0$, we say that the 0-vector is V's contact-space, and that V is not tangent to C. For any V^0, $V^0V^{0\prime}=V^0$. If V^0 is V's contact-space, and $V\neq V^0$, then V^0 has an inverse (in fact, many inverses) $\overline{V}$ in V such that $\overline{V}\leqq V^{0\prime}$ and $\overline{V}$ is not tangent to C.

* The greater simplicity and symmetry of the newer view-point is clearly indicated by a comparison of the definitions just given for "join" and "intersection" with the more general ones, so specialized as to apply to the space A. The join $U\oplus V$ is the space W such that (a) $U\leqq W$ and $V\leqq W$ and (b) $U\leqq X$ and $V\leqq X$ imply $W\leqq X$. The intersection UV is the space W such that (a) $W\leqq U$ and $W\leqq V$ and (b) $X\leqq U$ and $X\leqq V$ imply $X\leqq W$. The sub-spaces of A, including A and 0, are partially ordered by the relation "$\leqq$."

The intersection of V_k with C consists of all the isotropic vectors in V_k. These vectors form a cone C_V which has rank k if and only if V is not tangent to C. More precisely, if V^0 is V's contact-space and $\overline{V}_m$ is an inverse of V^0 in V, then C_V consists of (a) the space V^0, (b) a cone $C_{\overline{V}}$ in each $\overline{V}_m$, each $C_{\overline{V}}$ having rank m and a signature independent of $\overline{V}_m$, and (c) the join of V^0 with each space $\overline{V}^0$ on each $C_{\overline{V}}$.

Remark. Although (b) and (c) have both been given to exhibit fully the degeneracy of C_V, they are redundant. As $\overline{V}$ varies, (a) and (b) exhaust the isotropic vectors of V. Also, as $\bar{x}^0$ varies over the 1-spaces on a particular $C_{\overline{V}}$, the spaces $V^0 \oplus \bar{x}^0$ contain all the isotropic vectors of V.

The space $\overline{V}$ can be expressed as the join of a positive and a negative space. Since $C_{\overline{V}}$ is nondegenerate, a set of mutually polar non-isotropic principal axes may be chosen for it. Vectors along these axes form a basis for $\overline{V}$. Let k_1 of them be positive, and let the positive space $\overline{V}_{k_1}^+$ be their join; let (the remaining) k_2 of them be negative, and let the negative space $\overline{V}_{k_2}^-$ be their join. The signature of $C_{\overline{V}}$ is $k_1 - k_2$. If k_0 is the dimension of V's contact-space V^0, then $V_k = V_{k_0}^0 \oplus \overline{V}_{k_1}^+ \oplus \overline{V}_{k_2}^-$, and $k = k_0 + k_1 + k_2$. Combined with any basis for V^0, the bases for $\overline{V}^+$ and $\overline{V}^-$ form a mutually polar basis for V.

Example. In a 7-space, let e_i, $(i = 1, \cdots, 7)$, denote the vector whose every component is 0, save the ith, which is 1, and, for each i, let E_i denote the 1-space $t^i e_i$, $-\infty < t^i < +\infty$. Let C be the cone

$$x^2 + y^2 + z^2 + t^2 - u^2 - v^2 - w^2 = 0.$$

Let V_2^0 be the space whose basis vectors are $e_1 + e_6$ and $e_2 + e_7$, and let $V_5 = V_2^0 \oplus E_3 \oplus E_4 \oplus E_5$; that is, V_5 contains all vectors $(t^1, t^2, t^3, t^4, t^5, t^1, t^2)$. Then V_2^0 is V_5's contact-space. An inverse of V_2^0 in V_5 is $\overline{V}_3 = E_3 \oplus E_4 \oplus E_5$. The cone $C_{\overline{V}}$ has the equation $t^{3^2} + t^{4^2} - t^{5^2} = 0$. The spaces $\overline{V}^+$ and $\overline{V}^-$ are $\overline{V}_2^+ = E_3 \oplus E_4$ and $\overline{V}_1^- = E_5$.

The integers k_0, k_1, and k_2 satisfy certain inequalities. Let d^0, d^+, and d^- denote the maximum possible dimensions of isotropic, positive, and negative spaces. Then $d^0 = n_2$, $d^+ = n_1$, $d^- = n_2$. If $V_k = V_{k_0}^0 \oplus V_{k_1}^+ \oplus V_{k_2}^-$, then k_1 cannot exceed the lesser of $n_1 - k_0$ and $k - k_0$, and k_2 cannot exceed the lesser of $n_2 - k_0$ and $k - k_0$.

Thus V falls into one of eight types, according as the various k's equal or differ from 0. The type $k_0 = k_1 = k_2 = 0$ is trivial. If the equation of C_V in V is $t \cdot t = 0$, and $k_1 = k_2 = 0$, then $t \cdot t \equiv 0$. In two of the remaining six types, $t \cdot t$ is indefinite; in the four others, it is positive definite, negative definite, positive semi-definite, and negative semi-definite. Using sums, joins, and joins of sums of the E_i, it is easy to realize each type of space; in other words, a space may be constructed having a prescribed character and dimensions consistent with the above inequalities.

Finally, the integers k_0, k_1, and k_2 completely characterize a class of spaces equivalent under the group of Lorentz transformations of A which leave C fixed.

By letting $k_0=0, 1, \cdots, n_2$ and, for each k_0, counting the number of values k_1 and k_2 may have, it is seen that there are $\frac{1}{6}(n_2+1)(n_2+2)(3n_1-n_2+3)$ distinct classes of equivalent spaces.

ON THE MAXIMA AND MINIMA OF BERNOULLI POLYNOMIALS

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Being in need of rather precise information about the extreme values of the ordinary Bernoulli polynomial $B_n(x)$ on the unit interval $0 \leqq x \leqq 1$, I was rather surprised to find no account of this problem in the very considerable literature on $B_n(x)$. An instance of a situation in which such information is essential is afforded by the equation

$$n \sum_{1 \leqq \lambda \leqq t} (t-\lambda)^{n-1} = B_n(t) - B_n(t-[t]),$$

in which $B_n(t)$ is exhibited as approximating the sum to within an error bounded by the extreme values of $B_n(x)$ on the unit interval. This problem arose also in another connection in conversation with a seismologist, so that it occured to me that a simple treatment of it might be of use to workers in other fields, such as statistics and interpolation theory.

Of course, the problem depends directly on the consideration of the derivative of $B_n(x)$, and since

$$B_n'(x) = nB_{n-1}(x), \tag{1}$$

the roots of the Bernoulli polynomials are involved. It must be said that the roots of $B_n(x)$ in the unit interval have been discussed on several occasions. A good treatment of $B_n(x)$ in the interval is given by Nörlund.*

We adopt here the familiar notation for the Bernoulli numbers,

$$B_\nu = (B+1)^\nu = \sum_{s=0}^{\nu} \binom{\nu}{s} B_s,$$

so that $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_3=0$, $B_4=-1/30$, $B_5=0$, $B_6=1/42, \cdots$, and define Bernoulli polynomials by

$$B_n(x) = (B+x)^n = \sum_{\nu=0}^{n} \binom{n}{\nu} x^\nu B_{n-\nu},$$

so that

$$B_0(x) = 1,\ B_1(x) = x - \tfrac{1}{2},\ B_2(x) = x^2 - x + \tfrac{1}{6}, \cdots.$$

Excellent graphs of $B_n(x)$ in the unit interval for $n \leqq 8$ are given by Davis.†

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* Differenzenrechnung, Berlin, 1924, pp. 17–23. There is a more detailed account in Nörlund's Memoire sur les polynomes de Bernoulli, Acta Mathematica, vol. 43, 1922, pp. 121–131, where the reader will find proofs of several of the relations quoted in what follows.

† H. T. Davis, Tables of the Higher Mathematical Functions, vol. 2, 1935, p. 185.

The problem of the maxima and minima of $B_n(x)$ is quite simple when n is even. In fact by (1),

$$B'_{2k}(x) = 2kB_{2k-1}(x),$$

and for $k>1$, $B_{2k-1}(x)$ has precisely three simple roots in the unit interval at $0, \frac{1}{2}$, and 1. The extreme values of $B_{2k}(x)$ are attained at the ends of the interval, where

$$B_{2k}(0) = B_{2k}(1) = B_{2k} = (-1)^{k-1}|B_{2k}|,$$

and in the middle, where

$$B_{2k}(\tfrac{1}{2}) = -(1 - 2^{1-2k})B_{2k} = (-1)^k(1 - 2^{1-2k})|B_{2k}|.$$

A glance at $B_2(x) = x^2 - x + \frac{1}{6}$ shows that it also has these characteristics with the slight difference that the slope of the curve is not zero at the end-points of the interval. From these facts we have the following:

THEOREM 1. *Let M_n and m_n denote the maximum and minimum of the Bernoulli polynomial $B_n(x)$ for $0 \leqq x \leqq 1$. Then for n even we have*

$$\begin{aligned} M_{4h} &= B_{4h}(\tfrac{1}{2}) = (1 - 2^{1-4h})|B_{4h}|, & (h > 0),\\ M_{4h+2} &= B_{4h+2}(0) = B_{4h+2}, &\\ m_{4h} &= B_{4h}(0) = -|B_{4h}|, & (h > 0),\\ m_{4h+2} &= B_{4h+2}(\tfrac{1}{2}) = -(1 - 2^{-1-4h})B_{4h+2}. & \end{aligned}$$

The case of $n = 2k+1$ is more difficult simply because no exact formula exists for the roots of $B_{2k}(x) = 0$. From the relation

$$B_n(1-x) = (-1)^n B_n(x)$$

it follows that B_{2k+1} is skew-symmetric about the line $x = \frac{1}{2}$ and so, at any rate,

$$M_{2k+1} = -m_{2k+1} = |B_{2k+1}(r_{2k})|,$$

where r_{2k} stands for that simple root of $B_{2k}(x) = 0$ which lies in the interval $0 < x < \frac{1}{2}$. Nörlund (p. 131) remarks that

$$\tfrac{1}{6} < r_{2k} < \tfrac{1}{4}, \tag{2}$$

since

$$2B_{2k}(\tfrac{1}{6}) = (1 - 2^{1-2k})(1 - 3^{1-2k})B_{2k}$$

and

$$B_{2k}(\tfrac{1}{4}) = -2^{-2k}(1 - 2^{1-2k})B_{2k} \tag{3}$$

differ in sign. He states there that $r_{2k} \to \frac{1}{4}$ as $k \to \infty$. We shall prove a more precise result:

$$\tfrac{1}{4} - 2^{-2k-1}\pi^{-1} < r_{2k} < \tfrac{1}{4}. \tag{4}$$

In fact, let us define θ by the equation

$$\theta = 2\pi(\tfrac{1}{4} - r_{2k}).$$

Then by (2),

$$0 < \theta < \frac{\pi}{6}, \tag{5}$$

so that

$$\theta < \frac{\pi}{3} \sin \theta. \tag{6}$$

To prove (4) it is sufficient to show that

$$\theta < 2^{-2k}. \tag{7}$$

To this effect we find it simpler to use the Fourier expansion of $B_{2k}(x)$, (rather than its Taylor expansion):

$$B_{2k}(x) = (2k)!(-1)^{k-1}\sum_{n=1}^{\infty} (2n\pi)^{-2k}2 \cos 2n\pi x, \qquad (0 < x < 1).$$

Setting $x = r_{2k} = \frac{1}{4} - \theta/2\pi$ we obtain, after simplifying,

$$B_{2k}(r_{2k}) = (2l)!(-1)^{k-1}\left\{ \sum_{\nu=0}^{\infty} (-1)^\nu[(4\nu + 2)\pi]^{-2k}2 \sin (2\nu + 1)\theta \right.$$
$$\left. + \sum_{\nu=0}^{\infty} (-1)^\nu(4\nu\pi)^{-2k}2 \cos 2\nu\theta \right\}.$$

Setting the right member equal to zero, and cancelling unwanted factors, we find that θ must satisfy the trigonometric equation

$$\sin \theta = 2^{-2k} \cos 2\theta + 3^{-2k} \sin 3\theta - 4^{-2k} \cos 4\theta - 5^{-2k} \sin 5\theta + \cdots . \tag{8}$$

At first sight we have, on crudely taking absolute values,

$$\sin \theta < 2^{-2k} + 3^{-2k} + 4^{-2k} + \cdots < 2^{-2k} + \int_2^\infty x^{-2k}dx = \frac{2k+1}{2k-1} 2^{-2k}. \tag{9}$$

This gives some idea as to what inequalities will be needed in a closer examination of (8). These are

$$\cos 2\theta < 1, \ \sin 3\theta < 3 \sin \theta, \ \cos 4\theta > 1 - 8 \sin^2 \theta, \ \sin 5\theta > 0;$$

this last follows from (5). Then (8) becomes

$$\sin \theta < 2^{-2k} + 3^{-2k+1} \sin \theta - 4^{-2k} + 2^{-4k+3} \sin^2 \theta + 6^{-2k} + \int_6^\infty x^{-2k}dx. \tag{10}$$

Using the series for $\sin \theta$ with (6) and (9), we have

$$\theta < \sin\theta + \frac{\theta^3}{6} < \sin\theta + \frac{1}{6}\left(\frac{\pi}{3}\sin\theta\right)^3 < \sin\theta + \frac{1}{6}\left(\frac{\pi(2k+1)}{3(2k-1)}\right)^3 8^{-2k}.$$

Returning to (10) we can write, with the help of (9),

$$\theta - 2^{-2k} < \frac{2k+1}{2k-1}2^{-2k}3^{-2k+1} - 4^{-2k} + \left(\frac{2k+1}{2k-1}\right)^2 2^{-8k+3} + \frac{2k+5}{2k-1}6^{-2k} + \frac{1}{162}\left(\pi\frac{2k+1}{2k-1}\right)^3 8^{-2k}. \tag{11}$$

It is obvious that for all k sufficiently large, the right member of (11) is negative. To show that it is negative for all $k \geqq 3$, we proceed to estimate the various terms as follows:

$$\frac{2k+1}{2k-1}2^{-2k}3^{-2k+1} \leqq \frac{21}{5}6^{-2k},$$

$$\left(\frac{2k+1}{2k-1}\right)^2 2^{-8k+3} \leqq \frac{1}{8}\left(\frac{7}{5}\right)^2 8^{-2k} < \frac{1}{4}8^{-2k},$$

$$\frac{2k+5}{2k-1}6^{-2k} \leqq \frac{11}{5}6^{-2k},$$

$$\frac{1}{162}\left(\pi\frac{2k+1}{2k-1}\right)8^{-2k} \leqq \frac{1}{162}\left(\frac{7\pi}{5}\right)^3 8^{-2k} < \frac{3}{4}8^{-2k}.$$

Hence we have, from (11),

$$\theta - 2^{-2k} < \frac{32}{5}6^{-2k} + 8^{-2k} - 4^{-2k} < \frac{37}{5}6^{-2k} - 4^{-2k}$$

$$< 4^{-2k}\left[9\left(\frac{2}{3}\right)^{2k} - 1\right] \leqq 4^{-2k}\left(\frac{64}{81} - 1\right) < 0.$$

This proves (7), and hence (4) for $k \geqq 3$. That (4) is true also for $k=1$ and 2 may be verified directly from the values of r_2 and r_4 given below.

A good deal more can be obtained from (8) besides the inequality (4). In fact, if we assume, as is obviously justified by (10), that

$$\sin\theta = 2^{-2k} + O(4^{-2k}), \tag{12}$$

then (8) may be written

$$\sin\theta = 2^{-2k} + 2^{-2k}3^{-2k+1} - 4^{-2k} + 6^{-2k} + O(8^{-2k}); \tag{13}$$

and since

$$\theta = \sin\theta + O(\sin^3\theta) = \sin\theta + O(8^{-2k}),$$

we have at once

$$\theta = 2^{-2k}(1 - 2^{-2k} + 4\cdot 3^{-2k}) + O(8^{-2k}),$$

and hence the asymptotic formula for r_{2k} is

$$r_{2k} = \frac{1}{4} - \frac{1}{2\pi}(4^{-k} - 16^{-k} + 4\cdot 36^{-k}) + O(64^{-k}). \tag{14}$$

More exact results than this may be found by returning to (8), using (13) instead of (12) and taking more terms. For a particular k this iterative process is perhaps the simplest way of calculating r_{2k} when k is at all large. Even when $k=4$, (14) gives

$$r_8 = .2493803505,$$

whereas more exactly

$$r_8 = .2493803839.$$

From the fact that the root r_{2k} of $B_{2k}(x)$ tends rapidly to $\frac{1}{4}$ as $k\to\infty$, one might be apt to think that $B_{2k}(\frac{1}{4})$ tends to zero. On the contrary, by (3) and Stirling's theorem,

$$\left| B_{2k}(\tfrac{1}{4}) \right| \sim 4^{-k}\left| B_{2k} \right| \sim \sqrt{8\pi k}\left(\frac{k}{2\pi e}\right)^{2k},$$

a quantity which tends to ∞ with more than exponential rapidity.

To find the upper bound for M_{2k+1} we note first that the curve $y=B_{2k+1}(x)$ has no point of inflection near $x=\frac{1}{4}$. In fact its second derivative $2k(2k+1)B_{2k-1}(x)$ has no root between $x=0$ and $x=\frac{1}{2}$. Hence the tangent line drawn to the curve $y=\left|B_{2k+1}(x)\right|$ at the point $(\frac{1}{4}, \left|B_{2k+1}(\frac{1}{4})\right|)$ lies above the curve between $x=\frac{1}{4}-2^{-2k-1}/\pi$ and $x=\frac{1}{4}$. Hence the maximum point, which lies between these limits, is below that point on the tangent line whose abscissa is $x=\frac{1}{4}-2^{-2k-1}/\pi$.

The equation of this tangent line is obviously

$$y = \left| B_{2k+1}(\tfrac{1}{4}) \right| - (x - \tfrac{1}{4})\left| B'_{2k+1}(\tfrac{1}{4}) \right|. \tag{15}$$

Now

$$\left| B_{2k+1}(\tfrac{1}{4}) \right| = (2k+1)\cdot 4^{-2k-1}\left| E_{2k} \right|,$$

where E_n is the nth Eulerian number, and

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61, \cdots,\ E_{2k+1} = 0,$$
$$(E+1)^\nu + (E-1)^\nu = 0, \qquad (\nu > 0).$$

Using this with (1) and (3) in (15), with $x=\frac{1}{4}-2^{-2k-1}/\pi$, we obtain at once

$$M_{2k+1} < (2k+1)\cdot 2^{-4k-2}\left\{ \left| E_{2k} \right| + \frac{2}{\pi}(1 - 2^{-2k+1})\left| B_{2k} \right| \right\}. \tag{16}$$

Using the well known identities of Euler,

$$(17)\qquad \begin{aligned} |E_{2k}| &= (2k)!2^{2k+2}\pi^{-2k-1}(1 - 3^{-2k} + 5^{-2k} - \cdots),\\ |B_{2k}| &= (2k)!2^{-2k+1}\pi^{-2k}(1 + 2^{-2k} + 3^{-2k} + \cdots),\end{aligned}$$

we derive easily the inequalities

$$(18)\qquad \begin{aligned} |E_{2k}| &< (2k)!2^{2k+2}\pi^{-2k-1}(1 - 3^{-2k} + 5^{-2k}),\\ (1 - 2^{-2k+1})|B_{2k}|\frac{2}{\pi} &< (2k)!2^{2k+2}\pi^{-2k-1}4^{-2k}, \qquad (k > 1).\end{aligned}$$

Substituting these in (16) we get the simple inequality

$$M_{2k+1} < 2(2k+1)!(2\pi)^{-2k-1}(1 - 3^{-2k} + 4^{-2k} + 5^{-2k}) < 2(2k+1)!(2\pi)^{-2k-1}.$$

It is interesting to note that from Theorem 1 and from (18), the inequality

$$(19)\qquad M_n < 2n!/(2\pi)^n$$

holds for all n not of the form $4h+2$, while the inequality

$$m_n > -2n!/(2\pi)^n$$

holds for all n not of the form $4h$. In these exceptional cases one must supply the factor $\zeta(n)=\sum_{\nu=1}^{\infty}\nu^{-n}$, and replace the inequality sign by equality. Hence if we neglect a factor of order $1+2^{-n}$, the maxima and minima of all Bernoulli polynomials are given in absolute value by the same function of n. In conclusion we give a short table of the roots r_{2k} of $B_{2k}(x)=0$ and of M_{2k+1}. Formulas (14) and (16) and Theorem 1 render a more extensive table virtually unnecessary.

$2k$	r_{2k}	$2k+1$	M_{2k+1}
2	.2113248654	3	.04811252243
4	.2403351888	5	.02445819087
6	.2475407162	7	.02606511426
8	.2493803839	9	.04755056164
10	.2498447170	11	.13249665844
12	.2499611530	13	.52356641061

Comparing the final entries of this table with the results obtained from formulas (14) and (16), we find the latter values to be

$$r_{12} = .24996115328, \qquad M_{13} < .52356641134.$$

Even formula (19) gives

$$M_{13} < .5235667.$$

A FORMULA FOR APPROXIMATE COMPUTATION OF A TRIPLE INTEGRAL

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In the present paper the following approximate formula is derived and discussed:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} u(x, y, z)dxdydz = \frac{4}{225}\left[91\sum u_6 - 40\sum u_{12} + 16\sum u_{24}\right]. \tag{1}$$

The integrand $u(x, y, z)$ is a function of x, y, z; the region of integration is the cube bounded by the six planes

$$x = \pm 1, \qquad y = \pm 1, \qquad z = \pm 1.$$

Formula (1) is, evidently, a three-dimensional counterpart of similar one-dimensional and two-dimensional formulas as given by Gauss and by Burnside.* The points which are used in the right member of (1) to obtain the local values u_6, u_{12}, and u_{24} of the integrand lie on the surface of the cube, and consequently we may refer to (1) as a *boundary type* or a *surface type* formula.

The demand for a formula of this type came originally from an industrial company, in connection with an applied problem which arose in the research and development laboratories of this company. The interior of a certain cube was inaccessible, while its surface was exposed and could be used to obtain by direct measurements the values of the integrand at any point on the surface. The mathematical part of the problem was to determine a distribution of points on the surface that would give the most accurate approximation to the triple integral in the left member of (1).

No surface type formula whatsoever is adequate to give a precise result in case the integrand $u(x, y, z)$ is a polynomial of a degree equal to or exceeding six. Indeed, the example

$$u(x, y, z) = (x - 1)(x + 1)(y - 1)(y + 1)(z - 1)(z + 1) \tag{2}$$

presents a polynomial of degree six which vanishes identically at every point on the surface of the cube. Hence, every surface type formula will give the value zero as the approximation to the triple integral, while the exact value of the triple integral with (2) is readily found to be $-64/27$. This example shows that the best that we may expect from a surface type formula is exactness for polynomial integrands of a degree not exceeding five. Formula (1) does possess this degree of accuracy. Consequently, the natural limit of exactness for surface type formulas is reached by (1), and no further essential improvement of the accuracy is conceivable.

In choosing the distribution of points on the surface of the cube to pick up the particular values of the integrand to be used in the right member of (1),

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* For literature on approximate computation of double integrals, see E. T. Whittaker and G. Robinson, The Calculus of Observations, Second edition, 1937, p. 375.

it is natural to make use of the symmetric structure of the cube and to select the points in such a way that any transformation of the cube by the extended group G_{48} of the cube (the extended octahedral group) should transform the set of points selected into itself (an automorphism of the set). Since reflections in the coördinate planes belong to the group, such a selection of points will automatically make the formula to be established exact for any function $u(x, y, z)$ which is odd with respect to some one of the arguments x, y, z. The coefficients in the right member of the formula will have to be chosen so as to make the formula exact for all such monomial integrands which are of a degree not exceeding five and in which no odd exponent is present at all. Since the group G_6 of permutations of the three elements x, y, z is a sub-group of the G_{48} considered, there exist but four essentially independent monomials of the kind described, namely,

(3) $$u = 1, \qquad u = x^2, \qquad u = x^4, \qquad u = x^2y^2.$$

Figure 1 shows a face of the cube with the fundamental region of G_{48} shaded.

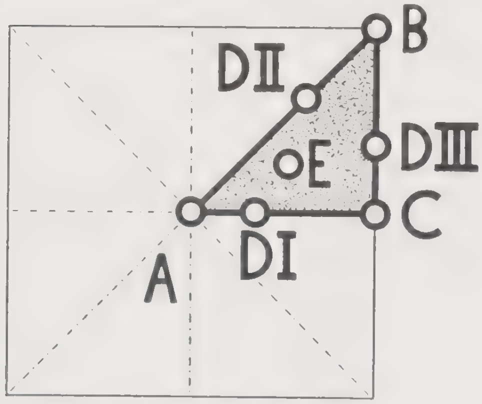

FIG. 1

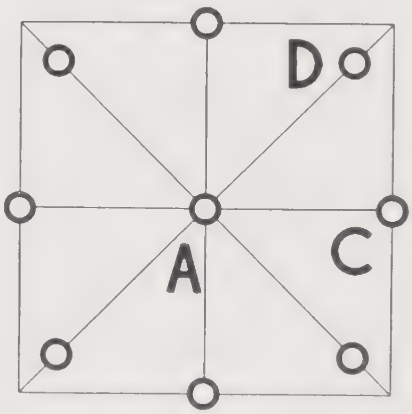

FIG. 2

With respect to this fundamental region, we find five distinct kinds of points on the face:

(A) The center of the face, A. This point and its transforms by G_{48} constitute a set of 6 points.

(B) The vertex of the face, B. This point and its transforms constitute a set of 8 points.

(C) The midpoint of the side of the face, C. This point and its transforms constitute a set of 12 points.

(D) Any point on the boundary of the fundamental region, D, other than A, B, or C. Points of this type may be classified as points $D(\mathrm{I})$, $D(\mathrm{II})$, and $D(\mathrm{III})$ according as they lie on AC, AB, or BC. A point of the type D and its transforms constitute a set of 24 points.

(E) Any point in the interior of the fundamental region, E. A point E and its transforms constitute a set of 48 points.

Suppose we choose the following composition of points within a fundamental region: a, b, c, d, e points of types A, B, C, D, E, respectively, where $a, b, c=0$ or 1, and $d, e=0$ or any positive integer. The total number of points generated by this selection and appearing on the whole of the surface of the cube will then be

$$n = 6a + 8b + 12c + 24d + 48e. \tag{4}$$

The problem is to determine the minimum value n which admits exact solutions for all four monomials (3) simultaneously. The numbers representable by the linear form (4) may now be arranged in order of increasing magnitude as far as 42. In cases where the value of d involved is different from zero it is necessarily one, because $d=2$ gives $n \geqq 48$, contrary to $n<43$. The value $d=1$ means that just one point of the type D is used, which gives rise to three distinct formulas according to the triple character of the points of type D. In the following table, the cases with $d=1$ are marked by the sign (I, II, III) to remind us of the three varieties present in each such case:

$n = 0$	$n = 24$ (I, II, III)
$= 6$	$= 26$
$= 8$	$= 30$ (I, II, III)
$= 12$	$= 32$ (I, II, III)
$= 14$	$= 36$ (I, II, III)
$= 18$	$= 38$ (I, II, III)
$= 20$	$= 42$ (I, II, III)

For each one of these arithmetically possible values of n, an exhaustive attempt was made by the author to set up a corresponding formula which would be exact for all four cases (3) simultaneously. The first success encountered was with $n=42$, type (II), and the formula obtained is formula (1) of the present paper. All the preceding 22 cases with $n<42$ and the two cases of types (I) and (III) with $n=42$ failed to give rise to a formula as desired for various irremovable algebraic reasons, such as appearance of complex solutions, or of points located outside of the proper surface of the cube on the extended faces. Some cases led to simultaneous algebraic equations with intrinsic contradictions in the system.

The minimum value of n is given by formula (4), using the following values:

$$a = 1, \ b = 0, \ c = 1, \ d = 1 \text{ (type II)}, \ e = 0.$$

Hence, $n=6+12+24=42$.

The computation of coefficients in the right member of (1) and the determination of the proper location of the point D on the diagonal AB are shown in the appendix. Now $AB=\sqrt{2}$, $AC=BC=1$, and it is shown in the appendix that $AD=\frac{1}{2}\sqrt{5}=1.118034$.

Figure 2, which is drawn to scale, shows one face of the cube with all the points that were used appearing on it. In (1), $\sum u_6$ denotes the sum of the six

values of $u(x, y, z)$ determined at the six points of type A (the six centers of the six faces of the cube), $\sum u_{12}$ denotes the sum of the twelve values of $u(x, y, z)$ determined at the twelve points of type C (the twelve midpoints of the twelve edges of the cube), and $\sum u_{24}$ denotes the sum of those twenty-four values of $u(x, y, z)$ which are determined at the twenty-four points of type D.

Appendix. To derive the coefficients in the right member of (1), let us first write the formula with indeterminate coefficients L, M, N as

$$L\sum u_6 + M\sum u_{12} + N\sum u_{24} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} u\,dxdydz. \tag{5}$$

The formula is to be exact for the integrands 1, x^2, x^4, x^2y^2. The corresponding values of the triple integral are 8, 8/3, 8/5, 8/9, respectively, and (5) becomes, correspondingly,

$$6L + 12M + 24N = 8, \tag{6}$$

$$2L + 8M + (8 + 16x^2)N = \frac{8}{3}, \tag{7}$$

$$2L + 8M + (8 + 16x^4)N = \frac{8}{5}, \tag{8}$$

$$4M + (16x^2 + 8x^4)N = \frac{8}{9}, \tag{9}$$

where x stands for the abscissa of D (Fig. 2). The coördinates (x, y, z) of D evidently have the values $(x, x, 1)$. Figure 3 may be used to verify equations

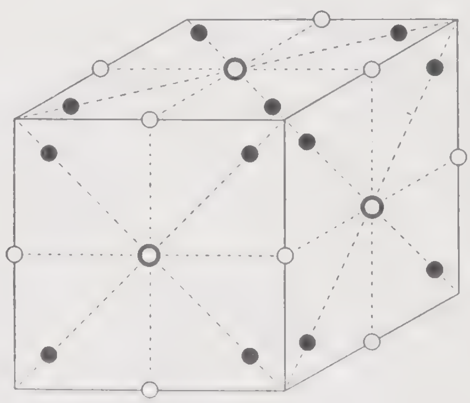

FIG. 3

(6) to (9). The three different kinds of points A, C, D are illustrated by distinct patterns in Figure 3. Counting the points contributing to each term and considering their coördinates, equations (6) to (9) are readily verified. To solve

the system (6) to (9) for L, M, N, x, let us solve for x first. Subtracting (8) from (7), we have

$$(16x^2 - 16x^4)N = \frac{16}{15}\cdot \tag{10}$$

Subtracting (9) from (7), we have

$$2L + 4M + (8 - 8x^4)N = \frac{16}{9}\cdot \tag{11}$$

Multiplying (11) by three and subtracting the product from (6), we have

$$24x^4N = \frac{8}{3}\cdot \tag{12}$$

Using this in (10) to eliminate the term $-16x^4N$, we obtain

$$x^2N = \frac{8}{45}\cdot \tag{13}$$

Dividing (12) by (13), we find

$$x^2 = \frac{5}{8}; \quad \text{thus} \quad x = \sqrt{\frac{5}{8}}.$$

Hence,

$$AD = x\sqrt{2} = \frac{1}{2}\sqrt{5}.$$

After x has been found, the determination of L, M, N is reduced to solving a linear system. The results are

$$L = \frac{364}{225}, \qquad M = -\frac{160}{225}, \qquad N = \frac{64}{225}\cdot$$

This completes the derivation of formula (1).

Summary. Formula (1) is exact for polynomial integrands $u(x, y, z)$ whose degree is less than six, and approximate for other types of integrands. There exists no surface type formula which is exact for polynomials of degree six or higher than six; any other surface type formula with the number of points used equal to or less than 42, is essentially less accurate than (1) because its range of exactness would not reach as far as to include polynomials of degree five.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

DIAGNOSTIC TESTING PROGRAM IN PURDUE UNIVERSITY

1. Formal Algebraic Manipulations

M. W. KELLER, D. R. SHREVE, AND H. H. REMMERS, Purdue University

In this paper are presented the factual findings of a diagnostic test on the algebraic abilities of students entering first-semester freshman mathematics courses in Purdue University. An accounting is given, in tabular form, of the types and frequencies of errors which these students made in attempting to perform basic manipulations with algebraic expressions.

The test to be discussed here is one of a series of three diagnostic tests and twelve achievement tests, constructed by the authors, and administered with the coöperation of Dr. R. H. Downing, of the Department of Mathematics. The diagnostic tests were given to 202 engineering freshmen, and to 75 School of Science students, enrolled by regular, non-selective registration procedure in the classes taught by Drs. Keller, Shreve, and Downing. Check on the Orientation (Placement) Test scores indicated that these students were a fairly representative sample of the student body. This sample was approximately 20 per cent of the first-semester freshman mathematics enrollment.

For reference there is given in the list titled "Sample Test Problems" a set of 16 problems chosen from the test to be discussed here. The problems are divided into 5 classes, according to type. The first 14 problems of the test sample list were matched by approximately equivalent "mates," so that the consistency of error of individual students could be observed. The problems of each of the fourteen pairs were designed to involve the use of the same manipulative processes, and in the judgment of the authors to be of equal difficulty; the coefficient of correlation determined by scoring odd-numbered problems against even-numbered problems was 0.91. The problems listed as number 15 and number 16 were not paired with other problems, but are of approximately equal difficulty, judged by student successes, although they involve distinctly different technique difficulties.

The figures in parentheses following the problems in the "Sample Test Problems" give first, the number of engineers who worked both problems of the pair correctly; second, the number of School of Science algebra course students who worked both problems of the pair correctly; and third, the total number of students who worked both problems of the pair correctly.

NUMBER OF CORRECT ANSWERS TO SAMPLE TEST PROBLEMS

(By 202 Freshman Engineers, 75 School of Science students, 277 students)

Group A. Addition and Subtraction

1.	$3a-2b+(1-2b-3a)$	(151, 54, 205)
2.	$(a-b)-(-2b+a)$	(150, 45, 195)
3.	$(3a-2b)-7(2a-b)$	(104, 32, 136)

Group B. Multiplication of Binomials

4.	$(2b+5a)(2b-5a)$	(158, 51, 209)
5.	$(A+2b)(a+3b)$	(150, 37, 187)

Group C. Factoring

6.	w^2-25x^2	(173, 62, 235)
7.	$3a^2+a-2$	(152, 45, 197)
8.	$2+9a+9a^2$	(168, 48, 216)
9.	$5b^2-9b-2$	(150, 39, 189)
10.	$2a-32a^3$	(79, 26, 105)

Group D. Completing the Square

11.	$x^2-Bx+81$ What must B equal if the polynomial	(140, 36, 176)
12.	$y^2+14y+B$ is to be a perfect square?	(137, 32, 169)

Group E. Fractions

13.	$\dfrac{x-2}{x+2}\div\dfrac{x-5}{x+2}$	(133, 41, 174)
14.	$\dfrac{x^2+3x}{7-x}\cdot\dfrac{7+x}{x+3}$	(90, 26, 116)
15.	$\dfrac{1}{x^2-y^2}-\dfrac{4}{x+y}$	(64, 11, 75)
16.	$\dfrac{5}{a-b}-\dfrac{2}{b-a}$	(60 16, 76)

CODE CLASSIFICATION OF ERRORS

1. Error in signs introduced in removing parentheses preceded by a positive sign.
2. Error in signs introduced in removing parentheses preceded by a negative sign.
3. Error in sign in addition of terms.
4. Numerical error in addition of terms.
5. Failing to combine like terms to simplify an expression.
6. Failing to distribute multiplication. For example, $(3a-2b)-7(2a-b)=3a-2b-14a-b$.
7. Dropping a term or a symbol, as $3a-2b+(1-2b-3a)=3a-2b-2b-3a$.
8. Failing to remove parentheses to simplify. For example, $(3a-2b)-7(2a-b)=-7(a-b)$; the student subtracts $2a$ from $3a$, and $-b$ from $-2b$.
9. Multiplying polynomials where algebraic addition is indicated. For example, $(a-b)-(-2b+a)=a^2-3ab+2b^2$.
10. Multiplying terms where addition is indicated. For example, $2a^2+3a=6a^3$.
11. Adding where multiplication is indicated. For example, $(2b+5a)(2b-5a)=4b$.
12. Miscopying.
13. Misunderstanding, misreading, or confusing symbols or operations. For example, $2a^3=6a$, or a division sign "$\div$" is read "$+$."
14. Multiplying polynomials "term by term." For example, $(A+2b)(a+3b)=Aa+6b^2$.
15. Failure to discriminate between unlike symbols. For example, $(A+2b)(a+3b)=a^2+5ab+6b^2$.
16. Combining unlike terms. For example, $x^2+3x=4x^2$.

17. Numerical error in multiplication.

18. Error in exponent in multiplication.

19. Multiplying by incorrect rule or wrong formula. For example, $(2b+5a)(2b-5a) = 4b^2+20ab+25a^2$.

20. Error in signs in factors. For example, the factors in problem 9 are given as $(5b-1)(b+2)$.

21. Numerical error in factoring, or error in exponent in factors.

22. Factoring "term by term." For example, the factors of problem 8 are given as $(2+9+9a)(1+a+a)$.

23. Factoring by wrong formula. For example, the factors in problem 6 are given as $(w-5x)^2$.

24. Failing to factor completely. For example, the factors in problem 10 are given $2a(1-16a^2)$.

25. Stating that the expression cannot be factored.

26. Apparent failure to understand what is meant by the instructions "factor the expression." For example, the factors of problem 7 are written $a(3a+1)-2$.

27. Arbitrarily changing signs.

28. Attempting to complete the square by factoring. For example, in problem 12, the factors $(y+8)(y+6)$ give a third term $+48$.

29. Failing to isolate the coefficient of the linear term in attempting to complete the square. For example, the solution in problem 11 is stated $Bx=18$, or $B=18x$.

30. Attempting to apply the rule: "Take half the coefficient of the second term; square it; and add it on."

31. Failing to cancel common factors. For example,

$$\frac{x^2+3x}{7-x}\cdot\frac{7+x}{x+3} = \frac{x^3+10x^2+21x}{21+4x-x^2}.$$

32. Unwarranted cross-multiplication, as in problem 14 where the fraction is written

$$\frac{x^3+6x^2+9x}{49-x^2}.$$

33. In dividing by a fraction, failing to invert the divisor.

34. Arbitrarily multiplying all parts of the fraction. For example, after obtaining the correct result in problem 14, the student may then write out the product $x(7+x)(7-x)$.

35. Dropping the denominator, or a factor of the denominator, of a fraction.

36. Adding or subtracting the parts of a fraction. For example, in problem 13, $(x-2)/(x-5) = 2x-7$.

37. Adding numerators and denominators indiscriminately in combining fractions. For example, in problem 16 the student may write

$$\frac{5-2}{a-b-b+a}.$$

38. Canceling unlike polynomial factors of numerator and denominator of a fraction.

39. "Canceling" term by term. For example

$$\frac{x^2+7x}{7-x} = x+x.$$

40. Miscellaneous errors, or errors in which the student's work is incomprehensible.

41. Answer omitted. No distinction is made here between the omission of an entire problem, and the omission of an answer after an unsuccessful attempt to work the problem.

In the following Tables, which summarize results, the entries may be explained thus: By Problem pair 1 is meant the pair of problems consisting of Problem 1, Group A of the Sample Test Problems, and the matched or equival-

ent problem of the test which is not given in this paper. Under Error class 1 the entry 20 means that there were 20 errors of classification 1 (error in signs introduced in removing parentheses preceded by a positive sign) on the problem pair 1,—a total of 20 errors of this type on the two problems.

TABLE 1

Errors in Problems of Addition and Subtraction (Group A)

Problem pair	Error class													
	1	2	3	4	5	6	7	8	9	10	11	12	40	41
1	20		14	16	3		4	15	4	5		2	5	19
2	8	23	8	2	8		2	1	48	1		3	8	16
3	15	40	18	6	15	25	6	3	67	2	6	4	11	20
Sub-totals	43	63	40	24	26	25	12	19	119	8	6	9	24	55

TABLE 2

Errors in Problems of Multiplication of Binomials (Group B)

Problem pair	Error class										
	7	10	11	14	15	16	17	18	19	40	41
4	3		5				29	9	19	5	17
5	15	11	15	10	42	19	15	17	2	12	8
Sub-totals	18	11	20	10	42	19	44	26	21	17	25

TABLE 3

Errors in Problems of Factoring (Group C)

Problem pair	Error class												
	7	13	16	20	21	22	23	24	25	26	27	40	41
6	5	8	1		2	3	7		2	2	3	6	17
7			6	21	16	1			3	7		8	49
8	1	1	3	4	8	7	5		12	10		8	29
9		1	3	32	7	4			4	10		4	54
10	19	2	5	18	10	23	8	114	9	5	13	10	87
Sub-totals	25	12	18	75	43	38	20	114	30	34	16	36	236

TABLE 4

Errors in Problems of Completing the Square (Group D)

Problem pair	Error class					
	13	28	29	30	40	41
11	8	25	17	58	2	67
12	14	23	5	57	7	63
Sub-totals	22	48	22	115	9	130

TABLE 5. PART I

Errors in Manipulating Fractions (Group E)

Problem pair	Error class												
	4	5	6	7	8	10	12	13	14	16	17	18	23
13	5		1	1		1	7	20	11	2	2	2	
14		1	3	5		1	5		8		18	7	6
15	1	1	1	10		4	5	6		60	29		
16	2	8	2		7	1	2	2	2		13	1	
Sub-totals	8	10	7	16	7	7	19	28	21	62	62	10	6

TABLE 5. PART II

Errors in Manipulating Fractions (Group E)

Problem pair	Error class											
	27	31	32	33	34	35	36	37	38	39	40	41
13		22	9	11	9	4	18	4	1	24	14	41
14	4	72	24		18	5	4		24	52	28	71
15	6					36	2	8	1	25	25	39
16	32	21	1		2	25		37		26	9	40
Sub-totals	42	115	34	11	29	70	24	49	26	127	76	191

This first algebra test on formal manipulations was followed by a second algebra test on the solutions of simple algebraic equations, on which the authors hope to report in a subsequent paper.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The Department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

DISTANCES BETWEEN POINTS OF THE CANTOR SET

J. F. RANDOLPH, Cornell University

The Cantor "middle third" set* consists of exactly those points on the unit interval that are left after the middle open third of the interval is removed, then the middle open third of each of the two remaining thirds is removed, then the middle open third of each of the 4 remaining ninths is removed, *etc.* For each positive integer n after n steps have been taken in this process there are 2^n intervals each of length $1/3^n$ left on the unit interval; the $(n+1)$th step is to remove the middle open third of each of these 2^n intervals.

Thus from an interval of length unity has been removed a countably infinite number of open intervals the sum of whose lengths,

$$(1/3) + 2(1/3^2) + 2^2(1/3^3) + \cdots + 2^n(1/3^{n+1}) + \cdots,$$

is unity. Consequently the remaining set of points, *i.e.*, the Cantor set, is pretty thinly scattered over the unit interval. Nevertheless, the following theorem is proved in this note.†

THEOREM 1. *The set of distances between points of the Cantor set fills the unit interval, i.e., given any number c, $0 \leq c \leq 1$, there are two numbers x and y representing points of the Cantor set such that*

$$y - x = c.$$

In this note the ternary representation of numbers is used. For example,

$$21.1201 = 2 \cdot 3 + 1 + 1/3 + 2/3^2 + 0/3^3 + 1/3^4.$$

Using the ternary notation, any point of the unit interval is represented as $.\gamma_1\gamma_2\gamma_3 \cdots$, where each γ_n is 0, 1, or 2 and for some points there are two such representations. Thus the number $1/3 = .1$ may also be written $1/3 = .022 \cdots$. Also, $2/3 = .2$ but might appear as $2/3 = .122 \cdots$. However, any number with the digit 1 in the first place which is not the last significant digit and not all succeeding digits are 2 (as $.102 = 1/3 + 2/27$) lies in the open middle third of the unit interval and so does not represent a point in the Cantor set. Thus $.1$ and $.122 \cdots$ are the only members with the digit 1 in the first place that represent points of the Cantor set, and these numbers may be written without the use of

* Georg Cantor, Mathematische Annalen, vol. 21, 1883, p. 590.

† This theorem was first proved using geometric methods and published in an obscure Polish journal, Wektor, in 1917 by H. Steinhaus.

the digit 1. By applying similar arguments to 1/9 and 2/9, 7/9 and 8/9, *etc.*, one will see that

The Cantor set consists of exactly those points whose ternary representation may be written without the use of the digit 1.

Theorem 1 may then be stated in the following way:

THEOREM 1′. *If* $c=.\gamma_1\gamma_2\cdots\gamma_n\cdots$, *where each* γ_n *is* 0, 1, *or* 2, *then there exist two numbers* $x=.\alpha_1\alpha_2\cdots\alpha_n\cdots$, *and* $y=.\beta_1\beta_2\cdots\beta_n\cdots$, *where each* α_n *and each* β_n *is* 0 *or* 2 *unless it is the last significant figure when it may be* 1, *such that* $y=x+c$.

The proof is made after introducing a notation suggested by Professor W. A. Hurwitz. Generically let a δ denote a succession (which may contain no elements) of 0's and 2's. Thus the number

$$.021220110200\cdots$$

would be indicated as $.\delta_1 1\delta_2 1\delta_3 1\delta_4$, where δ_1 is 02, δ_2 is 220, δ_3 contains no elements, and δ_4 is $0200\cdots$.

With δ_i known, let δ_i' be the succession of 0's and 2's obtained from δ_i by replacing each 0 by a 2 and each 2 by a 0. Thus for the example just given, δ_2' is 002.

Also, let $(0)_i$ be the succession of zeros obtained from δ_i by keeping each 0 and replacing each 2 by a 0.

We now distinguish three cases.

Case 1. Here $c=.\gamma_1\gamma_2\cdots$ contains the digit 1 an even (so of course finite) number of times (or not at all) and may therefore be written

$$c = .\delta_1 1\delta_2 1\cdots\delta_{2n-1}1\delta_{2n}1\delta_{2n+1}.$$

Let

$$x = .(0)_1 0\delta_2' 2\cdots(0)_{2n-1}0\delta_{2n}' 2(0)_{2n+1},$$

and

$$y = .\delta_1 2(0)_2 0\cdots\delta_{2n-1}2(0)_{2n}0\delta_{2n+1};$$

then $0\leqq x\leqq 1$, $0\leqq y\leqq 1$ and we will see that $y=x+c$.

For, symbolically, $\delta_{2n+1}+(0)_{2n+1}=\delta_{2n+1}$. Now by adding the last 1 of c to the last 2 of x we obtain 0 with 1 to carry. The sum of the last digits of δ_{2n} and δ_{2n}' is 2 which with the 1 we had to carry gives 0 with 1 to carry. Proceeding with the addition we continue to get 0 with 1 to carry as long as the digits of δ_{2n} last. We thus have 1 to carry over to the digit 1 before δ_{2n} which with the 0 before δ_{2n}' of x gives 2 with none to carry. Consequently we proceed with δ_{2n-1} exactly as we started with δ_{2n+1} and therefore see that $y=x+c$.

Case 2. The digit 1 appears an odd number of times among the γ's in c, *i.e.*,

$$c = .\delta_1 1 \delta_2 9 \cdots \delta_{2n-1} 1 \delta_{2n}\, 1 \delta_{2n+1} 1 \delta_{2n+2}.$$

Let

$$x = .(0)_1 0 \delta_2' \, 2 \cdots (0)_{2n-1} 0 \delta_{2n}' \, 2 (0)_{2n+1} 1 (0)_{2n+2},$$

and

$$y = .\delta_1 2 (0)_2 0 \cdots \delta_{2n-1} 2 (0)_{2n} 0 \delta_{2n+1} 2 \delta_{2n+2},$$

where the digit 1 is permissible in x since it is the last significant figure. The only difference in the definitions of x and y here and in case 1 is in the last two symbols. Here also it is seen that $y = x + c$.

Case 3. The digit 1 appears an infinite number of times in c, so

$$c = .\delta_1 1 \delta_2 1 \delta_3 1 \delta_4 1 \cdots .$$

Let

$$x = .(0)_1 0 \delta_2' \, 2 (0)_3 0 \delta_4' \, 2 \cdots ,$$

and

$$y = .\delta_1 2 (0)_2 0 \delta_3 2 (0)_4 0 \cdots .$$

Let $c_n = .\delta_1 1 \delta_2 1 \cdots \delta_{2n-1} 1 \delta_{2n} 1$, so that $\lim c_n = c$. For similar definitions of x_n and y_n from x and y respectively, $\lim x_n = x$ and $\lim y_n = y$. Since $y_n = x_n + c_n$ (from case 1), we thus have $y = x + c$.

Given a number c, $0 < c < 1$, the numbers x and y representing points of the Cantor set such that $y = x + c$ need not be unique, *i.e.*, there are rules other than those given in the proof of Theorem 1′ that yield numbers of the desired type. For example, if $c = .021201$ we may have

$$\begin{array}{r} .021201 \\ \underline{.001000} \\ .022201 \end{array} \quad \text{or} \quad \begin{array}{r} .021201 \\ \underline{.201000} \\ .222201 \end{array} \quad \text{or} \quad \begin{array}{r} .021201 \\ \underline{.000022} \\ .022000. \end{array}$$

However, care must be taken or we would proceed as follows:

$$\begin{array}{r} .021201 \\ \underline{.202001} \\ 1.000202, \end{array}$$

and obtain numbers x and y whose fractional parts may be written using only 0's or 2's, but $y > 1$ and therefore is not in the Cantor set of the unit interval.

The reader may prove for himself the following:

THEOREM 2. *There exist numbers x and y representing points of the Cantor set such that $y + x = c$.*

THEOREM 3. *Every point (except the end-points) of the unit interval is midway between two points of the Cantor set.*

ON THE EXPLICIT SOLUTION OF SIMULTANEOUS LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

B. Z. LINFIELD, University of Virginia

In the January 1940 issue of this MONTHLY (pp. 35–38) there is a note by Professor J. S. Frame, *On the explicit solution of simultaneous linear differential equations with constant coefficients.*

The object of this note is to bring out that just as

$$X = e^{rt}\left[I + (A - rI)t + (A - rI)^2 \frac{t^2}{2!} + \cdots + (A - rI)^{m-1} \frac{t^{m-1}}{(m-1)!}\right]C$$

is a solution of the set of simultaneous *differential* equations with constant coefficients

$$dX/dt = AX,$$

so

$$X = (1 + r)^t\left[I + \frac{A - rI}{1 + r} t + \left(\frac{A - rI}{1 + r}\right)^2 \frac{t^{(2)}}{2!} + \cdots + \left(\frac{A - rI}{1 + r}\right)^{m-1} \frac{t^{(m-1)}}{(m-1)!}\right]C$$

is a solution of the set of simultaneous *difference* equations with constant coefficients

$$\Delta X \equiv X(t + 1) - X(t) = AX,$$

where—in either case—the scalar polynomial $\det(A - tI)$ has a factor $t - r$ of multiplicity m, and C is any matrix satisfying $(A - rI)^m C = 0$.

More generally, there is no difficulty in writing down the solution of

$$\Delta_h X \equiv \frac{X(t + h) - X(t)}{h} = AX \tag{1}$$

for any value of $h \neq 0$, namely,

$$X = (1 + rh)^{t/h}\left[I + \frac{A - rI}{1 + rh} t + \left(\frac{A - rI}{1 + rh}\right)^2 \frac{t^{(2h)}}{2!} + \cdots + \left(\frac{A - rI}{1 + rh}\right)^{m-1} \frac{t^{(\overline{m-1}h)}}{(m-1)!}\right]C. \tag{2}$$

Here

$$t^{(ih)} = t(t - h)(t - 2h) \cdots (t - \overline{i-1}h),$$

$$\Delta_h t^{(ih)} = i t^{(\overline{i-1}h)}.$$

From this we obtain Professor Frame's solution by letting $h \to 0$, for then $(1+rh)^{t/h} \to e^{rt}$ and $t^{(ih)} \to t^i$.

Since the equations (1) are linear, the sum of two or more solutions is a solution. Therefore the sum of the solutions corresponding to the distinct factors $t-r_i$ of $\det(A-tI)$ is a solution. However, it is apparent from the form of (2) that it gives no solution when $1+rh=0$.

For example, using the same matrix A that Professor Frame used in his illustration,* a solution of (1) is

$$X = M(t, h) \cdot \begin{Vmatrix} 2c_1 - 2c_2 + 3c_3 \\ c_1 + 6c_2 - c_3 \\ 2c_1 + 3c_2 + 3c_3 \\ 2c_1 - 2c_2 + 6c_3 \end{Vmatrix} (1 + 2h)^{t/h} + \begin{Vmatrix} c_4 \\ 0 \\ 0 \\ c_4 \end{Vmatrix} (1 - h)^{t/h},$$

where

$$M(t, h) = I + \frac{A - 2I}{1 + 2h} t + \left(\frac{A - 2I}{1 + 2h}\right)^2 \frac{t(t - h)}{2}$$

$$= I + \begin{Vmatrix} -1 & -2 & 3 & -2 \\ 1 & 3 & -1 & -1 \\ 2 & 3 & 0 & -2 \\ 2 & -2 & 6 & -5 \end{Vmatrix} \frac{t}{1 + 2h} + \begin{Vmatrix} 1 & 9 & -13 & 8 \\ -2 & 6 & -6 & 2 \\ -3 & 9 & -9 & 3 \\ -2 & 18 & -22 & 11 \end{Vmatrix} \frac{t(t - h)}{2(1 + 2h)^2}.$$

In particular, when $h=1$, the part of the solution of (1) having the factor $(1-h)$ clearly drops out, and the remaining part has only three arbitrary constants, c_1, c_2, c_3, with the factor 3^t. When $h=-1/2$, the part having the factor $(1+2h)$ drops out, and the remaining part has only one arbitrary constant, c_4, with the factor $(3/2)^{-2t}$.

The procedure of deriving the solutions of

$$\frac{dX}{dt} = F(X)$$

from the solutions of

$$\Delta_h X = F(X),$$

by letting $h \to 0$, is that of deriving the solution of the scalar equation

$$\frac{d}{dt} f(t) = rf(t)$$

from the solution of

$$\Delta_h f(t) = rf(t), \qquad f(t) = (1 + rh)^{t/h},$$

by letting $h \to 0$, which shows how the *exponential function* e^{rt} could have been discovered and, to some extent, how it actually was discovered and studied.

* There is a misprint in equation (9) of Professor Frame's paper; the $8r$ should be $4r$.

Moreover, it has the essential idea of the original derivation of *Taylor's expansion*, and throws considerable light on numerous situations which cannot be considered here.

Note by the Editor. The application of Frame's method to both differential and difference equations is not surprising if one considers the abstract nature of the process. Let $x, y, \cdots$ represent functions, and let θ be any linear operator on these functions; that is, $\theta(ax+by)=a\theta x+b\theta y$ if a and b are constants. Then Frame's result is essentially the following:

If the functions $x_{a,i}$ satisfy the recursion relations,

$$(\theta - a)x_{a,0} = 0, \qquad (\theta - a)x_{a,i} = x_{a,i-1}, \qquad (i = 1, 2, \cdots), \tag{A}$$

for any constant a, and if r is an m-fold root of $\det(A-tI)=0$, *then the matric equation*

$$\theta X = AX \tag{B}$$

has a solution

$$X = \sum_{i=0}^{r-1} (A - rI)^i x_{r,i} C,$$

where C is any solution of

$$(A - rI)^m C = 0.$$

This theorem essentially reduces the solution of equation (B) to that of the equations (A). If $\theta=d/dt$, the simplest solutions of (A) are $x_{a,i}=t^ie^{at}/i!$; if $\theta=\Delta$, the corresponding solutions are $x_{a,i}=t^{(i)}(a+1)^t/i!(a+1)^i$.

Considerations of this kind have been used in the theory of integral equations.

R. J. W.

A METHOD OF FINDING PRIME NUMBERS

JOHN LOTKA, Demarest, N. J.

THEOREM. *Make a list of integers of the forms $(6m-1)q-m$ and $(6m+1)q+m$, where m and q are positive integers; denote the numbers thus obtained as class 1. Similarly, make a list of integers of the form $(6m-1)q+m$ or of the form $(6m+1)q-m$ (both forms yield the same integers); denote the numbers so obtained as class 2. Denote all other integers as class 3.*

Now denote all integers found in class 1 only (and not in both class 1 and 2) as n_1, denote all integers found in class 2 only (and not in both class 1 and 2) as n_2, and denote all integers found in class 3 as n_3. Then all prime numbers, from 5 upwards, are found by forming them according to the formulas $6n_1-1$, $6n_2+1$, $6n_3-1$, and $6n_3+1$. This not only yields all primes, from 5 upwards, but yields nothing but primes.

A table of integers of classes 1, 2, 3, with the prime numbers corresponding thereto, begins as follows:

Class 1				4				8	9					14	15				19	20
Class 2						6					11		13			16				20
Class 3	1	2	3		5		7			10		12					17	18		
$6n_1-1$				23				47	53					83	89				113	
$6n_2+1$						37					67		79			97				
$6n_3-1$	5	11	17		29		41			59		71					101	107		
$6n_3+1$	7	13	19		31		43			61		73					103	109		

Note that 20 is in both class 1 and class 2, so that neither $6\cdot 20-1$ nor $6\cdot 20+1$ is a prime.

Proof. It is well known that all primes, from 5 upwards, are of the form $6n\pm 1$. Some numbers of this form are composite. Obviously, an integer of the form $6n\pm 1$ cannot be divisible by 3, hence composite numbers of this form must be of the form $(6m\pm 1)(6q\pm 1)$, that is, one or more of the following forms:

$$(6m+1)(6q+1) = 6(6mq+q+m)+1,$$

$$(6m-1)(6q-1) = 6(6mq-q-m)+1,$$

$$(6m+1)(6q-1) = 6(6mq+q-m)-1,$$

$$(6m-1)(6q+1) = 6(6mq-q+m)-1.$$

In other words, $6n+1$ is composite only if n is of the form $6mq+q+m$ $=(6m+1)q+m$, or of the form $6mq-q-m=(6m-1)q-m$, or of both of these forms; and $6n-1$ is composite only if n is of the form $6mq+q-m=(6m+1)q-m$, in which case n is also of the form $6mq-q+m=(6m-1)q+m$.

All other integers of the form $6n\pm 1$ are necessarily primes, that is to say,

(1) the numbers $6n-1$ in which n is *only* of the form $(6m+1)q+m$, or the form $(6m-1)q-m$, or *only* of both of these forms;

(2) the numbers $6n+1$ in which n is *only* of the form $(6m-1)q+m$, which yields the same integers as $(6m+1)q-m$;

(3) all other numbers of the forms $6n-1$ and $6n+1$; that is to say, all numbers of the form $6n\pm 1$ in which n is not of any of the four forms $(6m\pm 1)q\pm m$.

Procedure in determining integers of classes 1 and 2. (A) As to class 1, since $(6m+1)q+m=(6q+1)m+q$, and $(6m-1)q-m=(6q-1)m-q$, it is sufficient to list those integers of the forms $(6m-1)q-m$ and $(6m+1)q+m$ in which $q\geqq m$.

For	$q=m=1$	2	3	4	5	r
we find	$(6m-1)q-m=4$	20	48	88	140	$2r(3r-1)$,
and	$(6m+1)q+m=8$	28	60	104	160	$2r(3r+1)$.

From this we establish the following table of values for $(6m-1)q-m$ for certain values of m and q:

TABLE 1a

m	$q =$	1	2	3	4	5	6	7
1	$5q - 1$	4	9	14	19	24	29	34
2	$11q - 2$		20	31	42	53	64	75
3	$17q - 3$			48	65	82	99	116
4	$23q - 4$				88	111	134	157
5	$29q - 5$					140	169	198
6	$35q - 6$						204	239

Similarly, for $(6m+1)q+m$, we establish the following:

TABLE 1b

m	$q =$	1	2	3	4	5	6	7
1	$7q + 1$	8	15	22	29	36	43	50
2	$13q + 2$		28	41	54	67	80	93
3	$19q + 3$			60	79	98	117	136
4	$25q + 4$				104	129	154	179
5	$31q + 5$					160	191	222
6	$37q + 6$						228	265

Note that in both Tables 1a and 1b, as well as in Table 2 below, the numbers form arithmetical series, horizontally as well as vertically.

(B) As to class 2, if we base the computation on the form $(6m-1)q+m$, Table 2 below yields the integers of class 2, since $(6m-1)q+m=(6q+1)m-q$.

TABLE 2

m	$q =$	1	2	3	4	5	6	7
1	$5q + 1$	6	11	16	21	26	31	36
2	$11q + 2$	13	24	35	46	57	68	79
3	$17q + 3$	20	37	54	71	88	105	122
4	$23q + 4$	27	50	73	96	119	142	165
5	$29q + 5$	34	63	92	121	150	179	208
6	$35q + 6$	41	76	111	146	181	216	251

Remarks. The method here described may be used to find primes from a given number upwards, without first computing the primes smaller than such a given number. For instance, if we wish to compute the primes greater than 200, say in the interval from 200 to 300, we divide 200 and 300 by 6 and thus find the numbers 33 and 50 as the approximate limits of the range in which to determine the required integers of classes 1, 2, and 3. For class 1 we find the integers $5q-1=34$, 39, 44, and 49; $11q-2=42$; $17q-3=48$; $7q+1=36$, 43, 50; and $13q+2=41$. For class 2 we find the integers $5q+1=36$, 41, 46; $11q+2=35$, 46; $17q+3=37$; $23q+4=50$; $29q+5=34$; $35q+6=41$; and $41q+7=48$.

This yields the following table of integers of classes 1, 2, and 3 in the range of 33 to 50 and the corresponding primes in the range of 200 to 300:

Class 1	34		36			39		41	42	43	44				48	49	50
Class 2	34	35	36	37				41					46		48		50
Class 3					38		40					45		47			
$6n_1-1$						233			251	257	263					293	
$6n_2+1$		211		223									277				
$6n_3-1$					227		239					269		281			
$6n_3+1$					229		241					271		283			

Note by the Editor. Professor B. W. Jones has pointed out that Mr. Lotka's method of determining the primes was mentioned by W. L. Kraft in 1801 (see Dickson, *History of the Theory of Numbers*, vol. 1, p. 426). The method seems to be of sufficient interest, however, to warrant its republication, especially since Kraft's paper is difficult to obtain in most places.

R. J. W.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Die Zahl π und der Kreis. (Eine Studie für junge Freunde der Mathematik.) By Franz Hennecke. Hamburg, Boysen and Maasch, 1938. 58 pages. RM 1.50.

A Brief Course in Trigonometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Company, 1940. 8+118+17 pages. $1.50.

The Chicago College Plan. Revised and enlarged after ten years of operation of the plan. By A. J. Brumbach. Chicago, University of Chicago Press, 1940. 13+413 pages. $3.00.

Elements of Analytic Geometry. Second edition. By C. E. Love. New York, The Macmillan Company, 1940. 12+159 pages. $1.75.

The Journal of Unified Science (Erkenntnis). Vol. 8, no. 1–3. The Hague, W. P. Van Stockumandon. Chicago, University of Chicago Press, 1939. 193 pages. $3.75.

Tables of the Exponential Function e^x. Compiled by A. N. Lowan, M. Pfeferman, G. Blanch, F. G. King, W. Kaufman, and M. Abramowitz. New York, Work Projects Administration, 1939. 535 pages.

An Introduction to Analytical Geometry. Vol. I. By A. Robson. Cambridge, University Press, 1940. 14+409 pages. $2.50.

Differential and Integral Calculus. By R. R. Middlemiss. New York and London, McGraw-Hill Book Company, 1940. 10+416 pages. $2.50.

The Physical Sciences. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, Inc., 1940. 17+754 pages. $5.00; school, $3.75.

College Algebra. Revised edition. By N. J. Lennes. New York, Harper and Brothers, 1940. 12+432 pages. $2.25.

Tables of the First Ten Powers of the Integers from 1 to 1000. Project for the Computation of Mathematical Tables, United States Works Progress Administration for the City of New York, 1939. 82 pages.

Metric Differential Geometry of Curves and Surfaces. By E. P. Lane. Chicago, University of Chicago Press, 1940. 8+216 pages. $3.00.

A Course of Analysis. Second edition. By E. C. Phillips. Cambridge, University Press; New York, The Macmillan Company, 1939. 8+361 pages.

Lattice Theory. (American Mathematical Society Colloquium Publications, vol. 25.) New York, American Mathematical Society, 1940. 6+155 pages. $2.50.

The Theory of Group Characters and Matrix Representations of Groups. By D. E. Littlewood. Oxford, Clarendon Press, 1940. 8+292 pages. $5.50.

Advanced Calculus. By C. A. Stewart. London, Methuen and Co., Ltd., 1940. 18+523 pages. 25s.

REVIEWS

Statistical Mathematics. By A. C. Aitken. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 7+153 pages. 4/6s.

This book is a survey of the mathematical basis of statistics. The contents include a discussion of probability, and the importance of probability theory in statistical analysis; frequency distribution; correlation, partial correlation and regression, practical curve fitting; probability and frequency of two variates; probability distribution of statistical coefficients obtained from sampling, and comments on the problem of estimation. The book presents the essential features of the recent mathematical theories in the field of statistical analysis. Every new idea is illustrated by an example and where practical these are followed by well chosen exercises for the purpose of illustrating the theory and teaching the method.

There are two distinct levels of mathematical statistics apparent in the number of texts available on the subject. The one level involves elementary mathematics as a basis. It is limited to the use of devices which facilitate the description and analysis of data, and is designed principally to provide an introduction to most of the applied fields. The second level is intended for those of mathematical maturity equal to that of college students who have had algebra up to the binomial theorem, infinite series, and probability, elementary calculus, and the elements of mathematical statistics of level one.

The book in question presumes the reader has a knowledge of mathematics comparable to that of the second level of preparation. I personally would select the book as a text in college courses only for the best of those students whose mathematical preparation is limited to the minimum of the second level. It is,

however, recommended for self-study on the part of those of more mathematical maturity who desire to obtain a knowledge of the recent advances in mathematical statistics. It is a contribution toward the underlying theory involved in recent important methods of statistical analysis.

C. C. WAGNER

Mathematical Clubs and Recreations. By S. I. Jones. Nashville, Tenn., Kingsport Press, 1940. 14+236 pages. $2.75.

Prefacing his book with an ancient Irish love charm, a Latin sentence for which he quotes C. C. Keys' most free translation, "God fills the universe of Mathematics and controls its operations," the author presents aims and ideals for mathematics clubs. He quotes in full several articles published by high school teachers on such subjects as Creating Mathematical Atmosphere in the Classroom, The Math Teacher's Aim, Appreciation of Beauty. A discussion of topics suitable for high school students is followed by a constitution of a typical college organization and the lengthy original (now superseded) constitution of Pi Mu Epsilon. A good, brief list of books suitable for high school club libraries concludes the first third of the book.

There remains a compilation of mathematical amusements, quaint problems, multiplication oddities, and curious tables. The number rhymes begin with, One, two, button my shoe. The games include hop scotch and jackstraws which may be considered better adapted to children of the second grade. However, many of the problems are quite suitable for first and second year high school students. The sections on Perpetual Calendar, Magic Squares, and Calendar Magic, together with Curious Tables and Fallacies, would furnish entertainment for several of their meetings.

The college student will find very little to test his mathematical ability but, if he will solve the concluding half dozen Bible Tests, he will know more Biblical history than most college seniors.

HELEN B. OWENS

Elements of Analytic Geometry. Second edition. By C. E. Love. New York, The Macmillan Company, 1940. 12+159 pages. $1.75.

The question as to the amount of time that should be devoted to a first course in analytic geometry, preceding the calculus, depends on the ulterior purpose. To many students it is a fascinating subject, well suited to develop skill in manipulation. On the other hand, many of the more interesting parts have but little use in the calculus.

Of late years the tendency is to shorten the course in analytics, restricting it to the bare essentials for later use. The present volume is written with that in view. The first edition was written in 1931; the present book is rather an abridg-

ment of the third edition of the author's larger treatise on Analytic Geometry (1938).

The choice of subject-matter is limited. It seems to the reviewer that the chapter on the locus of an equation, before the straight line is treated, is too hard and not sufficiently convincing at this stage. The treatment of the distance of a point from a line is consistent, but the average student will be confused as to the reason for the rule for signs.

A short chapter on algebraic curves is a laudable innovation.

The examples are numerous and well chosen. The practice of introducing theorems as exercises is a commendable one. It is frequently used throughout the book.

The part on Solid Geometry is short, but covers the fundamental properties of figures of the first and second orders.

The style is that of the other books by the same author. The figures and press-work are excellent.

VIRGIL SNYDER

A Brief Course in Trigonometry. By D. R. Curtiss and E. J. Moulton. Boston, D. C. Heath and Company, 1940. 8+118+17 pages. $1.50.

While recognizing the demands for brevity, the authors believe that the older division of early college mathematics into distinct courses is wise, and have written on that basis. Only intermediate algebra and plane geometry are presupposed.

Coördinates are introduced at the outset and used freely throughout. The addition theorem for sines and cosines is proved first for acute angles all in the first quadrant, and later generalized by a method not heretofore published. Inverse trigonometric functions are usually designated by arc sin x, *etc.*, but the other notation is sometimes employed. No previous knowledge of logarithms is assumed. Cologarithms and vectors are both explained, but each with a star (for possible omission).

A very generous number of exercises is provided; answers to odd-numbered ones only are given, but the others may be obtained from the publishers. The figures and press-work are both attractive.

VIRGIL SNYDER

Die Zahl π und der Kreis. (Eine Studie für junge Freunde der Mathematik.) By Franz Hennecke. Hamburg, C. Boysen, 1938. 58 pages. RM 1.50.

As the sub-title implies, this book is written for the elementary student. The author presupposes only a knowledge of the relation of π to the area and circumference of a circle, and a familiarity with similar triangles and proportions. Frequent use is made of the binomial theorem which is proved in an appendix. Also included in the appendix are formulas for tan $(\alpha+\beta)$, and for the sum of kth powers of the first n positive integers.

The object of the book is to develop series for π. This is done from two standpoints.

The first approach is strictly elementary. From considerations of area and arc length and without using the language of the calculus, the author first derives numerical series for π; then using the method of undetermined coefficients he develops the familiar Maclaurin series for the functions $(\arcsin x + x\sqrt{1-x^2})/2$ and $\arcsin x$, without identifying them as such. These series are evaluated for various values of x.

Definitions of function, derivative, and integral are given and integration is applied to obtain series associated with π. In order to find the area bounded by the curve $y=\sqrt{2rx-x^2}$, the x-axis, and the ordinate $x=X$, the integrand in $\int\sqrt{2rx-x^2}\,dx$ is expanded by the binomial theorem and the series integrated termwise. The same method is applied to obtain the series for $\arcsin x$ and $\arctan x$. Identities such as $\pi/4 = \arctan(1/2) + \arctan(1/3) = 4\arctan(1/5) - \arctan(1/239)$ are used in connection with the series for $\arctan x$ to illustrate the desirability of securing rapidly converging series for computation. The value of π is given to ten significant figures.

The book should not be recommended to elementary students without a word of caution. The concept of limit is repeatedly used but nowhere is a definition of limit given. It is also disconcerting to find the following (page 15):

$$\lim_{n=\infty} \frac{2k-1}{2n} = \frac{2k}{2n} - \frac{1}{2n} = \frac{k}{n} - 0 = \frac{k}{n},$$

it being understood that k is a fixed positive integer. Numerous typographical errors occur but are easily detected.

G. B. THOMAS

Living Mathematics. By R. S. Underwood and F. W. Sparks. New York, McGraw-Hill Book Company, 1940. 9+365 pages. \$2.25.

In the preface the authors state that they had in mind two goals in writing this book: first, "the production of a college text-book which would provide enough of the conventional subject-matter to meet practical credit-transfer requirements" and second, "highlighting for non-specialists the interest that is inherent in mathematics itself and · · · fostering an appreciation of its place in modern life."

To the end that the first goal may be reached, algebra is taken from its beginnings through quadratic equations and the use of synthetic division to obtain approximate zeros of polynomials in a single variable, numerical trigonometry and logarithms, the usual analytic geometry of the straight line and conic sections, an introduction to differentiation and integration, and a taste of probability. Additional topics are compound interest and annuities, and some theory of numbers.

The originality of the text lies in its attempts to reach the second goal. One of the best means to this end is the conversational and even, at times, breezy

tone of the book. A fine example of this is their "Golden rule of algebra. Always do to the left side what you do to the right side, and vice versa." (This is followed by a more precise explanation.) The development of topics, especially that of algebra, is very slowly and carefully worked out. The introduction to the calculus is as thorough as such a brief treatment can be, the stress being on the understanding of the fundamental principles and a feeling for the usefulness of the topic rather than a development of techniques.

Since the authors have two goals, neither is quite as well met as it might be by a book with a single goal. If the first were the only one, the topics in the theory of numbers would perhaps better yield to further discussion of other topics. If the second were the only one, time could be saved for more useful things by a mere mention of conic sections and their uses. The reviewer feels that the chapter on the calculus would be lost on anyone who did not go on to further mathematics; only a brilliant student could grasp the real significance of calculus from such a brief treatment however well it was given.

A reviewer can always find what he considers flaws in a book—flaws which may be virtues to others: the titles to the sections usually give little clue to their contents. It would have been more enlightening to emphasize the fact that the commutative law and other similar laws are first laws of operations with numbers and, for that reason, laws of algebra. In numerous places one has the feeling that it would have been much better if the authors had let themselves go to a greater extent. Perhaps they dared not.

The authors seem to have a leaning toward the first of their goals, but their eye toward the second results in a much more interesting and rich book than the usual text in freshman mathematics. This leaven together with the style in which the book is written should result in more interested and careful reading by the student. And it should be a pleasure to teach.

B. W. Jones

College Algebra. Second edition. By N. J. Lennes. New York, Harper and Brothers, 1940. 12+432 pages. $2.50.

The first edition of this book was published in 1928 and reviewed in this Monthly in 1931 by R. M. Mathews, now deceased. Ordinarily a second edition does not receive notice in this Monthly; but in this case the time interval since the first was published has been so long, and the contents of the first edition have been so radically changed, that the present comments should be regarded as applying to a new book.

The first half is given to a concise, well written review of elementary algebra. Graphical processes are introduced early and used freely, as are determinants of the second and third orders. Complex numbers are discussed before the general quadratic equation. The first topic in the theory of equations is the determination of approximate values of roots of numerical equations. Unusual emphasis is put upon progressions as applied to laws of interest, discount, annuities, *etc.*,

including various tables. A chapter on infinite series is included. No work is given on statistics. The chapter on determinants is unusually long and full.

Answers to exercises are not given. There is a full index. One commendable feature is the inclusion of an extensive historical sketch; another is the generous lists of exercises, mostly not formal, but verbal, frequently involving a stiff test in ingenuity for their solution. In particular, 46 cumulative reviews are provided for systematic use at assigned intervals.

The printing and press-work are excellent; the style is attractive.

VIRGIL SNYDER

Mathematics in Action, Book Three. By W. W. Hart and Lora D. Jahn. New York, Heath and Company, 1940. 6+442 pages. $1.28.

The third volume of *Mathematics in Action* is in keeping with its predecessors and is designed to provide "a simplified and socialized course in general mathematics" for the last year of the junior high school.

Complete in itself, the text reviews the work studied in the earlier grades, especially consumer's arithmetic and topics in geometry. Along with extensions into the fields of algebra and geometry, the units cover the fundamentals of algebra—formulas, graphs, equations, and signed numbers— and elementary materials on polynomials, factoring, fractions, and fractional equations. Emphasis in geometry is placed on construction and measurement of the common plane and solid geometric figures.

Diagnostic tests, remedial sections, and cumulative reviews are amply present to insure and facilitate mastery of the subject-matter, while assistance in adopting the abundance of material to individual classes is provided in the form of starred sections and a teacher's manual.

The units are rich in content and make especially fine use of the inductive method of instruction. In a few cases, the reviewer felt the geometric definitions lacked rigor. For instance, "a trapezoid has two sides parallel," or "a circle is a curved line on a plane; all its points are at the same distance from an inside point called its center." This is not a serious drawback, however, and is more than compensated for by the many excellent features of the text.

R. A. HARRISON

Pandiagonal Magic Squares of Prime Order. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1940. 5+93 pages. $1.00.

Pandiagonal magic squares are perhaps the most interesting of all the various sub-classes of magic squares that have been treated from time to time. I believe this to be the first book devoted entirely to squares of this kind. As the title indicates, however, squares of *prime* order only are discussed.

The author divides his squares into two sub-classes which he calls Class I and Class II. Chapter I deals with squares of the first class, and a general formula for the total number of squares of order n of this class is derived. This formula is not new, but the fact that it can be derived so readily from Professor

Candy's point of view is of considerable interest. A general formula for the number of symmetric pandiagonal squares of Class I is also derived. The number given for the case $n=11$ at the top of page 12 and in the Summary should be multiplied by 100.

Chapter II deals with squares of Class II. Isolated examples of squares not belonging to Class I have been noticed previously but no systematic discussion seems to have been made. Professor Candy's treatment is thus of much interest in suggesting new problems. In particular, the problem raised in the fifth paragraph on page 23 appears to be worth further study. The *related* pairs of squares, such as numbers 4 and 8 on pages 24 and 25 are also worthy of note.

The third and last chapter deals with *semi-pandiagonal* magic squares; that is, those in which the broken diagonals parallel to *one* of the principal diagonals are not magic. General formulas for the number of such squares of prime order n are derived.

G. E. RAYNOR

Vector Analysis. With an Introduction to Tensor Analysis. By J. H. Taylor. New York, Prentice-Hall, 1939. 9+180 pages. $2.85.

This book is a terse elementary account of some fundamental aspects of Vector Analysis and of the rudiments of Tensor Analysis. Professor Taylor treats the subject as a branch of mathematics, rather than of physics. The presentation is lucid and orderly. Each topic is preceded by a list of references to rather more ambitious works in English, French, German, and Italian.

The book is highly original not only in view-point, but also in choice of material and character of exposition. The chapter on Vector Algebra contains the principle novelties of this book as compared with the usual tracts in English. The group postulates and the postulates for a real vector space are formulated. The norm and scalar product are defined in terms of a positive definite quadratic form and its polar. The formulas are developed for the case of three-dimensional space in terms of a non-orthogonal vector basis. Distributive operators are defined by a bilinear relation on pairs of "quantities." The chapter also includes material on affine transformations, linear dependence, the vector representation of infinitesimal rotations and areas, and applications to mechanics. The chapter on the differential calculus of vectors covers some elementary differential geometry of curves and surfaces in three dimensions, and has a good treatment of moving axes. The chapter on the integral calculus of vectors gives a formal presentation of Green's and Stokes's theorems and the theorem of the Rotational, and some applications to line integrals and the heat equation. The chapter on tensor analysis, though fragmentary, preserves the continuity of the developments. It is brought out that the affine group is replaced by a wider group of transformations, and that a local basis at a point may be selected in a tangent space so that the earlier vector ideas and terminology undergo natural extension.

The stress is on logical aspects rather than on the formal technique of com-

putation. Thus there is frequent reference to components, and the utility of vector methods in the details of calculations appears in a secondary rôle. In some cases directness has been sacrificed for novelty, as for instance in the proof of the distributive law for vector cross multiplication where integrals appear. Instead of observing that the postulates for a linear space define the vector concept, the postulates are in part introduced as consequences of a specialized definition of a vector in three space. Indeed, the formal generality attained by the abstract formulation may well affect a beginning student as prissy pedantry unless further illustration besides the tensor extension is provided. Some starred sections including, for instance, a brief reference to vector function spaces, would justify the postulational approach and indicate other aspects of the subject.

The book is admirably suited for Vector Analysis courses in a mathematics curriculum. Moreover, the book is recommended strongly for collateral reading in survey courses in mathematics.

D. G. Bourgin

MATHEMATICS CLUBS

Edited by E. H. C. Hildebrandt, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

FILMS

1. A most interesting experiment in the production of a classroom motion picture was carried on during the past year by a group of mathematics students at Vassar College under the direction of Professor Grace M. Hopper. The only materials required were heavy white drawing paper, such as that used in mechanical drawing classes, and India ink. The movie film cost four dollars. The equipment was borrowed from amateurs or from the science department. While experimental filming took about eight hours, the final negative was photographed in four. Clubs or groups interested in seeing this film and in conducting similar experiments should take advantage of Professor Hopper's offer to answer any questions regarding this type of project.

MATHECINEMATICS

Grace M. Hopper, Vassar College

The use of the motion picture to provide group projects in mathematics was demonstrated at the New York State Student Scientific Conference at Vassar College this spring. The exhibition aroused a great deal of interest and elicited a number of inquiries which this account of our experiment will attempt to answer.

The idea originated two years ago when a group of Vassar mathematics students had seen Walt Disney's *Snow White* and on the next day were obliged to test the accuracy of their use of the compass for inking. The suggestion was ventured that a plane curve could be animated, and an envelope construction

of the limaçon-cardioid series was selected for the purpose. When the thirty-eight drawings for the series had been allocated among the students and completed, the animation was accomplished with the aid of a drawing-board and a small movie camera. The drawing-board was set vertically in the shade on a very bright day. Tacking the successive drawings accurately proved difficult, but was accomplished by breaking the heads off two tacks to penetrate each drawing at the fixed points. As far as the students were concerned, the effort was a great success; the curve did move and change from a hyperbolic limaçon to a cardioid to an elliptic limaçon and finally to a circle. The picture was actually very crude; shadows appeared and disappeared, thumb tacks bobbed in and out, and the curves were very jumpy. The group showed it for their friends, and it was stored away with the verdict that probably something of the kind could be done if one had more time.

This year the Vassar Science Club was hostess to the scientific conference. After consideration of several projects, the mathematics group decided to remake the movie and to display it with an exhibition of drawings related to the Cartesians and a survey of their history and properties. The choice of the Cartesians for such a project was dictated by the considerations that it was possible for students from all classes to contribute, that it appealed to the interests of physics majors taking mathematics courses, that an abundance of different types of constructions and theorems was provided, and that bibliographical material was readily available.

One group of students traced the history of the curves from Descartes to Newton, Huygens, Quetelet, Sturm, Chasles, Cayley, Darboux, Sylvester, Casey, and Clifford. A second group, consisting largely of elementary students, examined the Cartesian and polar equations of the curves and the variations of their focal and other properties. A third group studied them as a special case of the bicircular quartics. A fourth gathered the geometric methods of construction, the use as a trisectrix, envelopes, roulettes, pedals, inverse curves, and anallagmatics of which the Cartesians appear as special cases. The descriptive geometry class constructed models of the space intersections and projections generating Cartesians. Others displayed certain cases as caustics by refraction and reflection. The material was then sifted and combined into two papers which were presented to the conference, illustrated by the drawings and models.

The movie, however, was the "chief attraction" of the exhibit. A set of drawings was made, illustrating the theorem: *The envelope of a variable circle whose center lies on a given base circle and which passes through a fixed point is a hyperbolic or elliptic limaçon or a cardioid according as the fixed point lies without, within, or on the base circle.* The fixed point was taken to the right (sixty thirty-seconds of an inch) of the base circle (of radius thirty thirty-seconds of an inch). With each drawing it was moved three thirty-seconds of an inch to the left until it reached the center of the base circle. The series of drawings could then be reversed so that the point moved out to the left of the base circle. Thus thirty-two careful drawings were made. The students then consulted every amateur

movie photographer on the campus, borrowing camera, equipment, and advice.

This time we used the regulation amateur titling equipment which includes a titling board on which the drawings can be fastened accurately, two fixed lamps for illumination, and a bracket for the camera. We used a 100 ft. Dupont film which has the advantage of affording both a positive and a negative for further cutting and experimentation. It is impossible to be dogmatic concerning exposures. Each drawing was "shot" for two seconds; timing and aperture were found to vary with the individual camera. Before taking the actual picture, several feet of film were exposed at a time and developed in a dark room until the best possible results were obtained. One difficulty with the usual spring-wind amateur camera is that it does not run at an even speed. Being slow to start, it causes over-exposure of the first two frames of each drawing. Cutting out these frames is impracticable since the splicing glue shows up against such a plain background. Our solution was to use the negative of the film which shows white lines against a black background, with the result that the "flashing" was much less noticeable. It is probable that cutting and splicing the negative would entirely eliminate awareness of the transitions between drawings.

However imperfect our results may have been, there is no doubt that the students who participated in this effort felt that all of it was distinctly worth while. The movie, although still very amateurish, is very effective and has made an impression on non-scientists as well as scientists. I hope that other groups will make similar attempts so that, by sharing experiences and results, a technique may be developed for wider presentation of mathematical subjects. I should be most glad to hear from any group interested in making such a movie or in seeing ours.

2. *The Isograph.* In the August-September 1939 MONTHLY (see page 451), the attention of readers of this department was called to the film *The Isograph* which the Bell Telephone Laboratories, through the coöperation of Dr. T. C. Fry and Mr. John Mills, had agreed to loan for use at meetings during the year. We are pleased to announce that they have again consented to make this film available to interested groups for another year. Reservations should be made as early as possible, preferably before November 30th, so that a schedule for showing and shipping this film may be arranged for the entire year. Arrangements should be made by addressing Miss L. E. Smith, Librarian, Bell Telephone Laboratories, 463 West Street, New York City.

Thirty-three clubs and departments showed this film and discussed the principle of this mathematical machine, during the year 1939–40. The mathematics clubs using it at one of their meetings included:

Archimedean Club, Winthrop College

Case Mathematics Club, Case School of Applied Science

Junior Mathematics Club, University of Chicago

Kappa Mu Epsilon, University of New Mexico

Mathematics Club, College of William and Mary

Mathematics Club, Connecticut College

Mathematics Club, Cooper Union
Mathematics Club, Oberlin College
Mathematics and Physics Club, Queens University, Kingston, Ontario
Mathematics-Physics Club, Haverford College
Mathematics Society, Trinity College, Washington, D. C.
Mathematics Club, University of Buffalo
Mathematics Club, University of Michigan
Oliver Mathematical Club, Cornell University
Pi Mu Epsilon, Duke University
Pi Mu Epsilon, Iowa State College
Pi Mu Epsilon, Syracuse University
Pi Delta Theta, University of Denver
Pi Mu Epsilon, Washington University

Departments of mathematics at the following schools had a discussion of this material following the showing of the film to their members:

Davidson College
DeKalb State Teachers College
Brooklyn Technical High School
University of Georgia
Los Angeles City College
University of Oklahoma
St. Mary's College, Winona, Minnesota
San Diego State College
University of Tennessee
State College of Washington
Worcester Polytechnic Institute

Robert Williamson, a senior in the Engineering School of Syracuse University, after its showing at the Pi Mu Epsilon meeting, made additional arrangements to have the material presented before the senior class in mechanical engineering. Later he gave a paper on the isograph at the New England Student Convention of the American Society of Mechanical Engineers held at Worcester, Massachusetts.

3. *Precisely So.* Clubs treating historical topics at some of their meetings may be interested in this film which traces the history of measurement from early times, and shows how measurements correct to a millionth of an inch are made and used in the automotive industry. This is a 2 real, 16 mm. sound film and requires approximately 20 minutes for showing. Requests and correspondence may be directed to the Department of Public Relations, General Motors Corporation, 1775 Broadway, New York City. There is no rental charge, but transportation charges must be paid to and from point of shipment.

4. *The Workers' Old-Age and Survivors Insurance Account.*

5. *Social Security Benefits.*

These two films may be used at meetings devoted to a discussion of actuarial and social-economic problems. The first describes the procedures involved in setting up and maintaining workers' old-age and survivors insurance accounts; the second shows the processes involved in filing a claim and receiving payments under the old-age and survivors insurance law. Both are 2 reels, 16 mm. sound, require approximately 20 minutes each, and were produced by the Social Security Board. Showings may be arranged without cost through regional Informational Service representatives of the Board, or by writing to the Informational Service, Social Security Board, Washington, D. C.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 436. *Proposed by E. H. Johnson, Emory University.*

A tall rectangular piece of furniture, of length a and width b, is moved down a hallway of width c, and goes through a door whose width, d, barely allows its passage into an adjacent room. If we neglect the thickness of the wall, it is easily seen by the comparison of similar triangles that $d=ab/c$. If the wall has a thickness h, find the value of d in terms of a, b, c, and h.

E 437. *Proposed by V. Thébault, Le Mans, France.*

For what kind of tetrahedron does the Monge point lie on the circumsphere? (The Monge point lies on planes perpendicular to the edges through the midpoints of the respective opposite edges.)

E 438. *Proposed by J. S. Frame, Brown University.*

If p is any odd prime, show that the decimal expansion of the fraction $1/p$ will repeat in $(p-1)/2$ digits or some factor thereof if and only if $p\equiv \pm 3^k \pmod{40}$.

E 439. *Proposed by J. H. M. Wedderburn, Princeton University.*

ABC is a triangle; lines are drawn external to it, parallel to AC and BC at distances which bear a fixed ratio to the lengths of AC and BC, respectively, making a parallelogram of which CD is one diagonal. If the length of CD is kept constant, show that the locus of C is obtained as follows. Draw two equal circles with centers A and B, and let a line, equal in length to the diameter of the circles, slide with its ends on the two circles; then C is on the locus of the midpoint of this line. (The radius of the circles is determined by any one point on the locus.)

E 440. *Proposed by H. S. M. Coxeter, University of Toronto.*

Prove that, for every integer $n>2$, there are from ten to thirteen n-digit numbers whose digits are the same as the last n digits of their cubes, and that for $n=6$ the thirteen numbers are $5\cdot 10^5\pm 1$, 10^6-1, 5^8, 10^6-5^8, $5\cdot 10^5\pm 5^8$, $5\cdot 10^5\pm(5^8-1)$, $2\cdot 5^8-1$, $10^6-2\cdot 5^8+1$, $2\cdot 5^8-5\cdot 10^5-1$, $15\cdot 10^5-2\cdot 5^8+1$. (G. Fistie, in *Sphinx*, 1935, p. 4, found only nine such numbers.)

SOLUTIONS

E 398 [1939, 652]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Given a triangle ABC, let O be the circumcenter, A' the projection of A on BC, M any other point of BC, and B_1, C_1 the respective projections of B, C on AM. Let lines through M, parallel to $A'C_1$ and $A'B_1$, meet AC and AB in points P and Q, respectively. Prove that the lines PQ and OM are perpendicular.

Solution by L. M. Kelly, University of Missouri.

Produce AM to meet the circumcircle in D. Join CD and BD. Since the quadrangles $ABDC$ and $ABA'B_1$ are cyclic,

$$\angle BCD = \angle BAD = \angle MA'B_1 = \angle BMQ.$$

Hence MQ and CD are parallel; similarly so are MP and BD. Suppose BD and CD meet AC and AB in P' and Q', respectively. Then $P'Q'$, being the polar of M with respect to the circumcircle, is perpendicular to OM. But, from similar triangles,

$$AQ/AQ' = AM/AD = AP/AP'.$$

Thus PQ is parallel to $P'Q'$, and perpendicular to OM.

Also solved by V. W. Graham.

E 401 [1940, 48]. *Proposed by A. H. Stone, Graduate College, Princeton.*

Fit together twenty-eight squares, of sides 2, 18, 22, 37, 38, 39, 41, 43, 49, 67, 72, 80, 85, 103, 116, 154, 164, 175, 178, 192, 200, 207, 215, 222, 230, 247, 422, 593, to make a single square (of side 1015). Is there any simpler solution to the problem of dissecting a square into several unequal squares?

I. *Partial Solution by Michael Goldberg, Washington, D. C.*

Using the packer's rule of packing the largest parts first, the arrangement shown in Figure 1 is obtained empirically. The solution depends on the existence of a rectangle which can be divided, in two distinct ways, into different squares. Intuitively this seems less probable than the division of a square into squares directly. But I know of no way of finding a simpler solution than the given solution without obtaining a complete census of all possible rectangles formed of fewer than thirteen squares. This census could be made by testing every possible arrangement of squares in the following manner. Designate, by a and b, the edges of a pair of adjacent squares of the arrangement to be tested. (See Fig. 2.) The edges of other squares are readily obtained. By adding new letters whenever necessary, all the squares may be labelled. Then, by equating different forms for the same edge, Diophantine equations are obtained. In the rectangle shown, equating the left and right edges we obtain

$$(12a - 2b) + (8a - b) = (3a + 2b) + (2a + b) + (3a + b),$$

whence $12a = 7b$. The solution $a = 7$, $b = 12$, gives a rectangle 104 by 105, formed

from ten different squares. Steinhaus, in *Mathematical Snapshots* (p. 9), refers to another nearly square rectangle, namely 32 by 33, formed from nine different squares.

Also partially solved by J. Rosenbaum, E. R. van Kampen, and R. J. Walker.

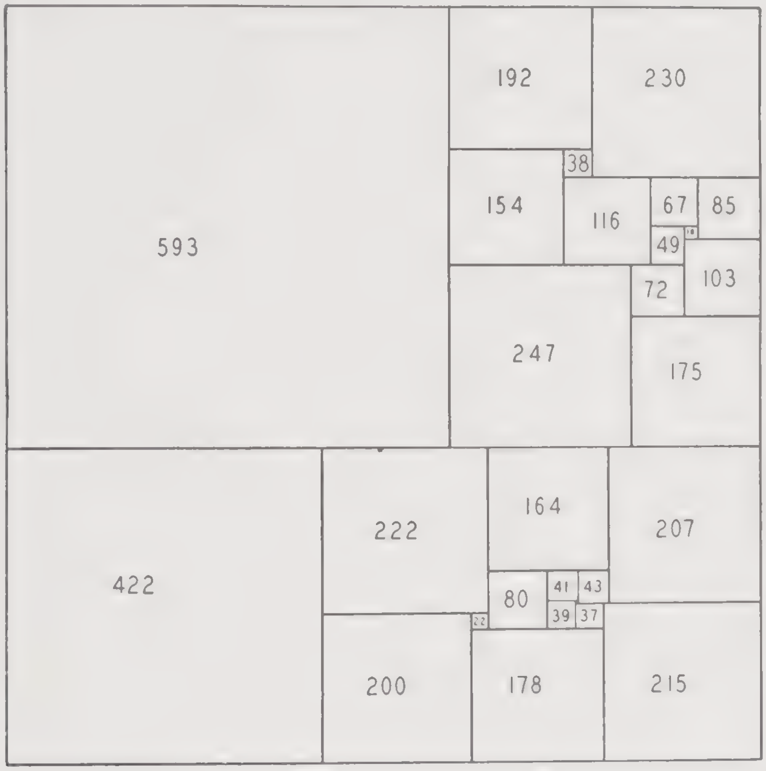

FIG. 1

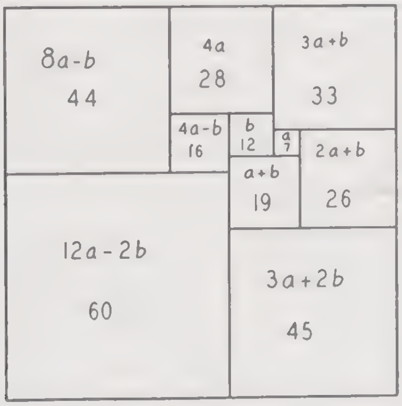

FIG. 2

II. *Solution by W. T. Tutte, Trinity College, Cambridge, Eng.*

Yes, there is at least one simpler solution to the problem of "squaring the square." The number of component squares is 26, and the side of the large square

is 608. For details, see R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, *On the dissection of rectangles into squares*, Duke Mathematical Journal, vol. 6, to appear in December, 1940.

Moreover, there is another solution which is neither simpler nor less simple, since the number of pieces is again 28, and the side of the large square is again 1015. The sides of the component squares are 13, 16, 17, 23, 30, 43, 47, 84, 92, 93, 119, 120, 142, 163, 165, 167, 177, 183, 188, 199, 215, 219, 261, 270, 280, 363, 372, 382.

Editorial Note. The above problem arose out of the joint work of Brooks, Smith, Stone, and Tutte on "perfect rectangles" (which is not "elementary" at all). All three of the above "perfect squares" were actually found by Tutte. The proposer remarks that Moroń's 32×33 is, as Steinhaus surmised, the simplest perfect rectangle. The only other one with nine pieces is 61×69; another with ten pieces is 47×65. The problem of "squaring the square" was formerly believed to be insoluble. (See Kraïtchik, *La Mathématique des Jeux*, p. 272.) An independent solution has been given by R. Sprague, "Beispiel einer Zerlegung des Quadrats in lauter verschiedene Quadrate," *Mathematische Zeitschrift*, vol. 45, 1939, p. 607; he uses 55 squares. After reading Goldberg's remarks, the proposer adds that it is true, though by no means obvious, that there are only a finite number of 13-piece perfect rectangles. Apropos of "nearly square" rectangles, like 32×33 and 104×105, it has been conjectured that there are infinitely many perfect rectangles whose sides differ by 1. Brooks has proved that there are infinitely many whose sides differ by 2. In fact, let $(n-1)\times(n+1)$ be one such (*e.g.*, 64×66); then we obtain another by doubling its dimensions and juxtaposing squares of sides 1, $2n-1$, $2n$, $2n+1$, so as to make $(4n-1)\times(4n+1)$.

E 402 [1940, 48]. *Proposed by Irving Kaplansky, Harvard University.*

If n, r, and a are positive integers, the congruence $n^2\equiv n \pmod{10^a}$ obviously implies $n^r\equiv n \pmod{10^a}$. (When such a number n has only a digits, it is called an automorphic number.) For what values of r does $n^r\equiv n \pmod{10^a}$ imply $n^2\equiv n \pmod{10^a}$?

Solution by the Proposer.

The implication does not hold when r is odd, for then $4^r\equiv 4 \pmod{10}$, while $4^2=16$; and it does not hold when $r\equiv 1 \pmod 5$, for then $21^r\equiv 21 \pmod{100}$, (since $21^5\equiv 1$), while $21^2=441$. Now let $r-1$ be prime to 10. Since $r-1$ is odd, $(n^{r-1}-1)/(n-1)$ is odd. Suppose, if possible, that $(n^{r-1}-1)/(n-1)$ is divisible by 5. Then $n^{r-1}\equiv 1 \pmod 5$. But by Fermat's Theorem, $n^4\equiv 1 \pmod 5$. Since $r-1$, being odd, differs by 1 from a multiple of 4, it follows that $n\equiv 1 \pmod 5$. Setting $n=5m+1$, we have

$$\begin{aligned}(n^{r-1}-1)/(n-1) &= \{(5m+1)^{r-1}-1\}/5m\\ &= \{(5m)^{r-1}+\cdots+5m(r-1)+1-1\}/5m\\ &\equiv r-1 \pmod 5,\end{aligned}$$

which contradicts the hypothesis that $r-1$ is prime to 10. Hence $(n^r-n)/(n^2-n)$ is prime to 10, and the congruence $n^r-n\equiv 0 \pmod{10^a}$ implies $n^2-n\equiv 0$ $\pmod{10^a}$. Thus the required values of r are all even numbers not ending in 6. (In the trivial case when $a=1$, the proviso against ending in 6 can be waived.)

Also solved by E. P. Starke.

E 403 [1940, 48]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

Show that the two conics

$$x^2 + 2xy + 3y^2 - 1 = 0, \qquad 2x^2 - 6xy - y^2 + 22 = 0$$

have the same director circle.

Solution by P. D. Thomas, Norman, Okla.

Applying the method of C. Smith (*Conic Sections*, p. 255) to the conic $ax^2+2hxy+by^2+c=0$, we see that the pair of tangents from (x', y') is

$$(ax^2 + 2hxy + by^2 + c)(ax'^2 + 2hx'y' + by'^2 + c) \\ = \{axx' + h(xy' + yx') + byy' + c\}^2.$$

These lines are perpendicular if the sum of the coefficients of x^2 and y^2 is zero, *i.e.*, if

$$(a + b)(ax'^2 + 2hx'y' + by'^2 + c) = (ax' + hy')^2 + (hx' + by')^2.$$

Hence the director circle, being the locus of (x', y'), is

$$(ab - h^2)(x^2 + y^2) + (a + b)c = 0.$$

This reduces to $x^2+y^2-2=0$, both when $a=1$, $h=1$, $b=3$, $c=-1$, and when $a=2$, $h=-3$, $b=-1$, $c=22$.

Also solved by M. W. Aylor, W. E. Buker, W. N. Huff, J. S. Leech, D. L. MacKay, C. W. Moran, Hazel E. Schoonmaker, L. W. Sheridan, E. P. Starke, C. W. Trigg, G. A. Williams, and another, most of whom rotated axes so as to put the two conics into the form

$$\frac{x^2}{1 + 2^{-1/2}} + \frac{y^2}{1 - 2^{-1/2}} = 1, \qquad \frac{x^2}{3\cdot 5^{1/2} + 1} - \frac{y^2}{3\cdot 5^{1/2} - 1} = 1.$$

E 404 [1940, 48]. *Proposed by V. Thébault, Le Mans, France.*

Determine the largest and smallest perfect squares which can be written with the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, used once each in both cases.

Solution by Alan Wayne, New York, N. Y.

By means of a table of squares of integers from 1 to 10^5 (constructed by the actuarial department of the Mutuelle Générale Française Vie, an insurance company located at Le Mans, France), V. Thébault has listed all the perfect squares having nine or ten different digits. (See *Mathesis*, vol. 52, 1938, pp. 121–124.)

There are 83 of the former, 87 of the latter. The first and last entries of the latter table are:

$$32043^2 = 10267\ 53849, \qquad 99066^2 = 98140\ 72356.$$

Also solved by W. E. Buker, C. W. Trigg, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3968. *Proposed by Frank Ayres, Jr., Dickinson College.*

Let the line through the vertex A_i, $(i=1, 2, 3)$, and parallel to the opposite side of the triangle $A_1A_2A_3$ meet the circumcircle in the point D_i. Show that: (1) The Δ_2 lines [3929, 1939, 601] of the pairs of points A_i, D_i intersect on the nine-point circle of $A_1A_2A_3$ midway between A_i and the orthocenter of the given triangle. (2) The Δ_2 lines of D_i intersect in the symmetrics of A_1, A_2, A_3 as to the nine-point center.

3969. *Proposed by Frank Ayres, Jr., Dickinson College.*

Let the line through the vertex A_i, $(i=1, 2, 3)$, and parallel to the opposite side of the triangle $A_1A_2A_3$ meet the circumcircle in the point D_i. Show that: (1) The orthocenter of $D_1D_2D_3$ lies on the join of the circumcenter and isogonal conjugate point of the nine-point center of $A_1A_2A_3$. (2) The join of the orthocenters of $D_1D_2D_3$ and $A_1A_2A_3$ is the image line of the Steiner point of the latter triangle.

3970. *Proposed by V. Thébault, Le Mans, France.*

Let $(H)\equiv A_1A_2A_3A_4A_5A_6$ and $(D)\equiv A_1\alpha_1A_2\alpha_2 \cdots A_6\alpha_6$ be a regular hexagon and a regular dodecagon inscribed in a circle (O). Show that: (1) The Simson lines Δ_1, Δ_2 of any point M of (O) with respect to the triangles $A_1A_3A_5$, $A_2A_4A_6$ are perpendicular and intersect at the midpoint of MO. (2) The consecutive sides of the pedal (H') of M with respect to (H) are parallel to Δ_1, Δ_2. (3) Opposite sides of the pedal (D') of M with respect to (D) are parallel to the bisectors of the angles between Δ_1 and Δ_2. (4) Two sides of (D'), separated by a side, are perpendicular. (5) If we denote by S_6, S_{12}, Σ_{12} the areas of (H), (D), (D') then $\Sigma_{12}=S_6+S_{12}/2$. (6) Extend (3) and (4) to pedal polygons of M with respect to a regular polygon of $6k$ sides, k being any integer.

SOLUTIONS

3848 [1937, 667]. *Proposed by P. Erdös, Budapest, Hungary.*

Let O be an arbitrary point in the interior of the triangle ABC, and let A', B', C' be the points in which AO cuts BC, *etc.* If $AA' \geqq BB'$ and $AA' \geqq CC'$, then $AA' \geqq OA' + OB' + OC'$, where equality holds only if $AA' = BB' = CC'$.

Solution by the Proposer.

Let O_1 and A'' be the orthogonal projections of O and A on BC, *etc.*, and set $\alpha = OO_1/AA''$, $\beta = OO_2/BB''$, $\gamma = OO_3/CC''$. It is well known that $\alpha+\beta+\gamma=1$; and evidently $OA' = \alpha \cdot AA'$, $OB' = \beta \cdot BB'$, $OC' = \gamma \cdot CC'$. Hence $OA' + OB' + OC' = \alpha \cdot AA' + \beta \cdot BB' + \gamma \cdot CC' \leqq (\alpha+\beta+\gamma) \cdot AA' = AA'$, and the proof is complete.

A solution of 3746 [1937, 400] follows immediately from the first equality in the last sentence above. For, in that problem BC is given as the greatest side, and then $BC > AA'$, BB', CC'. Hence $BC > OA' + OB' + OC'$, as was required in that problem.

Solved also by H. D. Ruderman.

Editorial Note. The barycentric coördinates of O are $\alpha:\beta:\gamma$; and, as is well known, O is then the centroid of masses α, β, γ placed at the vertices A, B, C. Hence by statics, $(\alpha+\beta+\gamma) \cdot OA' = \alpha \cdot AA'$; and we may for convenience take the total mass as unity, so that $OA' = \alpha \cdot AA'$, *etc.*

We regret that H. D. Ruderman's solution of 3746 was overlooked. He used in that solution the equality $OA'/AA' + OB'/BB' + OC'/CC' = 1$, given in Wentworth's *Plane Geometry*, p. 247, ex. 549; and he sketched a proof of it. The rest of the solution for either 3746 or 3848 then follows easily. For this reason we credit him with the solution of the present problem.

3871 [1938, 253]. *Proposed by V. Thébault, Le Mans, France.*

Let O, I, I_a, I_b, I_c be the centers of the circumcircle, inscribed circle, and the escribed circles of a given triangle ABC; and let Q_a, Q_b, Q_c, Q_a', Q_b', Q_c' be the intersections of the sides of the triangle with the interior and exterior bisectors of its angles. The parallels to the Euler line through the orthogonal projections on $Q_a'Q_b'Q_c'$, Q_bQ_c, Q_cQ_a, Q_aQ_b of an arbitrarily chosen point of that line meet respectively OI, OI_a, OI_b, OI_c in M, M_a, M_b, M_c. Show that the sum of the powers of O with respect to the circles with diameters IM, I_aM_a, I_bM_b, I_cM_c is zero.

Solution by the Proposer.

(1) Let a, b, c be the lengths of the sides of ABC; let R, r, r_a, r_b, r_c be the radii of the circumcircle, incircle, and the three escribed circles; let d, d_a, d_b, d_c be the lengths of OI, OI_a, OI_b, OI_c; and let G be the centroid. We know that the straight lines $\Delta \equiv (Q_a', Q_b', Q_c')$, $\Delta_a \equiv (Q_a', Q_b, Q_c)$, $\Delta_b \equiv (Q_a, Q_b', Q_c)$, $\Delta_c \equiv (Q_a, Q_b, Q_c')$ are perpendicular respectively to the straight lines OI, OI_a, OI_b, OI_c; and that their distances from O are (see *Note* by the proposer in *Mathesis*, 1937, p. 475)

$$\delta = R(R + r)/d, \qquad \delta_i = R(R - r_i)/d_i, \qquad (i = a, b, c).$$

If α, β, γ, Γ are the orthogonal projections of A, B, C, G on Δ, we have $A\alpha = bc/2d$, $B\beta = ca/2d$, $C\gamma = ab/2d$. We then deduce that

$$\begin{aligned} G\Gamma &= (A\alpha + B\beta + C\gamma)/3 = (bc + ca + ab)/6d \\ &= [p^2 + r(4R + r)]/6d, \qquad 2p = a + b + c. \end{aligned}$$

In the same manner we find that the distance of G from Δ_i is $G\Gamma_i = [(p-i)^2 - r_i(4R - r_i)]/6d_i$.

(2) More generally, the distances $M'M''$, $M'M_i''$ of the lines Δ, Δ_i from the point M' of the Euler line, OG, for which $k = OM'/OG$ are, first, $M'M'' = (1-k)\delta + kG\Gamma$, $M'M_i'' = (1-k)\delta_i + kG\Gamma_i$; then

$$\begin{aligned} M'M'' &= [6(1 - k)R^2 + 2(3 - k)Rr + k(p^2 + r^2)]/6d, \\ M'M_i'' &= [6(1 - k)R^2 - 2(3 - k)Rr_i + k\{(p - i)^2 + r_i^2\}]/6d_i. \end{aligned}$$

(3) The power of O with respect to the circle of IM as a diameter, M being the point of the problem, has for its value $(P) = OI \cdot OM = d\Delta'$, $\Delta' = M'M''$, so that

$$\begin{aligned} (P) &= [6(1 - k)R^2 + 2(3 - k)Rr + k(p^2 + r^2)]/6, \\ (P_i') &= [6(1 - k)R^2 - 2(3 - k)Rr_i + k\{(p - i)^2 + r^2\}]/6. \end{aligned}$$

Using the relations $r_a + r_b + r_c = 4R + r$ and $r^2 + r_a^2 + r_b^2 + r_c^2 = 16R^2 - (a^2 + b^2 + c^2)$, we find that

$$(P) + (P_a') + (P_b') + (P_c') = 0,$$

which is the desired result.

3891 [1938, 631]. *Proposed by J. R. Musselman, Western Reserve University.*

The Apollonian circles of a triangle meet in two points, the Hessian points h_1 and h_2. Show that the two Beltrami points (inverses in the circumcircle of the Brocard points) form with either Hessian point an equilateral triangle. Hence each Beltrami point is the center of the circle passing through h_1 and h_2 and the other Beltrami point. Naturally a Brocard point lies on each circle.

Solution by V. W. Graham, High School, Dublin, Ireland.

Let Ω, Ω', K denote the Brocard points and the symmedian point, respectively, and let B, B', $\overline{K}$ denote their inverses in the circumcircle (O) with center O. Since $OB \cdot O\Omega = OK \cdot O\overline{K}$, $K\Omega B\overline{K}$ is a cyclic quadrilateral; and therefore $\angle B\overline{K}K = \angle K\Omega O = 90°$. Hence OK is perpendicular to BB' at $\overline{K}$ the inverse of K, and thus BB' is the Lemoine line. The Hessian points are equidistant from this line so that $h_1\overline{K} = \overline{K}h_2$. Since O, K, h_1, h_2 is a harmonic range,

$O\overline{K}\cdot K\overline{K}=(h_1\overline{K})^2$, or $(O\overline{K}-OK)\cdot O\overline{K}=(h_1\overline{K})^2$. From $OK\cdot O\overline{K}=R^2$ and the known relation $(OK)^2=R^2(1-3\tan^2\omega)$, we have $OK/O\overline{K}=1-3\tan^2\omega$. From these equations we find $h_1\overline{K}=\sqrt{3}\tan\omega O\overline{K}=\sqrt{3}\overline{K}B$. Hence $\angle\overline{K}Bh_1=60°$; and, similarly, $\angle\overline{K}B'h_1=60°$. Thus the triangles $BB'h_1$ and $BB'h_2$ are equilateral. Since $O\Omega\cdot OB=Oh_1\cdot Oh_2=R^2$, Ω lies on the circle through h_1, h_2, B with its center at B'. A similar result is true for the other set of points.

Solved also by F. C. Gentry and O. J. Ramler.

Editorial Note. The other two solutions were analytic.

There is an Apollonian circle for each vertex of a triangle ABC, the circle (k_1) with center k_1, for A passes through A and has for a diameter the segment I_1I_1' cut from BC by the internal and external bisectors of angle A. Thus (k_1) is orthogonal to the circumcircle (O), and it is the locus of points P such that $b\cdot BP=c\cdot CP$, where a, b, c are the lengths of the sides of ABC. It follows immediately that the three circles intersect in two real distinct points h_1, h_2 forming an inverse pair with respect to (O). Each of the two points, say h_1, forms with A, B, C a quadrangle for which the product of the opposite sides is the same, that is, $a\cdot Ah_1=b\cdot Bh_1=c\cdot Ch_1$, and for this reason they are often called the isodynamic points for ABC. Obviously, the centers k_1, k_2, k_3 lie on the straight line k perpendicular to the common chord h_1h_2 at its midpoint $\overline{K}$. Since Ak_1 is antiparallel to BC with respect to angle A and k_1 is on BC, the polar of k_1 is the symmedian for A; and therefore k is the polar of the symmedian point K with respect to (O) and also with respect to ABC. Thus K, $\overline{K}$ form an inverse pair with respect to (O), and O, K, h_1, $\overline{K}$, h_2 are collinear. The circle $(\overline{K})$ on h_1h_2 as a diameter is orthogonal to (O), and the inverses of $(\overline{K})$ and k with respect to (O) are $(\overline{K})$ and the Brocard circle (OK) on OK as a diameter; hence (OK) and $(\overline{K})$ are orthogonal, and k is the radical axis of (OK) and (O). This gives $(\overline{K}h_1)^2=(O\overline{K})^2-R^2=R^4/(OK)^2-R^2$. It is easily shown that the pedal triangle for each of the points h_1, h_2 is equilateral. Also, if Ah_1 cuts (O) again in A_1, the triangle $A_1B_1C_1$ is equilateral. The isogonal conjugates h_1', h_2' of h_1, h_2 are of interest. Each side of the equilateral pedal triangle $X_1Y_1Z_1$ of h_1, say Y_1Z_1, is perpendicular to the corresponding straight line Ah_1'. Hence h_1' is a point such that the sides of ABC subtend at it angles of 120° or 60°; for this reason the points h_1', h_2' are called isogonic centers of ABC. If equilateral triangles are constructed outwardly upon the sides of ABC, for example $B\overline{A}C$, the three straight lines, such as $A\overline{A}$, meet in h_1'. If the equilateral triangles are constructed inwardly they give h_2'. Thus the isogonic centers lie on Kiepert's hyperbola; see the solutions of 3882, 3883 [1940, 403]. The Brocard pair of points Ω, Ω' belong to the class of isogonal conjugate pairs, and they have the distinction that the angles $CB\Omega$, $AC\Omega$, $BA\Omega$ are equal, and this common angle ω is called the Brocard angle for ABC. Among their other remarkable properties are the following: Ω and Ω' are on the circle (OK) so that $\angle\Omega'OK=\angle KO\Omega=\omega$ and $(OK)^2=R^2(1-3\tan^2\omega)$. The proofs are not simple. These equations with the one above for $\overline{K}h_1$ give the theorem of the problem as in the above solution.

3894 [1938, 631]. *Proposed by Walter Leighton, The Rice Institute.*

Given a polynomial equation

(1) $$f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0,$$

where the a_i are rational integers, find all roots of the form $(a \pm \sqrt{b})/c$, where a, b, c are integers, $c \neq 0$, and $\sqrt{b}$ is irrational (possibly imaginary).

Solution by the Proposer.

Suppose that $a+\sqrt{b}$ is a root of (1), where a and b are integers and where $\sqrt{b}$ is not rational (possibly imaginary). Then $a-\sqrt{b}$ is also a root and

(2) $$f(x) \equiv [(x-a)^2 - b]g(x),$$

where $g(x)$ is a polynomial of degree $n-2$, with rational integral coefficients. The following theorem is immediate:

If $a+\sqrt{b}$ is a root of (1), *where a and b are integers and $\sqrt{b}$ is irrational, then $(m-a)^2-b$ must divide $f(m)$ for every integer m. In particular, a^2-b must divide $f(0)=a_n$.*

If $b<0$, the final remark in the theorem reduces the possible values of each a and b to a finite set. The first statement in the theorem supplies a "sieve" for each m by means of which this finite set may be reduced in size.

The general case where $\sqrt{b}$ is either real or imaginary, but irrational, is only slightly more difficult to handle. Let all the integral divisors (positive and negative) of a_n be the set $\alpha_1, \alpha_2, \cdots, \alpha_k$. Then by the theorem, a and b must satisfy at least one of the equations

(3) $$b = a^2 - \alpha_j, \qquad (j = 1, \cdots, k).$$

A further application of the theorem with $m=1$* yields the fact that $(1-a)^2-b$ must divide $f(1)$. If the integral divisors (positive and negative) of $f(1)$ are the set $\beta_1, \beta_2, \cdots, \beta_p$, we have the further condition that a and b must satisfy at least one of the equations

(4) $$(1-a)^2 - b = \beta_i, \qquad (i = 1, 2, \cdots, p).$$

Combining (3) and (4) we have the result that a must satisfy one of the $k \cdot p$ equations

(5) $$2a = 1 + \alpha_j - \beta_i, \qquad (j = 1, \cdots, k;\ i = 1, \cdots, p;\ i, j \text{ independent}).$$

However, condition (5) yields at most a finite set of possible values a, and conditions (3) and (4) permit only a finite number of possibilities b. The theorem may now be applied with other values of m to this finite set of numbers a and b to reduce it further. The problem of finding all roots $a \pm \sqrt{b}$ of (1) is thus solved.

We turn now to the problem of finding all quadratic irrational roots of (1);

* Some other choice of m will often be advantageous. Clearly it is desirable that $f(m)$ have as few divisors as possible.

that is, all roots of the form $(a \pm \sqrt{b})/c$, where a, b, c are integers, $c \neq 0$, and $\sqrt{b}$ is irrational. This case can be reduced to the preceding one. To that end the substitution $y = a_0 x$ is first made in equation (1), reducing (1) to a polynomial equation of degree n in y with rational integral coefficients and leading coefficient unity:

$$g(y) \equiv y^n + b_1 y^{n-1} + \cdots + b_{n-1} y + b_n = 0. \tag{6}$$

It is well known that if $y^2 + py + q$ is a factor of $g(y)$, where p and q are rational, then p and q are integers. Hence all quadratic irrational roots of (6) are of the form $\frac{1}{2}(a \pm \sqrt{b})$, where a and b are integers and $\sqrt{b}$ is irrational. We now make the substitution $z = 2y$. The result is a polynomial equation

$$h(z) \equiv z^n + 2b_1 z^{n-1} + 2^2 b_2 z^{n-2} + \cdots + 2^{n-1} b_{n-1} z + 2^n b_n = 0.$$

The quadratic irrational roots of this equation are of the form $a \pm \sqrt{b}$ and the preceding theory may be applied.

Example. Let us set $b = -\beta$, $\beta > 0$, and consider the equation*

$$x^4 - 2x^2 - 8x - 3 = 0. \tag{7}$$

Then $a^2 + \beta$ must be one of the numbers 1, 3. Accordingly the possible values of a and β are

$$\beta = 1,\ a = 0; \qquad \beta = 2,\ a = 1,\ -1; \qquad \beta = 3,\ a = 0. \tag{8}$$

Observe that $f(2) = -11$ and hence that $(2-a)^2 + \beta$ must divide 11. The set (8) thus reduces to $\beta = 2$, $a = -1$. We try this possibility and find that $-1 \pm \sqrt{-2}$ are indeed roots of (7). The other two roots are now readily found to be $1 \pm \sqrt{2}$.

Editorial Note. See also the paper of Oystein Ore, Revista Ci., Lima, 41, 1939, pp. 587–592.

* L. E. Dickson, First Course in the Theory of Equations, p. 51.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The following persons have been awarded fellowships by the Guggenheim Foundation: Dr. Jesse Douglas, Assistant Professor Gordon Pall, Professor R. L. Wilder, Associate Professor A. F. Wintner.

Professor G. D. Birkhoff of Harvard University was elected Foreign Associate of the National Academy of Sciences of Lima.

Johns Hopkins University has conferred the doctorate of laws on Dr. Vannevar Bush, President of the Carnegie Institution, Washington, D. C.

Associate Professor A. W. Tucker, on leave of absence from Princeton University for the year 1940–41, is giving a series of six lectures on "The Tensor Topology of Networks" at Northwestern University this autumn.

Professor Hermann Weyl of the Institute for Advanced Study has been elected a member of the National Academy of Sciences.

Assistant Professor Max Astrachan of Antioch College has been made chairman of the department of mathematics.

At the University of Kansas Assistant Professors Wealthy Babcock and Florence L. Black have been promoted to associate professorships.

Dr. R. P. Bailey of Lafayette College has been promoted to an assistant professorship.

Dr. N. H. Ball of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. F. S. Beale of Lehigh University has been promoted to an assistant professorship.

Dr. E. F. Beckenbach of Rice Institute has been appointed an assistant professor at the University of Michigan.

Associate Professor A. H. Black of Lebanon Valley College has been promoted to a professorship.

Dr. Augusto Bobonis of the University of Puerto Rico has been promoted to an assistant professorship.

J. M. Boswell of the University of Kentucky has been appointed professor of mathematics and physics at Cumberland College.

Assistant Professor D. G. Bourgin of the University of Illinois has a leave of absence for the year 1940–41 and is at the Institute for Advanced Study.

Dr. A. T. Brauer of the Institute for Advanced Study will lecture at New York University during 1940–41.

Dr. Foster Brooks and Dr. Emma J. Olson of Kent State University have been promoted to assistant professorships.

Dr. A. A. F. Brown is now associated with the statistical department of Bamberger's Store, Newark, New Jersey.

Dr. G. W. Brown is associated with the research department of R. H. Macy and Company, New York, N. Y.

Dr. F. A. Butter of the University of Southern California has been promoted to an assistant professorship.

Dr. Claude Chevalley has been appointed assistant professor of mathematics at Princeton University.

Dr. J. A. Clarkson of the University of Pennsylvania has been promoted to an assistant professorship.

Assistant Professor H. M. Cox has been appointed Director of the Bureau of Instructional Research at the University of Nebraska.

Assistant Professor H. V. Craig of the University of Texas has been promoted to an associate professorship.

Associate Professor M. M. Culver of the University of Pittsburgh has been promoted to a professorship.

Dr. J. H. Curtiss of Cornell University has been promoted to an assistant professorship.

Dr. E. H. Cutler of Lehigh University has been promoted to an assistant professorship.

Dr. M. M. Day, Dr. R. J. Duffin, and Dr. R. H. Fox have been appointed to associateships at the University of Illinois.

Dr. Carl Denbow of Ohio University has been promoted to an assistant professorship.

At Michigan State College, Dr. P. L. Dressel, J. D. Hill, Dr. C. C. Hurd, and Dr. E. T. Welmers have been promoted to assistant professorships.

Assistant Professor J. E. Eaton of Hofstra College is now head of the department of mathematics.

Dr. Churchill Eisenhart of the University of Wisconsin has been promoted to an assistant professorship.

Assistant Professor H. T. Engstrom of Yale Universty has been promoted to an associate professorship.

Assistant Professor O. J. Farrell of Union College has been promoted to an associate professorship.

Assistant Professor E. J. Finan of the Catholic University of America has been promoted to an associate professorship.

At Cornell University Assistant Professors W. W. Flexner and B. W. Jones have been promoted to associate professorships.

Dr. K. O. Friedrichs of New York University has been promoted to an associate professorship.

Professor Orrin Frink, Jr., of Pennsylvania State College, on leave of absence for the first semester of 1940–41, is at the Institute for Advanced Study.

Associate Professor Gordon Fuller of Alabama Polytechnic Institute has been promoted to a professorship.

Dr. T. N. E. Greville of the University of Michigan is spending the year in actuarial work with the Bureau of the Census in Washington, D. C.

Professor V. G. Grove has been appointed chairman of the department of mathematics at Michigan State College.

Assistant Professor O. H. Hamilton of Oklahoma Agricultural and Mechanical College was promoted to an associate professorship in September 1939.

Dr. E. E. Haskins of Northeastern University has been promoted to an assistant professorship.

Dr. A. E. Heins of Purdue University has been promoted to an assistant professorship.

Dr. Olaf Helmer and Dr. H. E. Vaughan of the University of Illinois have been promoted to associateships.

Dr. Banesh Hoffmann of Queens College has been promoted to an assistant professorship.

Dr. J. T. Hurt of the Agricultural and Mechanical College of Texas has been promoted to an assistant professorship.

Dr. L. P. Hutchison of Indiana Technical College has been appointed an assistant professor at the Citadel, Charleston, S. C.

Assistant Professor Nathan Jacobson of the University of North Carolina has been appointed visiting associate professor at Johns Hopkins University.

Assistant Professor Glenn James of the University of California at Los Angeles has been granted a sabbatical leave for the first semester of the year 1940–41.

L. W. Johnson was appointed assistant professor at Purdue University. However, he was called to duty with the National Guard with the rank of Captain, and is located at Fort Sill, Oklahoma.

Professor L. C. Karpinski of the University of Michigan is on leave of absence for the first semester of the year 1940–41.

Assistant Professor Chosaburo Kato of Denison University has been promoted to an associate professorship.

Dr. Fulton Koehler has resigned his position at the University of Minnesota in order to enter the actuarial department of the Northwestern National Life Insurance Company.

Dr. O. E. Lancaster of the University of Maryland has been promoted to an assistant professorship.

Assistant Professors V. S. Lawrence, Jr., and J. B. Rosser of Cornell University have been promoted to associate professorships.

Dr. T. H. Lee of the University of Wisconsin has been appointed an associate professor at the University of South Carolina.

Associate Professor V. F. Lenzen of the University of California has been promoted to a professorship.

Assistant Professor Sophia H. Levy of the University of California has been promoted to an associate professorship.

Professor C. C. MacDuffee of the University of Wisconsin has been appointed to a professorship at Hunter College. He will fill the vacancy left by the retirement of Professor Lao G. Simons, chairman of the department.

Dr. W. C. McDaniel of Southern Illinois State Normal University has been promoted to an assistant professorship.

Assistant Professor J. D. Mancill of the University of Alabama has been promoted to an associate professorship.

Dr. S. Mandelbrojt has been appointed visiting lecturer at the Rice Institute.

Associate Professor Morris Marden of the University of Wisconsin Extension Division at Milwaukee is on leave of absence and is lecturing at the University of Wisconsin.

Assistant Professor W. T. Martin of the Massachusetts Institute of Technology has been granted a leave of absence to spend the year Princeton University.

Assistant Professor W. E. Mason of the University of California at Los Angeles has been promoted to an associate professorship.

After a two years' leave of absence spent at California Institute of Technology, Dr. A. B. Mewborn has returned to the University of Arizona as an assistant professor.

Professor B. J. Miller of St. Ambrose College, Davenport, Iowa, has received an appointment at St. Louis University.

Dr. G. E. Moore of the University of Illinois has been promoted to an assistant professorship.

Dr. T. W. Moore of the U. S. Naval Academy has been promoted to an assistant professorship.

Dr. W. K. Morrill of Johns Hopkins University has been appointed an associate in mathematics.

Assistant Professor W. R. Murray of Franklin and Marshall College has been promoted to an associate professorship.

Dr. W. H. Myers has been appointed assistant professor of mathematics at San José State College.

Assistant Professor Sara L. Nelson of Georgia State College for Women has been promoted to an associate professorship.

Dr. L. F. Ollmann of Texas Technological College has been appointed an assistant professor at the College of Wooster.

Assistant Professor F. W. Perkins of Dartmouth College has been promoted to an associate professorship.

Assistant Professor G. W. Petrie III of South Dakota State School of Mines has been promoted to an associate professorship.

Dr. H. R. Phalen has been appointed an associate professor at the College of William and Mary.

Professor H. R. Pyle of Earlham College has been appointed to a professorship at Whittier College.

Associate Professor C. B. Read of the University of Wichita has been promoted to a professorship and becomes head of the department.

Associate Professor D. E. Richmond of Williams College has been promoted to a professorship.

Dr. S. T. Sanders, Jr., of Delta State Teachers College has been appointed to a professorship at Southwestern Louisiana Institute.

Brother Louis de LaSalle Seiler of St. Mary's College, Winona, Minnesota, is now dean of studies.

Dr. R. W. Shephard is now Research Fellow, Bureau of Public Administration, at the University of California.

Dr. Seymour Sherman has accepted a stipend for the year 1940–41 from the Institute for Advanced Study.

Professor W. G. Simon, Dean of the Faculties of Arts and Sciences at Western Reserve University has been made vice president of the University.

Assistant Professor Ruth G. Simond of Hampton Institute has been promoted to an associate professorship.

Professor Marcus Skarstedt of Whittier College has been appointed Librarian at San Francisco Junior College.

R. E. Smith of the University of North Carolina has been appointed to a professorship at Atlantic Christian College, Wilson, N.C.

Associate Professor A. H. Sprague of Amherst College has been promoted to a professorship.

Associate Professor D. J. Struik of Massachusetts Institute of Technology has been promoted to a professorship.

Dr. A. E. Taylor of the University of California at Los Angeles has been promoted to an assistant professorship.

Dr. R. M. Thrall of the University of Michigan is spending the year 1940–41 in study on a fellowship at the Institute for Advanced Study.

Professor W. J. Trjitzinsky of the University of Illinois is on leave of absence for the year 1940–41, and is at Princeton University.

During 1940–41, Associate Professor A. R. Turquette of Florida Southern College is on leave of absence and is studying at Cornell University.

Associate Professor M. S. Vallarta of the Massachusetts Institute of Technology has been promoted to a professorship.

Professor H. S. Vandiver of the University of Texas is on leave of absence for the first semester of the year 1940–41.

Assistant Professor G. C. Vedova of St. John's College has been appointed an assistant professor at the University of Maryland.

Assistant Professor O. E. Walder of South Dakota State College has been promoted to an associate professorship.

Assistant Professor R. J. Walker of Cornell University is on leave of absence for 1940–41 and is at Princeton University for the year.

Dr. J. A. Ward of Tennessee Polytechnic Institute has been appointed associate professor and acting head of the department of mathematics at the Delta State Teachers College, Cleveland, Miss.

Assistant Professor Hassler Whitney of Harvard University has been promoted to an associate professorship.

Professor R. L. Wilder of the University of Michigan is on leave of absence for the year 1940–41 and is spending his time in reading and research at the University of Texas.

Dr. C. R. Wylie, Jr., of Ohio State University has been promoted to an assistant professorship.

Assistant Professor G. A. Yanosik of New York University has been promoted to an associate professorship.

Professor Oscar Zariski of Johns Hopkins University is on leave of absence for the year 1940–41 and is at Harvard University as a visiting lecturer.

Professor A. Zygmund, formerly of the University of Wilno, has accepted a position at Mount Holyoke College.

The following appointments to instructorships are announced:

University of Alabama: Dr. R. H. Bruck

Amarillo College: Dr. G. A. Whetstone

Antioch College: Dr. Leonard Tornheim

University of Arizona: Dr. F. E. Hohn

Bowling Green State University: Dr. Morris Hendrickson

Brown University: G. E. Forsythe

University of California at Los Angeles: Dr. Ralph Byrne, Jr.

University of Cape Town: Dr. J. S. de Wet

Carnegie Institute of Technology: Dr. R. F. Clippinger, Dr. A. D. Hestenes

University of Cincinnati: Dr. Gabriel Horvay

College of the City of New York: Dr. S. F. Barber

Cornell University: B. H. Bissinger, W. H. Durfee, Dr. Michael Golomb, N. G. Gunderson, R. R. R. Luckey

Drew School, San Francisco: S. E. Field

Duke University: Miss Mary E. Layne, part-time

Fenn College: Dr. H. W. Alexander

Georgia School of Technology: L. B. Williams

Gogebic Junior College, Ironwood, Mich.: K. F. McLaughlin

Harvard Graduate School of Engineering: Dr. H. W. Emmons in mechanical engineering

Harvard University: Dr. A. L. Whiteman

Hunter College: Dr. Annita Tuller

University of Illinois: Dr. P. H. Anderson, Dr. K. L. Nielsen

Indiana University: Dr. F. J. Weyl

University of Indiana Extension, East Chicago, Indiana: J. F. Paydon

Itasca Junior College, Coleraine, Minn.: Dr. H. L. Olson

Johns Hopkins University: Dr. R. B. Kershner

University of Maryland: Dr. H. E. Newell

Massachusetts Institute of Technology: Dr. G. B. Thomas, Jr.

University of Michigan: Dr. W. D. Duthie, Dr. Samuel Eilenberg, Dr. G. E. Hay, Dr. Wilfred Kaplan, Dr. A. V. Martin

Michigan State College: A. C. Cohen, Jr., Dr. J. F. Heyda, Dr. B. M. Stewart

University of Minnesota: Dr. J. M. H. Olmsted

University of Missouri: Dr. G. E. Schweigert, Dr. M. E. Shanks, Dr. J. V. Wehausen

Montana State College: Dr. H. M. Schwartz

Sophie Newcomb College of Tulane University: Miss Mary E. Ladue

University of North Carolina: Dr. Henry Wallman

North Carolina State College: Dr. A. M. Gelbart

Northwestern University: H. E. Burns, Dr. J. M. Dobbie, Dr. O. G. Harrold, W. N. Huff, Dr. W. T. Scott

University of Notre Dame: Dr. J. P. Nash

Ohio State University: Dr. Maxwell Reade, Dr. W. J. Schart

University of Pennsylvania: H. N. Laden

Princeton University: Dr. E. G. Begle, part-time, Dr. Brockway McMillan

Purdue University: Dr. H. L. Langhaar, Dr. A. W. McGaughey, C. D. Olds

Reed College: Dr. L. Louise Johnson, R. A. Rosenbaum

Rutgers University: Dr. J. H. Giese

University of Saskatchewan: Dr. D. C. Murdoch

Stanford University: Dr. T. C. Doyle, Dr. P. V. Reichelderfer

Suffolk University: F. X. Sutton

University of Texas: J. R. Foote, H. J. Jones, Dr. A. M. Mood, I. C. Roberts

Trinity College, Washington, D. C.: Dr. Mary Varnhorn

U. S. Naval Academy: Dr. W. E. Bleick

Vermont Junior College: D. F. Johnson

University of Washington: Dr. R. A. Beaumont, Dr. R. S. Phillips

University of Western Ontario: Dr. R. H. Cole

University of Wichita: Edison Greer

Williams College: W. D. Wray

Winthrop College: Dr. Katharine Hazard

University of Wisconsin: Dr. H. E. Goheen, Fred Kiokemeister, Dr. A. O. Lindstrum, Jr., J. W. Odle, Dr. C. B. Smith, Dr. S. M. Ulam

University of Wyoming: Dr. V. J. Varino

Mrs. Edward Fitch, the former Annie L. MacKinnon, professor of mathematics from 1896 to 1901 at Wells College, died on September 12, 1940, at Clinton, N.Y. She was a charter member of the Mathematical Association.

Dr. R. S. Martin of the University of Illinois died June 2, 1940.

Dr. George Rutledge died on September 21, 1940. He became an instructor in mathematics at Massachusetts Institute of Technology in 1915, and was made professor in 1934. He was a charter member of the Mathematical Association.

A RESOLUTION BY THE S.P.E.E.

At the Berkeley meetings of the Society for the Promotion of Engineering Education, held June 24–28, 1940, their Council passed the following resolution:

In various parts of the country there seems to be a movement to postpone and to abbreviate the courses in mathematics given in the secondary schools. This movement apparently does not recognize the fact that those courses are essential prerequisites for the future training of scientific and engineering students, and that the university has not postponed and cannot postpone the mathematical or the scientific and engineering instruction in the university, if its graduates are to enter those professional fields. Moreover, at the present time, for our own defense as a nation, it is suicidal not to develop the most thorough kind of training for engineers.

The members of the Conference on Mathematics of the Society for the Promotion of Engineering Education wish to go on record as recommending that there be no postponement in the mathematical education in the secondary schools of those students who are to seek careers in science and engineering. In particular they feel that it is essential that a full four-year program of mathematics be available in the high schools for capable students, beginning with a year of college preparatory algebra in the ninth grade. They feel that this subject should not be postponed and also that thorough work in trigonometry and solid geometry should be available.

This resolution in no way implies that university preparatory courses be required of all students. But this organization feels very strongly the importance of providing substantial courses in mathematics for those who need them in preparation for future work or for those who choose to elect them. We believe that to be effective these courses must begin with algebra at least as early as the ninth year, that is, the first of the last four years in the secondary schools.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va., April 20; Grove City, Pa., November 2.
Illinois, Bloomington, May 10–11.
Indiana, Richmond, May 3–4.
Iowa, Mt. Vernon, April 19–20.
Kansas, Wichita, March 31.
Kentucky, Lexington, April 27.
Louisiana-Mississippi, Oxford, Miss., March 8–9.
Maryland-District of Columbia-Virginia, Richmond, Va., May 11; Washington, D.C., December 7 or 14.
Michigan, Ann Arbor, April 26–27.
Minnesota, Mankato, May 4.
Missouri, Warrensburg, April 19.
Nebraska, Omaha, May 9–11.
Northern California, Berkeley, January 27.
Ohio, Columbus, April 5.
Oklahoma, Oklahoma City, February 16.
Philadelphia, November 30.
Rocky Mountain, Fort Collins, Colo., April 19.
Southeastern, Athens, Ga., March 29–30.
Southern California, Compton, March 2.
Southwestern, Tucson, Ariz., April 22–23
Texas, Dallas, March 29–30.
Upper New York State, Hamilton, May 11.
Wisconsin, Milwaukee, May 4.

Affiliated Organizations: The New England Association of Teachers of Mathematics
The National Council of Teachers of Mathematics

THE TWENTY-THIRD SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION

The twenty-third summer meeting of the Mathematical Association of America was held at Dartmouth College, Hanover, New Hampshire, on Monday and Tuesday, September 9–10, 1940, in conjunction with the summer meeting and colloquium of the American Mathematical Society, and the meeting of the Institute of Mathematical Statistics. Four hundred sixty were in attendance at the meetings, including the following one hundred and ninety-nine members of the Association:

C. R. Adams, Brown University
R. P. Agnew, Cornell University
R. C. Archibald, Brown University
H. E. Arnold, Wesleyan University
H. T. R. Aude, Colgate University

J. A. Benner, Lafayette College
Brother Bernard Alfred, Manhattan College
Felix Bernstein, New York University
Harry Birchenough, New York State College for Teachers
G. D. Birkhoff, Harvard University
R. P. Boas, Duke University
H. D. Boutelle, Massachusetts State College
J. G. Bowker, Middlebury College
C. B. Boyer, Brooklyn College
R. W. Brink, University of Minnesota
B. H. Brown, Dartmouth College
R. E. Bruce, Boston University
J. A. Bullard, University of Vermont
C. T. Bumer, Kenyon College
R. S. Burington, Case School of Applied Science
F. J. H. Burkett, Union College
Jewell Hughes Bushey, Hunter College
J. H. Bushey, Hunter College
A. D. Butterfield, University of Vermont

W. D. Cairns, Oberlin College
B. H. Camp, Wesleyan University
Mildred E. Carlen, Brown University
P. N. Carpenter, Grove City College
W. B. Carver, Cornell University
L. W. Cohen, University of Kentucky
Nancy Cole, Sweet Briar College
G. M. Conwell, St. Paul's School
J. A. Cooley, University of Tennessee
T. F. Cope, Queens College
A. H. Copeland, University of Michigan
Lennie P. Copeland, Wellesley College
Richard Courant, New York University
A. T. Craig, University of Iowa
A. R. Crathorne, University of Illinois
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University
J. H. Curtiss, Cornell University

D. R. Davis, New Jersey State Teachers College, Montclair
F. F. Decker, Syracuse University
L. S. Dederick, Aberdeen Proving Ground
A. H. Diamond, Oklahoma A. and M. College
C. E. Dimick, U. S. Coast Guard Academy
C. H. Dix, Socony Vacuum Oil Corporation
H. A. DoBell, New York State College for Teachers
Arnold Dresden, Swarthmore College

L. P. Eisenhart, Princeton University
G. C. Evans, University of California
G. W. Evans, Swampscott, Mass.

C. H. Fischer, Wayne University
W. B. Fite, Columbia University
H. T. Fleddermann, Loyola University, New Orleans, La.
M. M. Flood, Princeton University
L. R. Ford, Illinois Institute of Technology
Tomlinson Fort, Lehigh University
R. M. Foster, Bell Telephone Laboratories
A. H. Fox, Union College
J. S. Frame, Brown University
Philip Franklin, Massachusetts Institute of Technology
Orrin Frink, Jr., Pennsylvania State College
Thornton C. Fry, Bell Telephone Laboratories

M. G. Gaba, University of Nebraska
H. J. Gay, Worcester Polytechnic Institute
B. P. Gill, College of the City of New York
R. E. Gilman, Brown University
Michael Goldberg, Bureau of Ordnance, Navy Department

C. H. GRAVES, Pennsylvania State College
L. J. GREEN, Georgia School of Technology

E. H. HADLOCK, Boston University
D. W. HALL, Brown University
J. A. HAMILTON, Metropolitan Life Insurance Company
D. C. HARKIN, Brooklyn College
OLIVE C. HAZLETT, University of Illinois
E. R. HEDRICK, University of California at Los Angeles
ROBERT HENDERSON, Crown Point, N. Y.
E. H. C. HILDEBRANDT, New Jersey State Teachers College, Montclair
T. H. HILDEBRANDT, University of Michigan
EINAR HILLE, Yale University
T. R. HOLLCROFT, Wells College
GRACE M. HOPPER, Vassar College
E. V. HUNTINGTON, Harvard University
W. A. HURWITZ, Cornell University

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
FRITZ JOHN, University of Kentucky
R. A. JOHNSON, Brooklyn College
F. E. JOHNSTON, George Washington University
L. S. JOHNSTON, University of Detroit
B. W. JONES, Cornell University

MARK KAC, Cornell University
WILFRED KAPLAN, University of Michigan
E. S. KENNEDY, University of Alabama
L. S. KENNISON, Brooklyn College
W. L. KICHLINE, University of New Hampshire
J. R. KLINE, University of Pennsylvania

W. D. LAMBERT, U. S. Coast and Geodetic Survey
A. E. LANDRY, Catholic University of America
R. E. LANGER, University of Wisconsin
CAROLINE A. LESTER, New York State College for Teachers
MAYME I. LOGSDON, University of Chicago
W. R. LONGLEY, Yale University
C. I. LUBIN, University of Cincinnati

L. A. MACCOLL, Bell Telephone Laboratories
C. C. MACDUFFEE, Hunter College
SAUNDERS MAC LANE, Harvard University
H. M. MACNEILLE, Kenyon College
N. H. MCCOY, Smith College
E. J. MCSHANE, University of Virginia
L. C. MATHEWSON, Dartmouth College
W. E. MILNE, Oregon State College
E. B. MODE, Boston University
DEANE MONTGOMERY, Smith College
F. C. MOORE, Massachusetts State College
T. W. MOORE, U. S. Naval Academy
F. M. MORGAN, Clark School
W. K. MORRILL, Johns Hopkins University
MAX MORRIS, Case School of Applied Science
MARSTON MORSE, Institute for Advanced Study
F. C. MOSTELLER, Princeton University
E. J. MOULTON, Northwestern University
G. W. MULLINS, Columbia University
C. W. MUNSHOWER, Colgate University

C. V. NEWSOM, University of New Mexico
ABBA V. NEWTON, Hartwick College

C. O. OAKLEY, Haverford College
J. M. H. OLMSTED, Princeton University
ISAAC OPATOWSKI, University of Minnesota
OYSTEIN ORE, Yale University
F. W. OWENS, Pennsylvania State College
HELEN B. OWENS, State College, Pa.

GORDON PALL, McGill University
W. O. PENNELL, Exeter, N. H.
F. W. PERKINS, Dartmouth College
L. R. PERKINS, Middlebury College
O. J. PETERSON, Kansas State Teachers College, Emporia
C. G. PHIPPS, University of Florida
A. E. PITCHER, Lehigh University
HILLEL PORITSKY, General Electric Company
G. B. PRICE, University of Kansas
E. J. PURCELL, University of Arizona

E. D. RAINVILLE, University of Michigan
J. F. RANDOLPH, Cornell University
MINA S. REES, Hunter College
C. E. RHODES, Groton, N. Y.
HARRIS RICE, Worcester Polytechnic Institute
R. G. D. RICHARDSON, Brown University
R. F. RINEHART, Case School of Applied Science
J. F. RITT, Columbia University
ROBIN ROBINSON, Dartmouth College
SELBY ROBINSON, College of the City of New York
JOSEPH ROSENBAUM, Bloomfield, Conn.
HELEN G. RUSSELL, Wellesley College

S. T. SANDERS, Louisiana State University
I. J. SCHOENBERG, Colby College
W. E. SEWELL, Georgia School of Technology
I. M. SHEFFER, Pennsylvania State College
L. W. SHERIDAN, College of Mount St. Vincent
L. L. SILVERMAN, Dartmouth College
MARY EMILY SINCLAIR, Oberlin College
H. L. SLOBIN, University of New Hampshire
M. M. SLOTNICK, Humble Oil and Refining Company
CLARA E. SMITH, Wellesley College
W. M. SMITH, Lafayette College
VIRGIL SNYDER, Cornell University
JOSEPH SPEAR, Northeastern University
VIVIAN E. SPENCER, U. S. Bureau of Mines
M. H. STONE, Harvard University
ELIZABETH C. STRAYHORN, Western Kentucky State Teachers College
R. E. STREET, Rensselaer Polytechnic Institute
A. G. SWANSON, General Motors Institute
OTTO SZÁSZ, University of Cincinnati
GABOR SZEGÖ, Stanford University

J. D. TAMARKIN, Brown University
J. M. THOMAS, Duke University
MARIAN M. TORREY, Goucher College
J. I. TRACEY, Yale University
A. W. TUCKER, Princeton University
J. W. TUKEY, Princeton University

G. B. VAN SCHAACK, Michigan State College
H. E. VAUGHAN, University of Illinois
OSWALD VEBLEN, Institute for Advanced Study
JOHN VON NEUMANN, Institute for Advanced Study

R. J. WALKER, Cornell University
J. L. WALSH, Harvard University
WARREN WEAVER, Rockefeller Foundation
W. D. A. WESTFALL, University of Missouri
ANNA PELL WHEELER, Bryn Mawr College
P. M. WHITMAN, Harvard University
J. K. WHITTEMORE, Yale University
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, University of California at Los Angeles
D. V. WIDDER, Harvard University
C. E. WILDER, Dartmouth College
CLEMENT WINSTON, Railroad Retirement Board
W. D. WRAY, Cornell University

Those in attendance had comfortable quarters in the dormitories of Dartmouth College, easily accessible from the headquarters at Thayer Hall, which contained the place of registration, social parlors, and the cafeteria.

Tea was served Monday afternoon at the Graduate Club by the ladies of the faculty. A delightful violin and piano recital was given Monday evening by Raphael Silverman and Lily Dymont. During the evening moving pictures were shown of Dartmouth winter sports and campus life, and Professor Robin Robinson exhibited to small groups a collection of string models of ruled surfaces, especially with a view to exemplifying the ease with which such models can be constructed by mathematics teachers and their pupils.

A large number made the trip on Wednesday afternoon to Sugar Hill for an outstanding view of the Franconia Range, to Echo Lake and Franconia Notch, with the adjacent Old Man of the Mountain and the Cannon Mountain Tramway, to the Flume and Lost River.

Baker Memorial Library, the Carpenter Art Galleries, the Dartmouth College Museum were available during the week, not to speak of a visit to the Nigger Island and Pompanoosuc Railroad, a model constructed by students of the Thayer School of Civil Engineering.

The near-record number of three hundred twenty-four persons attended the joint dinner of the three organizations on Tuesday evening. Under the witty toastmastership of Professor B. H. Brown, speeches were given by Dean E. G.

Bill of Dartmouth College, Professor Hassler Whitney, Professor S. S. Wilks as president of the Institute of Mathematical Statistics, and Dean R. G. D. Richardson. Professor and Mrs. P. A. Smith gave a novel and pleasing musical number on a pair of old time "recorders." Due to an oversight, Professor Purcell failed to obtain an opportunity to present at the dinner a resolution of thanks; but at a later session the resolution was adopted for the three coöperating organizations, recognizing the hospitality of Dartmouth College, the good services of the local committee under the chairmanship of Professor Perkins, the kindness of the ladies of the faculty, Raphael Silverman and Lily Dymont, and Professor Robinson, as well as the generosity of Bell Telephone Laboratories for the installation and demonstration of its telephone equipment for carrying on calculations with complex numbers. This last feature required the services of experts for the week preceding the meeting, the use of two commercial wires connecting Hanover with New York City for three hours each day, and the supervision of Dr. Stibitz during those hours, while visiting mathematicians made individual use of the apparatus.

The American Mathematical Society held sessions for the reading of short papers Tuesday morning and afternoon, Wednesday morning, and Thursday morning and afternoon. Four colloquium lectures were given during the week by Professor G. T. Whyburn on "Analytic topology." On Thursday afternoon Professor Leonard Carlitz gave an invited address on "Arithmetic of polynomials in a Galois field."

The Institute of Mathematical Statistics held sessions Tuesday, Wednesday, and Thursday mornings, as well as a joint session with the Society Tuesday afternoon. The associated meetings of these organizations add a mutual strength to the attractiveness of the programs. The membership of the Institute overlaps that of the Association and the Society; and the collocation of these meetings affords opportunity both for those interests common to the organizations and for those special interests which are more suitable for separate programs.

The Mathematical Association held sessions on Monday forenoon and afternoon with a strong program arranged by a committee with Professor Gilman as chairman.

First Session of the Association

"Alexandria, a shrine of mathematics" by Professor R. E. Langer, University of Wisconsin.

"Simple examples of limiting processes in probability" by Professor B. H. Brown, Dartmouth College.

"Calculating with telephone equipment" by Dr. G. R. Stibitz, Bell Telephone Laboratories.

Second Session of the Association

"War preparedness among mathematicians" by Professor Marston Morse, Institute for Advanced Study.

"Mathematical methods in ballistics" by Dr. L. S. Dederick, Aberdeen Proving Ground.

"Some hydrodynamical problems related to ballistics" by R. H. Kent, Aberdeen Proving Ground.

"Mathematical problems useful for aviation" by Professor Richard von Mises, Harvard University.

The report of Professor Marston Morse, as chairman of the joint committee of the Association and the Society on war preparedness, was published in the September issue of the *Bulletin of the American Mathematical Society* with a shorter statement in the Monthly for August–September.

Because of the unusual character of the papers on this program, they will all appear, in full or in part, in early issues of the Monthly.

Meetings of the Board of Governors

Following meetings of the newly formed Executive Committee on September 7 and 8, the Board of Governors met under the new organization on Monday evening and Tuesday noon, with thirteen members present.

The following twenty-two persons were elected to membership on applications duly certified:

J. O. Blumberg, A.M.(Pittsburgh) Instr., Univ. of Pittsburgh, Pittsburgh, Pa.

E. L. Buell, A.B.(Syracuse) Teaching fellow, Massachusetts Inst. of Tech., Cambridge, Mass.

B. de J. Caraça, Prof., Inst. Superior de Cien., Econ., e Finan., Univ. Tec. de Lisboa, Lisbon, Portugal

R. H. Cole, Ph.D.(Wisconsin) Instr., Univ. of Western Ontario, London, Ont., Canada

Frances E. Falvey, A.M.(Southern Meth.) Head of Dept., Ward-Belmont Jr. Coll., Nashville, Tenn.

Abel Gauthier, A.M.(Columbia) Asst. Prof., Univ. of Montreal, Montreal, Canada

M. T. Goodrich, A.M.(Clark Univ.) Head of Dept., State Teachers Coll., Keene, N. H.

N. A. Hall, Ph.D.(Calif. Inst. of Tech.) Instr., Queens Coll., Flushing, N. Y.

H. J. Hamilton, Ph.D.(Brown) Instr., Pomona Coll., Claremont, Calif.

T. L. Koehler, A.M.(Pennsylvania) Asst. Prof., Muhlenberg Coll., Allentown, Pa.

Sister Teresa M. Madden, A.M.(Boston Coll.) Prof., Coll. of Our Lady of the Elms, Chicopee, Mass.

Nathan Newman, B.S.(C.C.N.Y.) Clerk, Grade 4, Railway Mail Service, Bronx, N. Y.

C. H. Nordstrom, A.M.(Lehigh) Instr., Michigan State Coll., East Lansing, Mich.

R. W. Price, A.M.(Columbia) Dean of Jr. Coll., Head of Dept., Cazenovia Seminary, Cazenovia, N. Y.

P. S. Pyuen, A.B.(Hawaii) Instr., Math. and Science, Konawaena High School, Kealakekua, Hawaii

K. C. Schraut, Ph.D.(Cincinnati) Instr., Univ. of Dayton, Dayton, Ohio

S. A. Stone, M.S.(New Hampshire) Instr., Univ. of New Hampshire, Durham, N. H.

L. W. Swanson, A.M.(Minnesota) Teaching asst., Univ. of Minnesota, Minneapolis, Minn.

G. H. Van Arkel, M.S.(Washington) Instr., Centralia Jr. Coll., Centralia, Wash.

Angeline Wilson, A.M.(Michigan) Instr., Jr. Coll., Grand Rapids, Mich.

Mary Jane Wilson, A.B.(Smith) Teacher, Williams Memorial Inst., New London, Conn.

W. E. Wilson, Ph.D.(Iowa) Prof., Mech., Colorado School of Mines, Golden, Colo.

The Secretary announced the names of four Regional Governors, elected by the membership in the various regions for the partial term through 1941: Region 5 (Alabama, Florida, Georgia, North Carolina, and South Carolina), J. M. Thomas; Region 7 (Kentucky, Ohio, and Tennessee), F. B. Wiley; Region 10 (Kansas, Missouri, and Nebraska), O. J. Peterson; Region 12 (Arizona, Colorado, New Mexico, Utah, and Wyoming), C. V. Newsom. By the plan adopted in December 1939, the other regions will elect their representatives at the end of 1940 or 1941.

At the recommendation of the Executive Committee the Board (1) elected W. B. Carver and R. E. Langer to serve with the Secretary-Treasurer as a Finance Committee, for the partial terms through 1941 and 1943, respectively, authorizing this committee for one year to make such current expenditures as seem necessary and to prepare a tentative budget; (2) voted twenty-five free reprints of the more important articles in the MONTHLY, beginning next January; (3) recommended a clarifying restatement of the duties of the Nominating Committee in the By-Laws, and appointed a Nominating Committee for the coming year; (4) voted a subsidy of \$500 per year for the next four years to *Mathematical Reviews*, it appearing that nearly one hundred members of the Association are obtaining the low rate of \$6.50 who could not otherwise obtain it; (5) proposed a general reconstruction of the subsidies of the Association; (6) voted to continue the Putnam Competition with a clarification of the conditions of the examination. In addition, they (7) appointed H. H. Mitchell and Mrs. Anna Pell Wheeler as the representatives of the Association on the Council of the American Association for the Advancement of Science for 1940; (8) appointed G. B. Price as the representative of the Association on the Board of the American Documentation Institute; (9) accepted the invitation of Cornell University to meet there in September 1942; (10) agreed to establish reciprocal relations with *Boletino Matematico* of Buenos Aires, whereby those associated with that South American journal may obtain the MONTHLY for \$4.00 instead of \$5.00, and may become members without the payment of the initiation fee, while the Association members may obtain *Boletino Matematico* for \$2.00 instead of \$2.50.

At the business session of the Association Monday afternoon two amendments to the By-Laws were adopted on recommendation of the Executive Committee and the Board of Governors. To the section on the Board of Governors was added: "and to authorize expenditures of funds of the Association." The section on the Finance Committee was amended to read: "There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of three members, of whom the Secretary-Treasurer shall be one." The last sentence of the section, on the score of simplicity, was omitted.

W. D. CAIRNS, *Secretary-Treasurer*

THE TWENTY-FOURTH ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-fourth* annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky on Saturday, April 27, 1940, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor H. H. Downing, chairman of the Section, presided.

There were fifty-six in attendance, including the following twenty-nine members of the Association: N. B. Allison, P. P. Boyd, M. C. Brown, W. B. Carver, L. W. Cohen, H. H. Downing, Charles Hatfield, Elizabeth J. Hines, Tryphena Howard, W. R. Hutcherson, E. D. Jenkins, Fritz John, F. Elizabeth LeStourgeon, Gasperine Milo, W. L. Moore, Sister Charles Mary Morrison, R. S. Park, Sallie E. Pence, D. W. Pugsley, J. K. Reckzeh, J. H. Simester, W. F. Smith, D. E. South, Guy Stevenson, J. A. Straw, Elizabeth C. Strayhorn, S. Helen Taylor, Mary E. Williams, H. A. Wright.

The Kentucky Section was very happy to have Professor W. B. Carver, president of the Association, at this meeting. At the luncheon meeting the following officers were elected for the next year: Chairman, H. A. Wright, Transylvania College; Secretary, D. E. South, University of Kentucky. It was voted to hold a joint meeting of the Kentucky Section and the Kentucky Council of Mathematics Teachers this fall, the date to be chosen later.

The following papers were presented:

1. "On infinite series" by Professor Fritz John, University of Kentucky.
2. "Inversion applied to conics" by Sister Charles Mary Morrison, Nazareth College.
3. "Some remarks on the Laplacian method of calculation of orbits" by Professor D. W. Pugsley, Berea College.
4. "Certain illustrations of non-linear correlations" by Dean S. Helen Taylor, Ashland Junior College.
5. "Some remarks on transformations that conserve areas" by Dr. Charles Loewner, University of Louisville, introduced by Professor Stevenson.
6. "Mathematical puzzles as a stimulant" by Professor W. B. Carver, Cornell University.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor John discussed proofs for the well known identities

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots=\log 2,\qquad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}\cdots=\frac{\pi}{4}$$

without the use of power series. The first relation can be derived from the in-

* Through an error in numbering the 1937 meeting in the report as printed in the MONTHLY for February 1938, the subsequent meetings have also been numbered incorrectly in their reports. These should be: 1937, 21st annual meeting; 1938, 22nd; 1939, 23rd.

tegral for log x. The second equation can be proved by counting the lattice points in a circle.

2. Sister Charles Mary developed the general theory of inverse figures and applied it to find the inverse of the straight line, the circle, the parabola, the equilateral hyperbola, the ellipse, and the hyperbola.

3. The substance of the remarks of Professor Pugsley grew out of work with Dr. Paul Herget of the University of Cincinnati, who made the extensions of formulas mentioned in the remarks. As an example of the Laplacian method the author had attempted a calculation of a preliminary orbit of 1938QA from three observations by the Laplacian method. Because the plane of the orbit was nearly in the ecliptic, the usual equations gave indeterminate or impossible results. The formulas were extended, as noted above, to include the use of a fourth observation. The author briefly indicated the bases of the Laplacian method and the means of extending the formulas, and gave a few data from the computation itself. For more details the reader is referred to two articles in the *Astronomical Journal No. 1113*, vol. 48, no. 13.

4. Dean Taylor examined data from extensive groupings in the field of education and found them to be non-linear. A study of certain correlations from education tests, sorted by the time-out-of-school element gives entirely contrary conclusions when its non-linear form is recognized and the eta coefficients used instead of the Pearson product-moment ratio. To test linearity a simple form of Blakeman's test was used. The most unusual data were from the metallurgy department of the American Rolling Mills; the data are non-linear and the regression curves hyperbolas.

5. Dr. Loewner proved the following two theorems, which permit generalizations for more than two dimensions and higher derivatives:

(1) Given a closed region D in the xy-plane bounded by a Jordan-curve with continuous derivatives up to the second order whose derivatives of the first order vanish at all points of the boundary of D, then there exists at least one point in D at which the derivatives of the second order are in a given proportion $f_{xx}:f_{xy}:f_{yy}=\alpha:\beta:\gamma$ provided that the given constants α, β, γ not being all zero, satisfy the condition $\alpha\gamma-\beta^2\geqq 0$.

(2) If the functions $x'=\phi(x, y)$, $y'=\psi(x, y)$ with continuous derivatives of the first order represent a transformation of D that conserves areas and leaves all points on the boundary of D fixed, then there exists at least one point in D at which

$$\psi_x:\frac{(\psi_y-\phi_x)}{2}:-\phi_y=\alpha:\beta:\gamma$$

provided that the given constants α, β, γ satisfy the conditions of theorem (1).

6. Professor Carver recommended the use of mathematical puzzle material to arouse the interest of indifferent students. Many students who take no interest in the more useful applications of mathematics will spend much time and effort on a clever puzzle, and be led into an understanding of the same mathe-

matical processes which are needed for handling more serious problems. A number of examples of such puzzles were presented involving mathematics of all grades, and references were given to further material of this kind.

D. E. SOUTH, *Secretary*

THE TWENTY-FIRST ANNUAL MEETING OF THE ILLINOIS SECTION

The twenty-first annual meeting of the Illinois Section of the Mathematical Association of America was held at Illinois Wesleyan University, Bloomington, on Friday and Saturday, May 10 and 11, 1940. Professor G. D. Gore, chairman of the Section, presided at all sessions.

The attendance at the two sessions was approximately seventy-five, including the following thirty-eight members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. G. Ayre, Ruth Mason Ballard, S. F. Bibb, D. R. Curtiss, H. T. Davis, J. E. Davis, W. M. Davis, Edna M. Feltges, Elinor B. Flagg, L. R. Ford, A. E. Gault, G. D. Gore, M. C. Hartley, M. R. Hestenes, Mildred Hunt, R. N. Johanson, P. W. Ketchum, E. C. Kiefer, J. M. Kinney, W. C. Krathwohl, Luise Lange, J. R. Mayor, E. B. Miller, C. N. Mills, E. J. Moulton, A. L. O'Toole, J. F. Paydon, Echo D. Pepper, J. W. Peters, E. W. Ploenges, Ruth B. Rasmusen, M. A. Sadowsky, E. W. Schreiber, H. A. Simmons, M. E. Wescott, Alice K. Wright.

At the annual business meeting the following officers of the Section were elected: Chairman, Mildred Hunt, Illinois Wesleyan University; Vice-Chairman, R. N. Johanson, Bradley Polytechnic Institute; Secretary-Treasurer, C. N. Mills, Illinois State Normal University. The next meeting of the Section will be held at Bradley Polytechnic Institute, Peoria, on May 9 and 10, 1941.

The Section adopted a petition to the Executive Committee of the Mathematical Association suggesting that "a portion of the annual dues of members of Sections be returned to the respective Sections and all initiation fees of new members secured by Sections be returned to their respective Sections."

The following eighteen papers were presented:

1. "Recreations connected with square arrays" by Dr. Ruth Mason Ballard, Wright Junior College.

2. "What is a million?—The story of a project" by Professor L. R. Ford, Armour Institute of Technology.

3. "Putting mathematical induction to work" by Professor J. E. Davis, Central Y.M.C.A. College, Chicago.

4. "Approximation by means of analogs to Laguerre polynomials" by Dr. M. E. Wescott, Northwestern University.

5. "Derivative lines" by Professor S. F. Bibb, Armour Institute of Technology.

6. "Concerning principles of quantitative comparison" by Dr. Luise Lange, Woodrow Wilson Junior College.

7. "Series with deleted terms" by Dr. I. E. Perlin, Armour Institute of Technology.

8. "Two metric quadrics" by Dr. Ruth B. Rasmusen, Woodrow Wilson Junior College.

9. "The theorem of the mean in differential calculus" by Professor Rufus Oldenburger, Armour Institute of Technology.

10. "Mathematics and the teeth" by Dr. G. E. Hay, Armour Institute of Technology, introduced by Professor Ford.

11. "The mathematical analysis of social patterns" by Professor H. T. Davis, Northwestern University.

12. "Mathematical Reviews" by Professor E. J. Moulton, Northwestern University.

13. "Mathematics of biology," illustrated, by Professor Nicolas Rashevsky, University of Chicago, introduced by Professor Gore.

14. "The logarithmic mean" by Professor W. M. Davis, Armour Institute of Technology.

15. "Equiareal orthogonal systems of curvilinear coördinates" by Dr. M. A. Sadowsky, Armour Institute of Technology.

16. "A continued fraction related to partition formulas" by Professor H. S. Wall, Northwestern University, introduced by Professor Moulton.

17. "Summation of series" by Professor A. L. O'Toole, Mundelein College.

18. "Remarks on an article by Professor L. R. Ford" by Professor H. A. Simmons, Northwestern University.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The recreations connected with square arrays discussed by Dr. Ballard are arrangements of n^2 counters composed of n equal sets, each set being distinguishable by some property such as color. The squares are to have no two counters of the same color in the same row or same column. When $n=4$ there are 4 fundamentally different and therefore in all 576 possible arrays. There are 2 basic arrangements with diagonals with no duplication of color, and therefore in all 48 possible such arrays. By fixing properly 4 of the counters any 3 of the basic arrangements can be uniquely determined. The problem for larger values of n, and for arrays of counters with 2 distinguishable features such as color and shape was demonstrated briefly.

2. Professor Ford described an exhibit prepared by a sophomore class for "open house" at Armour Institute of Technology. There were maps with circles containing a million square miles or a million acres, or showing distances from a million feet to a million rods. A time series showed what was going on in the world from a million centuries ago to a million seconds ago. The feature exhibit was a huge poster with exactly one million dots upon it, constructed in a short time with a typewriter and a mimeograph machine.

3. Professor Davis related the experiences of classes in college algebra and theory of equations in establishing properties of determinants. A determinant of

the second order is defined as $ad-bc$. The nth order determinant is for convenience described as a square array. Minor and cofactor of the element a_{ij} are defined as usual. Then the determinant is defined as the summation of the products of the elements of a row by their cofactors, which suspended definition is made complete by the definition of second order determinant. By use of this definition and mathematical induction all properties of determinants ordinarily encountered by undergraduates are established, including properties of certain bordered determinants (Laplace expansion a special case thereof), derivative of a determinant, *etc.*

4. The Laguerre polynomials are orthogonal with the weight function e^{-x} on the continuous range zero to infinity. Using a method essentially due to J. P. Gram, a set of Newton polynomials $\phi_n(x)$ can be obtained which are orthogonal with the weight function 2^{-x} over the range of all positive integers and zero. These polynomials are completely analogous in form and properties to the Laguerre polynomials, and reduce to them when the variable is made continuous under suitable limiting processes. The polynomials $\phi_n(x)$ can be used for approximating empirical data under a least square criterion following the method developed by Gram. For the population series of the United States from 1790 to 1930 Dr. Wescott stated these functions give as close agreement with the data as the Reed-Pearl logistic function over the first half range.

5. The normal form of the equation of a straight line in rectangular coördinates was immediately changed to a Clairaut's equation, the singular solution of which is represented parametrically by $x=f[p, n(p)]$, and $y=F[p, n(p)]$, where n is the normal from the origin and p the slope of a particular integral of the Clairaut equation. Not only are x and y expressible as line segments in terms of derivatives but it is shown that there are an unlimited number of other line segments depending upon $P(x, y)$ so expressible. In particular, Legendre's arc length formula was easily derived by Professor Bibb, and the radii of curvature of an infinite number of successive evolutes of the curve defined by $P(x, y)$ given. Finally, a method of solving particular differential equations was indicated.

7. This paper will appear in an early issue of the MONTHLY.

8. Suppose with each curve of section passing through a fixed, non-asymptotic tangent t at a point P of an analytic surface S in ordinary metric space is associated an osculating hyperbola whose projection on the tangent plane of S at P is a rectangular hyperbola. For a real, non-asymptotic direction on a real surface the locus of these osculating hyperbolas as the plane of section passing through t varies is either a hyperboloid of one sheet or a hyperbolic paraboloid. In the directions of the superosculating lines on the surface S the quadric is always a paraboloid. The locus of the osculating rectangular hyperbolas of the curves of section passing through a real, fixed, non-asymptotic tangent t at a point P of a real, analytic surface S is a hyperboloid of one sheet, or a hyperbolic paraboloid. At each point of an analytic minimal surface S for which the above two quadrics are defined, Dr. Rasmusen stated they have contact of the second order with the surface S.

9. Professor Oldenburger showed that the auxiliary function $F(x)$ so often used in proofs of the theorem of the mean of differential calculus for a function $f(x)$ may be defined to be the function satisfying the properties: (1) its derivative is the difference between the slope of the curve $y=f(x)$ and the slope of the secant line through two given points on this curve; (2) it vanishes at one of these points. This method of introducing $f(x)$ has certain obvious advantages over the usual method.

10. Dr. Hay obtained solutions for the equations of elasticity relating to the periodontal membrane, which is the membrane between the tooth and the jaw, in the case of a simple model of a tooth. These solutions permitted a determination to be made of the distribution of stress throughout the membrane surrounding a loaded tooth, also of the displacement of the tooth, of the location of its center of rotation, *etc.* The accuracy of these results and the extent to which they depended on experiment were discussed.

11. Professor Davis discussed four patterns observable in the mathematical and statistical analysis of economic data. The first was the behavior of the four variables in the equation of exchange, namely, money, its velocity, price, and trade. This equation appears to furnish a mathematical basis for appraising historical events, wherever data exist for evaluating the variables. The second pattern was supplied by the frequency function for incomes, a function widely different from that used to interpret ordinary skew-normal distributions. The third was the war cycle, which, in spite of certain irregularities, appears to throw much light upon the phenomena of war. The fourth pattern, based on Bernoulli's definition of the marginal utility of money, showed how price fluctuations can be accounted for by the maximization of an integral of utility (in the economic sense) in which the integrand is diminished by two factors measuring economic shocks and economic surpluses or scarcities.

13. Professor Rashevsky reveiwed the work of the University of Chicago group of mathematical biophysicists. The studies follow two distinct lines: the mathematical biophysics of the cell and that of the central nervous system. The first is based on the study of the effects of forces due to diffusion. For a systematic development of the theory a number of relations are derived such as relations between cell sizes and oxygen consumption, rates of elongation and constriction of dividing cells, rates of respiration at various oxygen tensions, *etc.* Comparison of the mathematical deductions with experimental data gives a good agreement. The second line of study starts with a few fundamental postulates in mathematical form, based on neurological data. From those a mathematical theory of reaction times, psychophysical discrimination, intensity discrimination, visual perception, and of other neuropsychological phenomena is developed and found again in agreement with experimental data.

14. Professor Davis pointed out that the logarithmic mean, $(b-a)/\ln(b/a)$, of two unequal positive real numbers, a and b, used in the solutions of certain heat transfer problems, is a special case of a more general mean $[\int_a^b x^k dx/(b-a)]^{1/k}$. Other special cases are the arithmetic mean and geometric mean. Application

of a well known theorem on mean values with an arbitrary function yields inequalities between the geometric, logarithmic, and arithmetic means.

15. Dr. Sadowsky showed that if selected parametric curves $u=u(nh)$, $v=v(mk)$, $(n, m=0, 1, 2, 3, \cdots)$, of an orthogonal system u, v give rise to a pattern of curvilinear rectangles in which all meshes are equal in area for any arbitrary choice of the positive constants h and k, the system u, v will be called an equiareal orthogonal system. *Examples*: Cartesian coördinates x, y with $x=nh$, $y=mk$; polar coördinates r, θ with $r=\sqrt{nh}$, $\theta=mk$; "logospiral" coördinates. The analytical criterion for preservation of areas is $\log EG=f_1(u)+f_2(v)$ with $F=0$ (f_1 and f_2 are arbitrary functions). A particular solution is given by trajectories intersecting an arbitrary one-parameter family of straight lines in the plane at $\pm 45°$. There is a one-to-one correspondence between equiareal orthogonal coördinates and plane plastic flow.

16. Professor Wall showed how a number of the identities of Euler involving the infinite product $\prod_{n=1}^{\infty}(1-x^n)$ can be derived from the continued fraction $1+nx/1+(1-n)wx/1+(1-w)wnx/1+(1-wr)w^2x/1+(1-w^2)w^2rx/1+\cdots$. The proof depends upon the fact that when $x=-1$ this continued fraction is equal to the infinite product $\prod_{n=0}^{\infty}(1-nw^n)$, and upon a recently found continued fraction identity.

17. Assuming that the purpose of higher education is to give students insight, and that generality is the soul of mathematics, Dr. O'Toole suggested that there may be psychological and mathematical disadvantages connected with the customary treatment of some elementary mathematical topics. For example, progressions are summed by special devices each of which applies to only one type of series; a general method, that any student of college algebra can understand, applicable to innumerable types of series, is not even mentioned. Such treatment may be devoid of culture and utility. Proof and applications of the fundamental theorem of summation were given.

18. In an article entitled *Cows and cosines* (this MONTHLY, Nov. 1939), Professor Ford obtained two formulas for determining an integer of interest. In one of these formulas there appeared a cosine that was useful in expressing the real part of a special linear combination of the conjugate imaginary roots of a real cubic. Professor Simmons indicated large classes of problems that can be solved by Professor Ford's procedure. It is apparent that such procedure can be used to solve: (a) infinitely many related problems in which no *cosine* is needed, when an equation analogous to the Ford cubic has only real roots; and (b) infinitely many related problems in which one or more *cosines* in the Ford rôle are needed. We find that in case (*b*) the *damping* referred to by Professor Ford is not generally to be expected, but that it occurs when the absolute values of the associated imaginary roots are all sufficiently small. We note also that if our analog of the Ford cubic has two equal roots, the Ford procedure no longer applies.

C. N. MILLS, *Secretary*

FUNCTIONAL DEPENDENCE IN THE CALCULUS OF PROPOSITIONS

WILLIAM WERNICK, New York University

1. In the two-valued calculus of propositions, a variable x_i may assume only the values 0 or 1; and a function F, of n variables $x_1, \cdots, x_n$, has, when constant values are assigned to the variables, either the value 0 or 1. A function of n variables is completely defined if we assign, to every constant set of arguments, a specific value for the function. Let $a_{1,1,\ldots,1}$ be the value assigned for $F(1, 1, \cdots, 1)$, and in general, a with the subscript consisting of the same sequence of 0's, and 1's, as in the constant set of arguments, for the value of F of that constant set. Then the function F is completely defined by the 2^n constant a's, which may be independently 0 or 1, depending on the particular definition of the function F.

This paper will define and discuss necessary and sufficient* conditions that a function F, however defined, be independent of, or dependent on any variable x_i, or any number of them. In section 5 a very simple sufficient condition will be given for a function of n variables to be dependent on all of them. The procedure will be to discuss fully the case for $n=2$, and then extend to general n.

2. A function of two variables (in the two-valued calculus) is completely defined by the following table:

x_1	x_2	$F(x_1, x_2)$
1	1	a_{11}
1	0	a_{10}
0	1	a_{01}
0	0	a_{00}

If $F(x_1, x_2)$ is to be independent of x_1, then it is NS that a change in x_1 produce no change in F; *i.e.*, it is NS that for all constant values of x_2, $F(1, x_2) = F(0, x_2)$ identically. That is, it is NS that

$$F(1, 1) = F(0, 1), \quad \textit{and} \quad F(1, 0) = F(0, 0),$$

or

$$a_{11} = a_{01}, \quad \textit{and} \quad a_{10} = a_{00},$$

or

(1) $$a_{11} - a_{01} = 0, \quad \textit{and} \quad a_{10} - a_{00} = 0,$$

i.e., that

(2) $$| a_{11} - a_{01} | + | a_{10} - a_{00} | = 0.\dagger$$

We call the left side of this equation A_1, and we have the NS condition that F be independent of x_1 is that

* "Necessary and sufficient" will be abbreviated to "NS."

† Without absolute values in (2) the condition becomes only necessary, but not sufficient.

$$A_1 = 0. \tag{3}$$

Note that A_1 is the sum of absolute values of the differences of all pairs of a's whose subscripts differ only in the first place.

In exactly the same way, we get the NS condition that F be independent of x_2 is that, for all values of x_1, $F(x_1, 1) = F(x_1, 0)$, *etc*., as above. This NS condition becomes

$$|a_{11} - a_{10}| + |a_{01} - a_{00}| = 0. \tag{4}$$

Calling the left member of (4) A_2, we have, as before, the NS condition that F be independent of x_2 is that

$$A_2 = 0. \tag{5}$$

Note that A_2 is the sum of the absolute values of the differences of all pairs of a's whose subscripts differ only in the second place.

Combining these conditions, we see the NS condition that F be actually dependent on both variables x_1, x_2 is that $A_1 \neq 0$, and $A_2 \neq 0$, *i.e.*, that $A_1 \cdot A_2 \neq 0$. Calling the left member of this equation, which relates of course only to a particular function F, the functional $A(F)$, the above statements become simply the following:

The NS condition that $F(x_1, x_2)$ depend on both variables is that

$$A(F) \neq 0; \tag{6}$$

and if $A(F) = 0$, then the function F is independent of one or both the variables in its argument.

3. Extending to n variables, we define A_i as the sum of the absolute values of differences of all pairs of a's, whose subscripts differ only in the ith place. Thus

$$A_i = \sum_{s_1, \cdots, s_n} \left| (a_{s_1, \cdots, s_n})_1 - (a_{s_1, \cdots, s_n})_0 \right|,$$

where each s has the range 0, 1 except that in $(a_{s_1, \cdots, s_n})_1$ we have $s_i = 1$ only, and in $(a_{s_1, \cdots, s_n})_0$ we have $s_i = 0$ only.

The NS condition that F be independent of x_i is that $A_i = 0$; and likewise, the NS condition that F be actually dependent on all its n variables is that every $A_i \neq 0$, *i.e.*, that $\prod_1^n A_i \neq 0$. Finally, calling the product on the left side of this inequality $A(F)$, which is defined only for a particular function F, we have the NS condition that $F(x_1, x_2, \cdots, x_n)$ depend on all its n variables is that

$$A(F) \neq 0. \tag{7}$$

4. There is another method which is simpler, *i.e.*, easier to apply, which we shall call method B in distinction to method A described above. We shall, as before, illustrate with two variables and extend to n.

In method A, we found the NS condition that F be independent of x_1 is that

(1) $$a_{11} - a_{01} = 0, \quad \textit{and} \quad a_{10} - a_{00} = 0.$$

Now, instead of equation (2), we write

(2′) $$(a_{11} - a_{01})^2 + (a_{10} - a_{00})^2 = 0.$$

Both (2) and (2′) give us the sum of two non-negative terms equal to 0, which can only be the case when each term is itself equal to 0, as required in (1). But here, equation (2′) becomes

$$\overset{2}{a_{11}} - 2a_{11}\cdot a_{01} + \overset{2}{a_{01}} + \overset{2}{a_{10}} - 2a_{10}\cdot a_{00} + \overset{2}{a_{00}} = 0,$$

or

(2″) $$\overset{2}{a_{11}} + \overset{2}{a_{01}} + \overset{2}{a_{10}} + \overset{2}{a_{00}} = 2(a_{11}\cdot a_{01} + a_{10}\cdot a_{00}).$$

But the a's are all either 0 or 1, so the square of any a is equal to that a. In that case, the left side of equation (2″) becomes simply $a_{11}+a_{01}+a_{10}+a_{00}$, *i.e.*, the sum of the four assigned constant values defining the function F. This sum is immediately found from the definition of F, and is a constant, say S, for any particular F. Equation (2″) becomes now

(2‴) $$S = 2(a_{11}\cdot a_{01} + a_{10}\cdot a_{00}).$$

Finally, calling the expression in parentheses B_1, we get the NS condition that F be independent of x_1 is that

(3′) $$S = 2B_1.$$

Note that B_1 is the sum of the products of all pairs of a's whose subscripts differ only in the first place. Similarly, the NS condition that F be independent of x_2 is that

(4′) $$S = 2B_2,$$

where $B_2=(a_{11}\cdot a_{10}+a_{01}\cdot a_{00})$ is defined as the sum of the products of a's whose subscripts differ only in the second place.

From these considerations, it is easily seen that a sufficient condition that neither (3′) nor (4′) hold is that S be odd, *i.e.*, a sufficient condition that $F(x_1, x_2)$ depend on both variables is that S is odd. Also, a necessary condition that F depend on fewer than two variables, *i.e.*, that $F(x_1, x_2)$ be independent of one or both its variables, is that S is even.

5. This result is quickly extended to n variables. We define B_i as the sum of the products of all pairs of a's whose subscripts differ only in the ith place; that is,

$$B_i = \sum (a_{s_1, \cdots, s_n})_0 \cdot (a_{s_1, \cdots, s_n})_1,$$

where the summation, as in the definition for A_i, is over the 2^{2^n-1} pairs of a's we get when the subscripts, with the exception of s_i, take independently, the value 0 or 1.

We have, analogously to (2′), (2″), (2‴), and (3′), the NS condition that $F(x_1, x_2, \cdots, x_n)$ be independent of x_i is that $S=2B_i$. Of course, if S is odd, this condition fails for every B_i, hence, the very simple sufficient condition that F be dependent on all its variables is that S is odd; and the corresponding necessary condition that F be independent of any or all its variables is that S is even.

MATHEMATICS AND THE SCIENCES*

TOMLINSON FORT, Lehigh University

"King David and King Solomon
 Led merry, merry lives;
They trifled with the wine cups
 And had very many wives;
But when old age came upon them
 They were filled with many qualms;
King Solomon wrote the Proverbs,
 King David wrote the Psalms."

It is said that it is a sure sign of age when a man stops working in his subject and starts talking about it. Well, the only thing that I can do to show that I am not entirely senile is to point to the program. I am to give another lecture tomorrow and that one is to be mathematics.

Speaking seriously, I do think that it is a good thing for us, so to speak, to stand off and look at our work from time to time and try to evaluate what we are doing. This talk does not propose to cover in any completeness so large an undertaking. I do propose, however, to examine critically some fundamental things. I most emphatically do not intend to make platitudinous and fulsome remarks about the beauty and usefulness of mathematics.

What is mathematics? What are the sciences? An easy way to proceed is to assume that each of us knows, for example, what is meant by mathematics. Mathematics is that subject which by common consent is taught in departments of mathematics in our schools and colleges. It includes geometry, algebra, calculus, *etc.* I believe that there are many people, even among mathematicians, who regard more detailed definition as hopeless, or, at least, useless. However, we mathematicians are given to striving for exact statement and it is only proper that we ask for a definition of mathematics and the sciences, or at least to decide in a fairly exact and restricted way as to what significance is to be attached to these terms.

If we begin to examine what actually is taught by those who profess and call themselves mathematicians or by those who, for example, call themselves physicists or astronomers, we are impressed by much overlapping and a confused and ill-marked boundary. At a recent meeting of the Society for the Promotion of

* Address delivered at the dinner of the Southeastern Section of the Mathematical Association of America at Athens, Georgia, March 29, 1940.

Engineering Education, there was a joint meeting of the section on mathematics and the section on mechanics. I rose to the occasion, saying that as mechanics really was mathematics I thought the joint meeting splendid and that such meetings should be held frequently. On being pushed, I gave a few reasons, satisfactory to me, as to why mechanics was mathematics. Whereupon one of the members present, who is an extremely good mathematician and also a distinguished electrical engineer, arose and said that he agreed with everything that I had said but that the argument which I advanced classified as mathematics the whole science of electricity quite as well as it did mechanics.

Then there is geometry. What subject is by common consent more usually classified as mathematics than geometry? Yet why should the study of the space in which we live not be classified as physics? Why is the mensuration of solids less physics than the study of light? Both can be studied experimentally. Both can be studied by means of equations. Both have a practical aim.

Many non-mathematicians attempt to separate their subjects from mathematics on the basis of aim. Mechanics is taught both by physicists and mathematicians. Physicists always make a great point of aim, and on a basis of aim contend loudly that mechanics is physics. I again become lost in confusion. Fourier wished to study the flow of heat (physics), but he developed the theory of trigonometric series.

But let us proceed and, at the same time, let us realize that we are not remaking the language and that a definition to be acceptable must not depart too far from common usage. Yet we strive to be precise. As a matter of fact, instead of laying down a definition of mathematics, I will quote some definitions that have been made and examine them briefly but critically.

I have heard it said that Charles Darwin gave the following. (He probably never did.) "A mathematician is a blind man in a dark room looking for a black hat which isn't there."

Not long ago I was chatting with a professor of philosophy when I was a bit startled by his telling me, in order to put me in my proper place in an argument, that mathematics was the science of measurement and that alone. I think he was speaking hurriedly. I also recently encountered this: "Mathematics is the science of quantity or magnitude." Heaven knows where this type of definition originated, but, like Shakespeare's Cleopatra, "Age cannot wither her nor custom stale her infinite variety."

Turning to what is to us more serious, we quote Benjamin Peirce, the great Peirce, who wrote in 1870,* "Mathematics is the science which draws necessary conclusions."

We can call this a great definition because for the last seventy years it has been the starting point of many discussions of the nature of mathematics. It is to be noticed that its emphasis is on method rather than on subject-matter or aim. The word "necessary" is not sufficiently precise, that is, we are not told

* Linear Associative Algebra; reprinted in the American Journal of Mathematics, vol. 4, p. 97.

what methods of drawing conclusions are allowable, and finally many of us will probably regard the whole definition as too broad.

Whitehead says,* "Mathematics in its widest significance is the development of all types of formal deductive reasoning."

This is an attempt to be more precise than Peirce. The words "all types" are satisfying. Deductive was probably intended by Peirce, but its specific mention is again satisfying. Yet there is with some a certain wishful thinking, a desire to include as part of mathematics the intuitional, inductive reasoning which always precedes formal proof in mathematical research. There is also again the possible criticism that the definition is too broad and will include more than is commonly understood as mathematics.

I next quote Morris R. Cohen, professor emeritus of philosophy at the College of the City of New York, an abundant and able writer on the foundations of science and mathematics. Cohen says,† "Mathematics as a pure formal science is indeed identical with logic." He further states, "It is of course to be expected that a new and radical thesis such as the identity of logic and pure mathematics will meet with really serious difficulties. But those that have so far come to light are of the kind that we may reasonably regard as problems that workers in this field must face, rather than as a justification for the abandonment of the whole enterprise." One trouble with Cohen's identification of mathematics with logic is that the necessity of defining mathematics is simply shifted to a necessity of defining logic. "We add nothing to Nature by calling it God."

Bridgman makes the following statement:‡ "Mathematics . . . is usually recognized to be properly a branch of logic."

Thus, according to Bridgman, mathematics is a subdivision of logic not coextensive with logic as according to Cohen. Also, so far as I have observed Bridgman's writings, he is even worse than Cohen when it comes to saying what logic is.

The definitions given have all put emphasis on method. A related but somewhat different type of definition is the now classical definition of Bertrand Russell:§

"Pure mathematics is the class of all propositions of the form—p implies q—where p and q are propositions containing one or more variables, the same in the two propositions and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: implication, the relation of a term to a class of which it is a member, the notion of 'such that,' the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth."

* Universal Algebra, p. vi.

† Reason and Nature, pp. 173 and 183.

‡ The Nature of Physical Theory, p. 47.

§ Principles of Mathematics, p. 3.

Here mathematics is regarded as a body of propositions (theorems). Apparently Russell regarded mathematics not as a process but as the results of that process. This point of view seems somewhat different from that of Peirce and Whitehead. As a definition the word "implies" may be construed inductively, although this would be contrary to Russell's mathematical procedure. Also, at a subsequent time Russell remarks,* "We shall assume that all mathematics is deductive."

However, it is the statement that mathematics uses the notion of truth with which I sharply disagree. Peirce and Whitehead say, mathematics only "implies." It is hard to be sure how much this notion of truth pervades Russell's thinking on the foundations of mathematics. The definition was made early in his career. Later he seems at times to have become more purely formalistic. One is reminded of his semi-jocular definition: "Mathematics is the subject in which one never knows what he is talking about nor if what he says is true." The word "true" is there again. As a matter of fact, the idea of truth pervades the notion of "Propositional Function" discussed so much by Russell and Keyser.

I started this discussion by some remarks on the possibility of defining mathematics by its subject-matter. I now return to this and give what is possibly the most striking of all definitions. This is from Study. It is to me one of the most interesting definitions of mathematics. Study says,† "Mathematics includes computation with natural numbers and everything that can be founded upon it, but nothing else."

This is a subject-matter definition as against the definitions of method which we have just been considering. At first glance, it seems much too narrow. However, the more I ponder it the less sure I am of this. It is particularly to be remarked that Study was the inventor of a kind of number system (the Study numbers) differing in essential respects from our ordinary integers.

I now come to what we can strictly call the symbolism school. First, Hilbert,‡ the great leader of this school, holds that mathematics is essentially the manipulation of symbols.

The most detailed of the symbolism definitions is the following from Professor C. I. Lewis of Harvard, the writer on symbolic logic:§

"A mathematical system is any set of strings of recognisable marks in which some of the strings are taken initially and the remainder derived from these by operations performed according to rules which are independent of any meaning assigned to the marks."

I will also given Poincaré's oft-quoted remark:‖ "Mathematicians do not study objects but the relations between objects. Matter does not engage their attention. They are interested in form alone."

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* Introduction to Mathematical Philosophy, p. 145.

† Mathematik und Physik, Braunschweig, 1923.

‡ Die Grundlage der Mathematik.

§ Survey of Symbolic Logic, pp. 355–356.

‖ Science and Hypothesis.

Here I again quote Russell's jocular definition: "Mathematics is the subject in which one never knows what he is talking about nor if what he says is true."

One cannot at this juncture fail to mention the intuitionists like Brouwer who a dozen years ago created such a furor in mathematical circles. Their general aim is to limit mathematics to a discussion of things that are, so to speak, intuitionally constructed. They exclude from proper consideration sets which are not constructed according to their intuition. From this they are sometimes called finitists. As has been said, to them "Mathematics is the language of the possible."

Leaving quotations and comments, I am inclined to the opinion that the majority of us at the present time hold to something resembling the symbolism view of mathematics. However, not all do. I recently heard a distinguished officer of the American Mathematical Society in a public lecture, after speaking briefly of Hilbert's ideas, remark with much disgust, "Well, if that is your idea of mathematics,—Oh well!"

According to the (as I believe) commonly held view, with which I agree, a mathematical structure consists of certain undefined terms (symbols), certain postulates upon them (that is, relations between them), an agreed logic (that is, method of drawing conclusions), and strings of theorems derived by means of the logic. This structure is built rather than discovered. Mathematics, according to this point of view, will consist of the process of laying the postulates and building the structures, or of the sum total of such structures completed and available, a dualistic meaning, just as many symbols in mathematics in one breath are symbols of operation and in the next breath represent the result of that operation. The expression 2/3 is either 2 divided by 3 or the fraction 2/3 which is a number. The great problem is the problem of consistency of the postulates. To establish consistency we exhibit an example of objects which satisfy the postulates, that is, give an existence theorem. Most of the time these examples lead back to the positive integers. Possibly Study is right.

To lend additional weight to this point of view, let us examine in a hurried way what is actually being done by mathematicians at present. Examine the programs of the Society or the pages of our mathematical journals.

First, Analysis. What is being done in analysis? Many things, of course, but it seems to me that very significant advances are in postulated spaces, Banach spaces, Hilbert spaces, function spaces, *etc.* A space is simply a set of elements subject to certain postulates. A little reflection will convince most mathematicians that classical analysis differs only in scope rather than in nature.

Second, Algebra. What is being done in algebra? Groups, rings, fields, lattices! Again, it is the same thing. The distinction from analysis becomes illusory. It is true that emphasis on limit, function, and continuity is lacking.

Geometry also starts with a set of undefined terms and proceeds as before. Geometries are n-dimensional, riemannian, euclidean, topological, *etc.*, *etc.* The classification of a mathematical system as geometry seems, shall I say, historical; that is, it has had a recognized origin in elementary geometry, and the term sticks with but little further reason.

The three great traditional branches of mathematics, algebra, geometry, and analysis, appear to be merging or at least drawing closer together, their boundaries becoming ever more illusory. The road that all are travelling is the road of postulates.

However, what of the Sciences? I see nothing added to mathematics by calling it a science. I purposely refrain from doing so, preferring a separate classification. Before going into this subject I must briefly discuss the subject of isomorphism. I will use the term in the following sense. Two mathematical structures are simply isomorphic or isomorphic if they are mathematically identical, the difference between them being one of language only. An example is the common dualism of projective geometry in which, for example, the system of lines and points in a plane is isomorphic to the system of points and lines in a way that is familiar to everyone. The mathematical structures are identical. Hence the term isomorphic as between mathematical systems is really meaningless. The systems are identical. The same can be said of the "Propositional Function" of Keyser and Russell. To be explicit, for the reason that mathematics is treating undefined terms and also for the reason that a single postulate outside a mathematical system is meaningless, the "Propositional Function" of Keyser and Russell seems to me to have no significance as a part of mathematics.

If now a system of objects of physical or other significance, not the bare bones of mathematics, can be put into one-to-one correspondence with the undefined terms of a mathematical system and if postulates (laws) can be made about them corresponding postulate to postulate (law to law) with the mathematical system, we say that the two systems are isomorphic, and applications of mathematics step into the picture. Theorems say, "If you do this, you will get that." At this point the "Propositional Function" as a method of describing isomorphism between physical phenomena, and mathematical systems achieves whatever justification there is for it as a term.

Have the sciences really been advanced by mathematics, or is mathematics one vast tautology capable of adding nothing, as has been claimed? It seems to me that this is merely a matter of the record. Nòne of us here will challenge it. The scoffer can always be cited such great achievements as the discovery of Neptune.

Physics consists essentially of measurements, what Eddington describes as pointer-readings. It was the custom of the older physical theorists to construct in imagination a simple mechanical model which would have much the same pointer-readings as the phenomena being studied. This was called understanding the phenomena. Certain laws governing this model were then observed, and the model was in an unconscious way made isomorphic to a mathematical system and conclusions drawn. Confidence in these conclusions was possibly greater than warranted but has been, usually, borne out by our experience. Supporting this view, I make one final quotation, from Thompson and Tait's *Natural Philosophy* written about 50 years ago:

"It has been long understood that approximate solutions of problems in the ordinary branches of Natural Philosophy may be obtained by a species of abstractions or rather limitations of the data such as enables us easily to solve the modified form of the question while we are well assured that the circumstances (so modified) affect the result only in a superficial manner."

The modern tendency is to dispense with the intermediate model. Once we had an aether, a kind of all-pervading medium which had properties which were studied. Now radiation phenomena are studied directly. A mathematical system is set up, hopefully isomorphic to the pointer-readings themselves. The model is gone. The most outstanding case where the model remains is in the solar system. Painlévé remarked that we state that the earth goes around the sun to satisfy women and children. Mark you, as a matter of fact we observe only motions on the dome of the sky. Is there only one model that will be isomorphic, or are there many, or why have a mechanical model at all? Possibly an abstract mathematical system, once we have postulates to correspond with observations, is best. You can call it a mathematical model if you like. Then there no longer would be any reason to say that the earth turns on its axis or revolves around the sun. If the aether is gone, why not the solar system? It may happen, although it is hard to believe a more convenient description for "women and children" will be found, or to believe that we shall be so impregnated with mathematics ourselves as to lose the solar system from our daily lives.

I have talked glibly of isomorphism and I now ask the question: Is it really possible to set up mathematical systems isomorphic to physical systems? I think that this question loses any meaning if the mathematical system contains the infinite in any way, and this includes the irrational and the continuous. These are mathematical fictions, constructions, and this alone. Human consciousness is not continuous and our experience of time is not continuous. To say that a simple law, $pv=\text{const.}$, is obeyed in physics is a statement without meaning unless we mean by obey, the simple fact of usefulness. Is space of experience euclidean or non-euclidean? I can see no meaning to this query other than as applied to the usefulness of the particular geometric system. Hence to me reality may change.

Mathematical systems involving the finite only, the non-continuous, the discrete may be isomorphic to certain physical systems. Beyond this we cannot say that there is isomorphism. The thoughtless person thinks that the equations which measured phenomena approximately satisfy are really obeyed, that the trouble is lack of precision in measurement. However, to say that physical phenomena really obey equations containing continuous variables is only making a noise. But to say that these equations are, so to speak, interpolation formulas which are useful on account of ease of handling, seems to me to approximate the truth.

It is sometimes said that the purpose of science is to describe. In reality I believe that its purpose is two-fold: to describe, and to forecast. We do not study

our daily life as a space-time continuum where "describe" is all that we want. Does mathematics help science in these two things? I think it has done so, and that the way has been the road of isomorphic systems, or rather the pseudo-isomorphic systems of which I have spoken.

The question is frequently asked as to why mathematical systems built without reference to applications are later applied in science. The answer seems to me to be somewhat as follows. Our minds are the product of experience. We dream human dreams. Our language is human language based on human experience. Any postulate that we can formulate is taken from our experience somewhere and framed in our language. Then, let a combination of postulates, ever so bizarre, be made, so long as they are non-contradictory; are we to be surprised if this set of postulates, each of which has been formed by experience, should as a whole be isomorphic with some portion of experience and useful in studying it? Why should we marvel if non-euclidean geometry is useful in studying space? I phrase this differently by saying: Why should we marvel if space is non-euclidean? Or why marvel if complex numbers are useful in studying alternating currents or what not? I shall never be amazed when a well built mathematical system is applied to any branch of science. It seems absurd to say, "Nature is mathematical." Rather man's mathematics is based on man's nature. Here I am an intuitionist, and I think that it is on the postulates that the intuitionists and formalists in mathematics, just as the mathematicians and the physicists, come together.

I have spoken primarily of physics. It is to be borne in mind that Newton, Euler, Bernoulli, Lagrange, Laplace, Fourier, *etc.*, the great giants who set the direction in which mathematics developed, were interested in physical science. If they had been interested in the biological and social sciences we probably should have had a very different body of mathematical doctrine from that which we now have. However, mathematics and its relation to the social and biological sciences seems to me to introduce no new fundamental idea. The more usual mathematics is being applied in both the social and biological sciences. Statistics is an example of a branch of mathematics which in its origin was aimed particularly at biology and the social sciences. Raymond Pearl says* that in his opinion biology is certain to be treated in the future by mathematical methods as is physics today. The economists are skeptical, but the rapid advance of the Econometric Society, now being joined by professional economists, shows a lack of confidence in their own skepticism. But this is not the topic of this talk. We are interested in fundamentals and, as all sciences are the study of experience, the rôle of isomorphism will be the same with each when and if they are studied by the mathematical method. Moreover, if I am right that the mathematics itself is a human structure, we must expect and acknowledge the frailties common to such structures.

* Bulletin of the American Mathematical Society, vol. 45, p. 223.

NOTES ON CURVATURE OF CURVES AND SURFACES

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Introduction. It is the principal purpose of this paper to show that the curvature of curves and the Gauss curvature of surfaces at elliptic points* are expressible by simple limiting processes which are at once intuitively appealing and (at least apparently) very different from the ones used in the customary definitions.† This is accomplished in sections 2 and 5. In sections 1 and 4 we state and investigate the conditions under which we shall say that curvature exists, and sections 3 and 6 are devoted to remarks on possible generalized definitions of curvature suggested by sections 2 and 5.

1. Curvature of plane curves. DEFINITION. *We shall say that a curve has curvature at a point Q on it if by translating the origin, O, to Q and properly orienting axes we can represent the part of the curve in some neighborhood of O by an equation of the form*

(1.1) $\quad y=f(x)$, *where $f''(0)$ exists and is finite, and $f(0)=f'(0)=0$.*

We take $|f''(0)|$ as the value, K, of this curvature.

That this definition is consistent with the ones customarily given is established by the following:

THEOREM 1.1. *In a neighborhood of O the curve* (1.1) *possesses arc length, s, of which $\phi(s)=\text{arc tan } f'(x)$ is a function for which $|\phi'(s_0)|=K$, where s_0 corresponds to $x=0$.*

Proof. First, $f'(x)$ exists and is bounded in a neighborhood N of $x=0$, and tends to zero with x. Let a be any negative number in N, and x a variable greater than a and in N. Since, for the difference Δx between any two numbers in (a, x), the ratio $\Delta y/\Delta x$ is bounded, $f(x)$ is of bounded variation in (a, x) and hence has arc length $s=s(x)=\sup\sum(\Delta x^2+\Delta y^2)^{1/2}$, where the summation is taken over all the sub-intervals Δx into which (a, x) is divided by a given mode of subdivision, and the supremum is taken over all such modes. Since the curve is continuous in N, we see that $s(x)$ is a continuous, as well as strictly increasing, function of x. Hence x is a like function of s, and we may write

* Or, rather, a generalization of the Gauss curvature at a class of points which, when the Gauss curvature exists, are elliptic.

† It may be of interest to compare Busemann and Feller, Krümmungseigenschaften Konvexer Flächen, Acta Mathematica, vol. 66, 1936, pages 1–47, especially pages 22–23.

Professor R. E. Gilman has called my attention to a considerable recent literature by Busemann and Feller on the differential geometry of convex surfaces in the large, in which the Gauss curvature (when it exists) is shown to be given by limiting processes involving the area which we denote in section 5 by A. The latest of these publications is Bemerkungen zur Differentialgeometrie der konvexen Flächen, III.

$$|\phi'(s_0)| = |[d \operatorname{arc} \tan f'(x)/dx]/[ds/dx]| \text{ at } x = 0, \tag{1.2}$$

provided that both derivatives exist and that $ds/dx \neq 0$.

The numerator of the principal fraction on the right-hand side of (1.2) exists because $f''(0)$ exists; and, since $f'(0)=0$, its value is $f''(0)$. On the other hand,

$$|ds/dx| = \lim_{x\to 0} [\sup \sum(\Delta x^2 + \Delta y^2)^{1/2}/|x|],$$

where the numerator of the ratio is defined with respect to the interval $(0, x)$ (or $(x, 0)$ in case x is negative) as s is with respect to the interval (a, x). Since for each Δx there exists a ξ such that $\Delta y = f'(\xi)\Delta x$ and $|\xi| < |x|$, we have

$$1 \leqq \sup \sum (\Delta x^2 + \Delta y^2)^{1/2}/|x| \leqq (1+\eta)^{1/2},$$

where $\eta = \sup [f'(\xi)]^2$ for $|\xi| < |x|$. Hence $|ds/dx| = 1$, and the conclusion follows.

2. **Evaluation of K by a different limiting process.** Let $f(x)$ be any function which is continuous in a neighborhood M of $x=0$, and let h and k be any two unequal numbers which lie in M. We shall consider the expression

$$\lim_{h,k\to 0} (12A/C^3), \tag{2.1}$$

where C is the chord joining the points $(k, f(k))$ and $(h, f(h))$, and A is the area algebraically between the chord and the curve, that is,

$$A = \left| [f(k) + f(h)](h-k)/2 - \int_k^h f(x)dx \right|.$$

THEOREM 2.1. *If a curve has curvature K at a point Q, then, assuming the representation* (1.1), *K is the value of the limit* (2.1) *with $k<0<h$.*

Proof. We use the form of Taylor's theorem given on p. 290 of Hardy.* First,

$$C = \{(h-k)^2 + [f(h) - f(k)]^2\}^{1/2} = |h-k| [1 + o(1)]^{1/2},$$

whence the limit (2.1) equals $\lim_{h,k\to 0} |12A/(h-k)^3|$, provided that the latter exists. But

$$A = \left| [f''(0) + o(1)][(h^2+k^2)/2][(h-k)/2] - \int_k^h [f''(0)x^2/2 + \eta(x)x^2/2]dx \right|,$$

where $\eta(x)$ tends to zero with x. Hence

$$A = |f''(0)(h-k)^3/12 + o[(h-k)^3]|,$$

and the conclusion follows.†

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* A Course of Pure Mathematics, seventh edition, Cambridge, 1938.

† It can be shown similarly that Theorem 2.1 remains valid if we replace the condition $k<0<h$ by the weaker condition that

$$(|h| + |k|)/|h-k| \text{ remains bounded.} \tag{2.2}$$

THEOREM 2.2. *Under the conditions of Theorem* 2.1, *we have* $\lim_{h\to 0}(12A/C^3) = K$, *with* $k=0$.

Proof. Take $k=0$ in the proof of Theorem 2.1.

Theorem 2.2 is also a corollary of the following:

THEOREM 2.3. *If, for a given function continuous near* $x=0$, *the limit* (2.1) *exists with* $k<0<h$, *then* $\lim_{h\to 0}(12A/C^3)$ *exists with* $k=0$ *and* h *positive or negative.*

Proof. The function $12A/C^3$ is continuous in k at $k=0$ for $h\neq 0$, and is continuous in h at $h=0$ for $k\neq 0$.

3. A possible generalized definition of curvature. We may say that *a curve has generalized curvature at a point Q on it if by translating O to Q and properly orienting axes we can represent the part of the curve in some neighborhood of O by an equation of the form*

(3.1) $y=f(x)$, *where* $f(0)=f'(0)=0$, $f(x)$ *is continuous, and the limit* (2.1) *exists with* $k<0<h$.

We take the value of the limit (2.1) *as the value of this curvature.*

In view of Theorem 2.1, this definition will be justified upon exhibition of a curve of the form (3.1) for which $f''(0)$ fails to exist. Let then $\phi(x)$ be a function for which $\phi''(0)$ exists and is finite, and $\phi(0)=\phi'(0)=0$, and define $f(x)=\phi(x)+x^3\psi(x)$, where $\psi(x)$ is a function continuous and without derivative in a neighborhood of $x=0$ within which $\phi'(x)$ exists. Then $f''(0)$ is undefined. To show that $f(x)$ is of the form (3.1), we need merely establish the existence of the limit (2.1).

For small h and k, we have

$$\begin{aligned} A &= \left| [\phi(k) + k^3\psi(k) + \phi(h) + h^3\psi(h)](h-k)/2 - \int_k^h [\phi(x) + x^3\psi(x)]dx \right| \\ &= A' + o[(h-k)^3], \end{aligned}$$

and

$$C = \{(h-k)^2 + [\phi(h) + h^3\psi(h) - \phi(k) - k^3\psi(k)]^2\}^{1/2} = C' + o(h-k),$$

where A' and C' are the area and chord functions for the curve $y=\phi(x)$. With $k<0<h$, then,

$$\begin{aligned} \lim_{h,k\to 0}(12A/C^3) &= \lim\{12A/(h-k)^3\}\{\lim[(h-k)/C]\}^3 \\ &= \lim\{12A'/(h-k)^3\}\{\lim[(h-k)/C']\}^3, \end{aligned}$$

and the last two limits exist, as is shown in the proof of Theorem 2.1.*

* Indeed, this particular $f(x)$ remains of the form (3.1) if the condition $k<0<h$ therein is replaced by the condition (2.2).

4. G-curvature of surfaces. DEFINITION. We shall say that *a surface has G-curvature at a point Q on it if by translating the origin, O, to Q and properly orienting axes we can represent the part of the surface in some neighborhood of O by an equation of the form*

(4.1) *$z=f(x, y)$, where $f(0, 0)=0$, $f(x, y)$ is continuous, and the function $F(r)=F(r, \theta)=f(r\cos\theta, r\sin\theta)$ is such that $F''(0)$ is finite, exists uniformly in θ, is continuous in θ, and does not vanish, and $F'(0)=0$ for all θ.*

We take $1/\left[\int_0^{2\pi}(1/K)d\theta/(2\pi)\right]$ as the value, G, of this curvature, where $K=K(\theta)$ is the curvature, $F''(0)$, of the curve $z=F(r)$ at $r=0$.

Thus G is the harmonic mean of K with respect to θ.

Whereas our definition of G-curvature is not based on the existence of the second partial derivatives of $f(x, y)$ at O, the usual definitions of surface curvature presuppose such existence and more. I believe that the following definition will not be inconsistent with the literature.

DEFINITION. We shall say that *Gauss curvature exists at a point Q on a surface if by translating O to Q and properly orienting axes we can represent the part of the surface in some neighborhood of O by an equation of the form*

(4.2) *$z=f(x, y)$, where $f(0, 0)=f_x(0, 0)=f_y(0, 0)=0$, $f_x(x, y)$ and $f_y(x, y)$ are differentiable at O, and one of these partial derivatives is continuous near O.**

The value of this curvature is $M\mu$, where M and μ are the maximum and minimum curvatures at O (shown to exist, below) *of the curves of vertical cross-section of the surface through O.*

The relation between Gauss curvature and G-curvature will be put into evidence by the following:

THEOREM 4.1. *If a surface has positive Gauss curvature at a point, it has G-curvature there, and the Gauss curvature equals G^2.*

Proof. We assume the representation (4.2). The conditions establish the finite existence of the four second partial derivatives at O, and hence the equality $f_{xy}(0, 0)=f_{yx}(0, 0)$. The differentiability of $f_x(x, y)$ and $f_y(x, y)$ at O insures their boundedness near O, and application of the mean value theorem to the difference $f(x+\Delta x, y+\Delta y)-f(x, y)$ therefore establishes the continuity of $f(x, y)$ near O. Also,

$$F'(r) = f_x(r\cos\theta, r\sin\theta)\cos\theta + f_y(r\cos\theta, r\sin\theta)\sin\theta,$$

from which we deduce, first, that $F'(0)=0$ for all θ, and, second, that

$$[F'(r) - F'(0)]/r = f_{x^2}(0, 0)\cos^2\theta + 2f_{xy}(0, 0)\cos\theta\sin\theta + f_{y^2}(0, 0)\sin^2\theta + \zeta,$$

where ζ tends to zero with r, uniformly in θ. Thus we see that $F''(0)$ exists uni-

* These conditions may be regarded as analogs of the conditions for curvature of plane curves given in §1.

formly in θ and obtain its value, a continuous function of θ. Since, finally, M and μ are of one sign,* $f(x, y)$ is of the form (4.1).

We have next to compare the values of the two curvatures. Now it can be shown that $K = F''(0)$ may be written in the form

$$M \cos^2 (\theta - \alpha) + \mu \sin^2 (\theta - \alpha) \tag{4.3}$$

for some α.† Hence

$$\int_0^{2\pi} (1/K)d\theta = \int_0^{2\pi} \{1/[|M| \cos^2 (\theta - \alpha) + |\mu| \sin^2 (\theta - \alpha)]\} d\theta = 2\pi(M\mu)^{-1/2}.$$

Thus $G^2 = M\mu$, as was to be proved.‡

5. Evaluation of G by a different process. Consider any surface which, in a neighborhood of O, is of the form

(5.1) $z = f(x, y)$, where $f(0, 0) = 0$, $f(x, y)$ is continuous, the function $F(r) = F(r, \theta) = f(r \cos \theta, r \sin \theta)$ is, for each θ, either strictly increasing or strictly decreasing for $r > 0$, and $F'(0) = 0$ for each θ.

Let $\bar{r}$ be such that the circle $x^2 + y^2 = \bar{r}^2$ lies in this neighborhood, and suppose for definiteness that $F(r, 0)$ is strictly increasing for $0 < r \leqq \bar{r}$. By the continuity of $f(x, y)$, then, $F(r)$ is strictly increasing for each θ, and $\inf_\theta F(\bar{r}) = \min_\theta F(\bar{r}) = E > 0$. Hence the plane $z = l$ for $0 < l < E$ is, for each θ, intersected by the curve $z = F(r)$ in exactly one point for $0 < r < \bar{r}$. We denote by $\rho = \rho(\theta)$ the corresponding value of r.

We now show that $\rho(\theta)$ is continuous. If it were discontinuous for $\theta = \theta_0$, then there would exist a number $p \neq 0$ and a sequence $\theta_i \to \theta_0$ for which $\rho_i = \rho(\theta_i)$ would tend to $\rho(\theta_0) + p = \rho_0 + p$. But $F(\rho_i, \theta_i) = l$, and $F(r, \theta)$ is continuous for $0 \leqq \theta \leqq 2\pi$ and $0 \leqq r \leqq \bar{r}$. Hence $F(\rho_0 + p, \theta_0) = l = F(\rho_0, \theta_0)$, contrary to the uniqueness of $\rho(\theta)$.

We may therefore speak of the area A cut out of the plane $z = l$ by the surface $z = f(x, y)$ for $x^2 + y^2 < \bar{r}^2$, and the volume V between the plane and the surface in this neighborhood. We are to consider the expression

$$\lim_{l \to 0} (4\pi V / A^2). \tag{5.2}$$

* A condition necessary and sufficient that this be the case is easily seen to be the following: $f_x^2(0, 0) f_y^2(0, 0) > [f_{xy}(0, 0)]^2$.

† Lest one suppose that the conditions for existence of G-curvature imply that K be of the form (4.3), which is so familiar in the differential geometry of "well-behaved" surfaces, we call attention to the paraboloidal surface

$$z = \phi(\theta) r^2 / 2, \tag{4.4}$$

which is of the form (4.1) for *any* continuous, non-vanishing $\phi(\theta)$ of period π, and for which $K = \phi(\theta)$.

‡ It may be of interest to compare this result with the fact that the so-called mean curvature of a surface is twice the mean of K with respect to θ. We have just shown that the Gauss curvature, when positive, is the square of the harmonic mean of K with respect to θ.

THEOREM 5.1. *If a surface has G-curvature at a point Q then, assuming the representation* (4.1), *G is the value of the limit* (5.2).

Proof. Taking, for definiteness, the constant sign of $F''(0)$ as plus, we deduce the existence of numbers b and B for which

$$0 < b < F''(0) < B, \quad (\text{all } \theta). \tag{5.3}$$

Since, moreover, $F''(0)$ exists uniformly and $F'(0)=0$ for all θ, there exists a number L such that

$$br < F'(r) < Br \quad \text{for} \quad 0 < r \leqq L, \quad (\text{all } \theta). \tag{5.4}$$

Using (5.4), the fact that $F(0)=0$, and the relations

$$d[F(r) - br^2/2]/dr = F'(r) - br, \qquad d[Br^2/2 - F(r)]/dr = Br - F'(r),$$

we have, finally,

$$br^2/2 < F(r) < Br^2/2 \quad \text{for} \quad 0 < r \leqq L, \quad (\text{all } \theta). \tag{5.5}$$

From (5.4) we see that representations of the form (4.1) are of the form (5.1), so that A, V, and $\rho=\rho(\theta)$ have meaning here.

We shall evaluate the limit (5.2) by showing it to be equal to the corresponding limit for the auxiliary surface

$$z = F''(0)r^2/2, \tag{5.6}$$

which is a special case of (4.4), pointed out above to be of the form (4.1). Let $R=R(\theta)$ be the (unique) positive value of r for which $F''(0)r^2/2=l$, $\overline{A}$ the area cut out of the plane $z=l$ by the surface (5.6), and $\overline{V}$ the volume between this plane and this surface.

Now the uniformity of existence of $F''(0)$ permits us to modify the development in §151 of Hardy* to the end that we may write

$$F(r) = F''(0)r^2/2 + \xi(r, \theta)r^2, \tag{5.7}$$

where ξ tends to zero uniformly in θ. Hence, since $F(\rho) = F''(0)R^2/2 = l$, we have

$$\begin{aligned} 0 &= F(\rho) - F''(0)R^2/2 = F''(0)\rho^2/2 + \rho^2\xi(\rho, \theta) - F''(0)R^2/2 \\ &= [F''(0)/2](\rho^2 - R^2) + \rho^2\xi(\rho, \theta), \end{aligned}$$

whence

$$R - \rho = 2\rho\xi(\rho, \theta)/[F''(0)(1 + R/\rho)].$$

From the left-hand part of (5.5), we see that ρ tends to zero uniformly with l. Hence so does ξ, so that $|R-\rho| < \rho\eta(l)$, where η tends to zero with l, independently of θ.

Using the right-hand part of (5.3), we have

* *Ibid.*

$$\overline{A} = (1/2)\int_0^{2\pi} R^2 d\theta \geqq \pi \min_\theta R^2 > 2\pi l/B.$$

Also,

$$\left| A - \overline{A} \right| \leqq (1/2)\int_0^{2\pi} \left| R^2 - \rho^2 \right| d\theta < \pi \max_\theta (R + \rho)\rho\eta(l).$$

From the left-hand parts of (5.3) and (5.5), then,

$$\left| A - \overline{A} \right| < (4\pi l/b)\eta(l).$$

It follows that $\lim_{l\to 0} [(A - \overline{A})/\overline{A}] = 0$.

Comparing V and $\overline{V}$, we have, first,

$$\begin{aligned} \left| V - \overline{V} \right| &= \left| \int_0^{2\pi}\int_0^{\rho} [l - F(r,\theta)] r\,dr\,d\theta - \int_0^{2\pi}\int_0^{R} [l - F''(0)r^2/2] r\,dr\,d\theta \right| \\ &= \left| \int_0^{2\pi}\int_0^{R} [-r^2\xi(r,\theta)] r\,dr\,d\theta + \int_0^{2\pi}\int_R^{\rho} [l - F(r,\theta)] r\,dr\,d\theta \right|. \end{aligned}$$

And since

$$\left| l - F(r,\theta) \right| \leqq \left| F''(0)r^2/2 - F(r,\theta) \right| = \left| \xi(r,\theta) \right| r^2$$

for $\rho \leqq r \leqq R$ or $R \leqq r \leqq \rho$, we have, further,

$$\begin{aligned} \left| V - \overline{V} \right| &\leqq \int_0^{2\pi}\int_0^{R} r^3 \left| \xi(r,\theta) \right| dr\,d\theta + \int_0^{2\pi} \left| \int_R^{\rho} r^3 \left| \xi(r,\theta) \right| dr \right| d\theta \\ &\leqq 2\int_0^{2\pi}\int_0^{(2l/b)^{1/2}} r^3 \left| \xi(r,\theta) \right| dr\,d\theta \leqq (4\pi l^2/b^2) \max \left| \xi(r,\theta) \right|, \end{aligned}$$

where the maximum is taken over $0 \leqq \theta \leqq 2\pi$ and $0 < r \leqq (2l/b)^{1/2}$. In view of the fact that $\xi(r,\theta)$ tends to zero with r, uniformly in θ, then, $\lim_{l\to 0}[(V - \overline{V})/\overline{A}^2] = 0$.

Finally, $\lim_{l\to 0} (4\pi V/A^2) = \lim_{l\to 0} (4\pi \overline{V}/\overline{A}^2)$, provided that the last limit exists. But

$$\overline{A} = \overline{A}(l) = (1/2)\int_0^{2\pi} R^2 d\theta = l\int_0^{2\pi} [1/F''(0)] d\theta = (2\pi/G)l,$$

and hence $\overline{V} = \int_0^l \overline{A}(t)dt = \pi l^2/G$. Thus $4\pi\overline{V}/\overline{A}^2 = G$, and the proof is complete.

COROLLARY 5.11. *If a surface has positive Gauss curvature at a point Q then, assuming the representation* (4.2), *the Gauss curvature is the square of the value of the limit* (5.2).

The question naturally arises: *Is it possible to generalize Theorem* 5.1 *by replacing the plane $z = l$ by the more general plane, $z = mx + ny + q$, where m, n, and q tend to zero, subject perhaps to certain restrictions*? The answer is *yes*, and we have the following:

THEOREM 5.2. *If a surface has G-curvature at a point Q then, assuming the representation* (4.1), *there exist quantities A' and V' analogous to A and V with respect to the plane $z=mx+ny+q$ for $q\cdot F''(0)>0$ and q and u sufficiently small, and* $\lim_{q,u\to 0}(4\pi V'/A'^2)=G$, *where* $u=(m^2+n^2)^{1/2}/q$.*

We omit the proof.

COROLLARY 5.21. *Theorem 5.2 remains valid if we replace the condition that the surface have G-curvature at Q by the condition that it have positive Gauss curvature there.*

6. A possible generalized definition of G-curvature. We may say that *a surface has generalized G-curvature at a point Q on it if by translating O to Q and properly orienting axes we can represent the part of the surface in some neighborhood of O by an equation of the form* (5.1), *and if the limit* (5.2) *exists. We take the value of the limit* (5.2) *as the value of the curvature.*

It is clear how we could construct a function of the form (5.1) for which $F'(r)$ would fail to exist for certain positive values of r however small and each θ, so that $F''(0)$ would be undefined for each θ, but for which the limit (5.2) would exist.

The desirability of such a generalized definition may, however, be questioned, since it would assign curvature to certain surfaces for which the generalized curvature (in the sense of §3) of normal sections would, in general, fail to exist; *e.g.*, to surfaces of the form (4.4) with $\phi(\theta)$ continuous and of period 2π but in general failing to satisfy the equality $\phi(\theta)=\phi(\theta+\pi)$.

* The condition $u\to 0$ means, of course, that the line of intersection of the plane $z=mx+ny+q$ with the xy-plane recedes to infinity.

EQUATIONS OF POLYGONAL CONFIGURATIONS*

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1. Introduction. This paper continues the studies made principally by V. Alaci and the author in the field of simply-discontinuous functions and semilinear equations, respectively, by means of which a representation of broken-line graphs as well as many other types is effected by equalities (equations) rather than inequalities.† The methods consist essentially in the use of two operators with which we shall later deal in some detail. We begin with some preliminary definitions.

2. Definitions. A collection of any finite number of line-segments in the plane we shall call a *generalized polygon*. Some or all of the lines may be infinite in length, either half-lines or full lines. The end-points of each of the line-segments of which the generalized polygon is composed we shall call *vertices*, and likewise label all points common to two or more lines of the configuration. We shall have occasion to distinguish between two kinds of vertices. We shall call the true end-point of a line-segment (no matter whether such a point may lie on another line-segment or not) a *true vertex* of the polygon. That is to say a true vertex is any point which is the end-point of some segment. The other points common to two or more segments shall be called *sub-vertices* when it is desirable to make such a distinction. When no distinction is necessary we shall refer, simply, to vertices. The portion of a line-segment between two adjacent vertices we shall call a *side* of the polygon.

3. The equation of a generalized polygon. We now proceed to write down the equation of a generalized polygon making essential use of the methods developed by Alaci. To this end we choose axes so that the join of no two true vertices is parallel to the y-axis. This orientation is always possible since there is only a finite number of vertices. Next, pass lines $L_i = x - k_i = 0$ through these true vertices in order from left to right (Fig. 1). Let R_1 be the region to the left of $L_1 = 0$, R_i the region between $L_{i-1} = 0$ and $L_i = 0$, R_{n+1} the region to the right of $L_n = 0$. We assume that the polygon is of such character that there are n true vertices. No region shall include its boundaries. By the very nature of things note that if region R_1 contains any line-segment, then that line extends to infinity and the same applies to the region R_{n+1}. Let N_{i-1} and N_{i+1} be the number of sides in R_{i-1} and R_{i+1}, respectively; let $u_{i1} = 0$, $u_{i2} = 0, \cdots, u_{ij} = 0$, be the equations of the j sides of the polygon in the non-vacuous region R_i; and in such a region set $\psi_i = u_{i1} u_{i2} \cdots u_{ij} \theta_i$, where $\theta_i = L_i$ if $N_{i-1} \leqq j < N_{i+1}$; where $\theta_i = L_{i-1}$ if

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* Presented to the Philadelphia Section of the Mathematical Association of America, December 2, 1939.

† See the author's papers: Semilinear equations, Tohôku Mathematical Journal, vol. 41, Part I, 1935; Equations of polygons, this MONTHLY, vol. 42, no. 8, 1935; and Sur les équations semilinéaires et leurs configurations géométriques, Bulletin Scientifique de l'École Polytechnique de Timişoara, Roumanie, tome 7, fasc. 3–4, 1937 in which there is a bibliography of the subject. Alaci's papers all occur in the last named journal.

$N_{i+1} \leqq j < N_{i-1}$; where $\theta_i = L_{i-1}L_i$ if $N_{i-1} > j < N_{i+1}$ (as in region R_h of Figure 1); and where $\theta_i = 1$ otherwise. In a vacuous region R_k set $\psi_k = L_{k-1}L_k$. Clearly θ_i

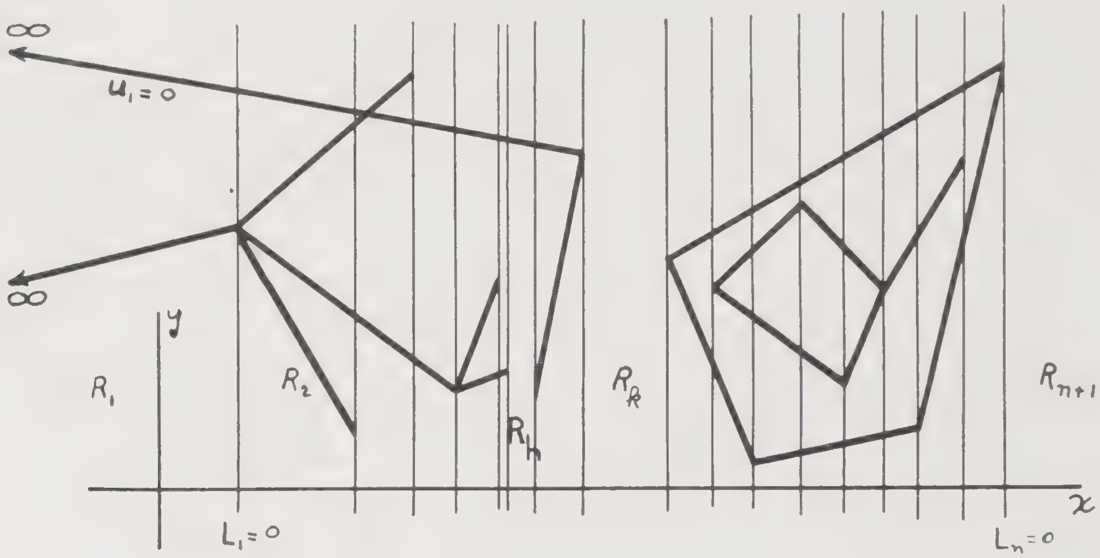

FIG. 1

and ψ_i are completely determined by any given polygon once axes are chosen. Finally, set

$$A(L) = \frac{2}{\pi} \int_0^\infty \frac{\sin L\alpha}{\alpha} \, d\alpha,$$

and recall that $A(L) = -1, 0, 1$ according as $L < 0, = 0, > 0$. Under these conditions we have, as the equation of the generalized polygon,

$$\psi_1 + \psi_{n+1} + \sum_{i=1}^{n} (\psi_{i+1} - \psi_i) A(L_i) = 0. \tag{1}$$

For in a non-vacuous region R_i, equation (1) reduces to $\psi_i = 0$ which is the equation of the sides of the polygon in that region. On $L_i = 0$, (1) reduces to the simultaneous system $L_i = 0$, $\psi_i + \psi_{i+1} = 0$ which is satisfied by and only by the points of the polygon lying on the line $L_i = 0$ no matter what the form of ψ_i or ψ_{i+1} which in turn depend upon the form of θ_i and θ_{i+1}. In a vacuous region R_k, (1) reduces to $\psi_k = L_{k-1}L_k = 0$ which is satisfied by no point of the region.

4. Operators. The main tool used in the development of equation (1) above is the operator $A(L)$. This integral is exceedingly useful here because it has the special property of taking on one of only three values, $-1, 0, 1$, and is used as a multiplier on the differences $(\psi_{i+1} - \psi_i)$ in order to control the signs of these terms. As a consequence, the appropriate cancellations and reductions in the various regions are accomplished. This particular operator has played an important rôle in Alaci's work in this field. The writer has made extensive use

rather of the operation of "taking absolute values," and in paragraph 5 adds to the theory of representing areas by means of absolute values of linear functions (semilinear equations). Other integrals with different properties might be used in order to give special effects. For example, the integral

$$B(L) = \int_0^\infty Le^{-Lx}dx = 1,\ 0,\ \infty, \text{ according as } L > 0,\ = 0,\ < 0;$$

or again,

$$C(L) = \int_0^\infty L^2e^{-L^2x}dx = 1,\ 0, \text{ according as } L \neq 0,\ = 0.$$

An "equivalent" form of $A(L)$ might be written as

$$A(L) = D(L) = \lim_{n=\infty} L^{1/(2n-1)} = 1,\ 0,\ -1, \text{ according as } L > 0,\ = 0,\ < 0.$$

Let $E(L)=|L|/L$; then, since $|L|=L\,A(L)$, we could write $A(L)=E(L)$ with the further understanding that $E(0)=0$ (although $\lim E(L)=1,\ -1$, according as L approaches zero through positive or negative values). Indeed, equation (1) could then be written with $E(L_i)$ replacing $A(L_i)$. It is interesting to note that, with $E(0)=0$, we could write

$$|L| = L\frac{2}{\pi}\int_0^\infty \frac{\sin L\alpha}{\alpha}d\alpha = L\,A(L),$$

or

$$A(L) = |L|/L = \frac{d}{dL}|L|;\text{*}$$

therefore

$$|L| = \int A(L)dL = \frac{2}{\pi}\int_0^L\int_0^\infty \frac{\sin t\alpha}{\alpha}d\alpha\,dt = +\sqrt{L^2},$$

and any one of these forms might be used for "absolute value of L."

There are still other operators that yield interesting results. The symbol $[L]$, *i.e.*, the greatest integer not greater than L, is one such. An equation of the form

$$u_0 + \sum m_i[u_i] = 0$$

may have a solution much as indicated in Figure 2. Figure 3 is the graph of $y=e^{[x]}$. Hardy, in his *Pure Mathematics*, has given some interesting examples making use of this and other functional concepts. One could, of course, manufacture quite arbitrarily any kind of an operator that he might want. For example, define W to be that operation which, when applied to a graph, reflects

* See Riabouchinsky, La fonction $|x|$, Bulletin de l'Institut Aérodynamique de Koutchino, fasc. 5, 1914.

that portion lying in the first quadrant into the second, the second into the third, *etc.* It is obvious that W is periodic of period not greater than four, and Figure 4 illustrates the case of applying W to the graph of the parabola $y-x^2+1=0$.

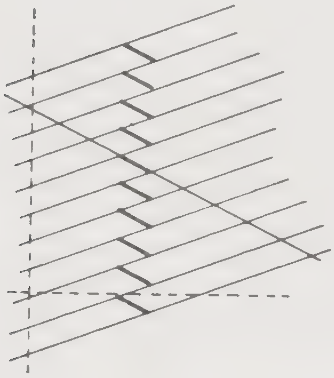

FIG. 2

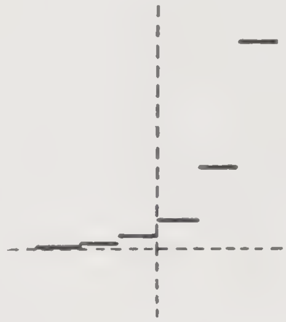

FIG. 3

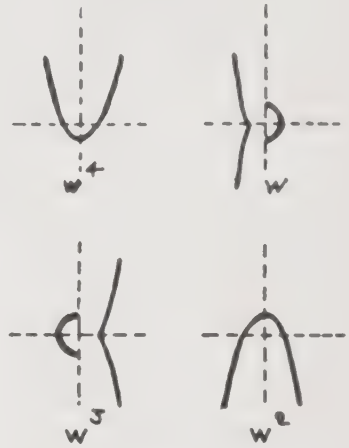

FIG. 4

5. Areas by means of absolute values. An important question still unanswered is the following: With n arbitrary lines as diagonals, how many areas formed by these lines can be represented by a semilinear equation? That is, how many regions of the plane cut off by the lines $u_i=0$ can be represented by the single equation

$$u_0+\sum_{i=1}^{n} m_i\,|\,u_i\,| = 0? \tag{2}$$

If $u_0\equiv 0$, we call (2) the reduced equation. The answer to the question is not forthcoming in a single theorem, since as yet too little is known about the way in which n lines divide the plane. There are many special cases. But the following statements, readily verifiable, throw some light on the situation.

Any one of the areas formed by any n lines can be represented by an equation of the form (2), namely,

$$\sum_{i=0}^{n} (\operatorname{sgn} R)u_i - \sum_{i=0}^{n} |\,u_i\,| = 0, \tag{3}$$

where $(\operatorname{sgn} R)$ indicates that the signs to be used in taking this (algebraic) sum are those of the u_i in R.

Only one area can be represented when $n=2$.

A necessary and sufficient condition that two areas be representable with $n=3$ is that the three lines be concurrent or parallel. But this does not imply that *any* two areas can be so represented. As a matter of fact, only complementary regions (regions having opposite sign-sets) are amenable to this treatment.

No closed region can be represented by means of a reduced equation in the

case of four lines, $n=4$. Here the trouble is a typical and ever present one: extraneous pieces automatically come in. Evidently, if $\sum m_i|u_i|=0$ represents R it also plots at the same time the region complementary to R which in this case is at infinity. Hence the additional lines, as illustrated in Figure 5.

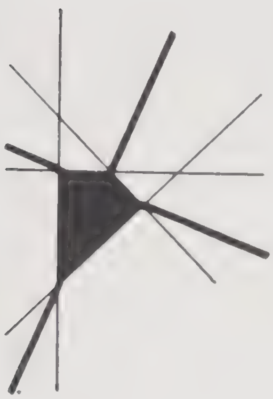

FIG. 5

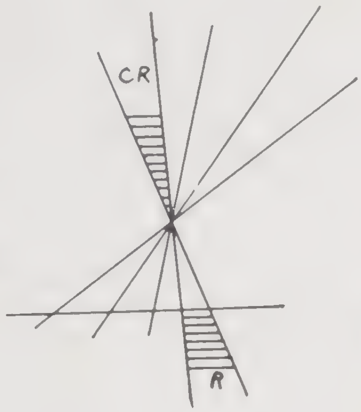

FIG. 6

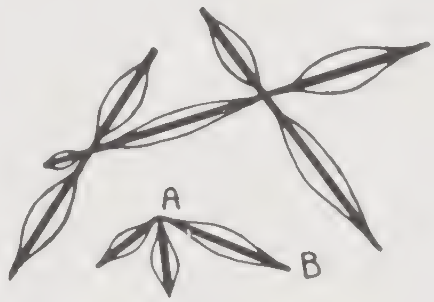

FIG. 7

Two but not more than two areas can be represented with four lines.

If no three of n lines are concurrent then, in order that two particular regions be representable, it is necessary (but not sufficient) that the sign-sets of the two regions differ in at least four signs. This means that once a region is chosen, then all adjacent regions must be discarded as areas of possible additional representation, as likewise all regions with any point in common with these adjacent regions. And, further, any remaining areas that differ in only three signs must be excluded.

It is impossible to represent any odd number of open regions, in the event two are complementary, by a reduced equation.

Not every pair of complementary open regions can be represented even with n large. Figure 6 illustrates this situation.

6. Other equations. With the definitions and orientation used in paragraphs 2 and 3 in mind we seek again an equation for an arbitrary set of lines in the plane. We first enclose each side AB of the polygon in a curve $f(x, y)=0$ such that (see Fig. 7):

a) The side AB is within $f=0$, having only the points A and B in common;

b) At A and at B the slope of $f=0$ is the same as that of AB; that is, A and B are cusps of the first kind;

c) No one of the curves $f_i=0$ shall enclose another nor shall it have more than one point in common with another, the one exceptional point being a vertex common to two sides;

d) The sign of f_i is negative within the loop $f_i=0$.

We thus isolate the sides of the generalized polygon in separate regions of the

plane determined by the auxiliary curves $f_i(x, y)=0$.* Write

$$\phi(x, y) = f_1 f_2 \cdots f_n,$$
$$\psi(x, y) = u_1 u_2 \cdots u_n,$$

where $u_i=0$ is the equation of the line coinciding with the ith side of the polygon. The equation of the generalized polygon is then

$$\psi + \phi - (\psi - \phi)A(\phi) = 0. \tag{4}$$

For, in regions of the plane where $\phi<0$, *i.e.*, within the loops, equation (4) reduces to $\psi=0$ which plots the sides of the polygon exclusive of vertices; in regions where $\phi>0$ (exterior to all loops), (4) reduces to $\phi=0$ which is a contradiction; and where $\phi=0$, on the loops themselves, we are led to the equation $\psi+\phi=0$ and these simultaneous conditions are satisfied only at the points common to the polygon and the loops, that is at the vertices.

A similar equation can be developed making use of the absolute value operator. The generalized polygon is given by

$$2|\psi| - \psi + \phi + |\phi| = 0. \tag{5}$$

For, in or upon the ith loop we have $f_i \leqq 0$ and therefore $\phi \leqq 0$; the equation reduces to $2|\psi| - \psi = 0$† which is satisfied only by the points of the ith side $u_i=0$ including vertices. Outside the loops, $\phi>0$ and the equation reduces to $\psi+2\phi=0$ if ψ is positive, to $\phi=0$ if $\psi=0$, and to $-3\psi+2\phi=0$ if ψ is negative. No one of these equations is satisfied by any point in the region. In the case of a single line-segment, equation (5) simplifies to $2|u| - u + f + |f| = 0$.

Making use of combinations of operators we get another equation whose graph is a generalized polygon. Examine

$$\sum \{u_i - u_i A(f_i) + f_i |A(\phi)| + f_i A(\phi)\} = 0. \tag{6}$$

Since this is somewhat more complicated in form than those previously given it will be instructive to check it with some care. In the ith loop, (6) reduces to $u_i=0$ which gives us the ith side. On the loop there results the system $f_i=0$, $u_i=0$ which will plot only vertices provided there are such vertices (on the ith loop and not on any other loop). If a vertex belongs to just two loops then we have the system $f_i=0$, $f_j=0$, $u_i+u_j=0$ which is satisfied by just that particular vertex. For a triple vertex, that is one belonging to exactly three loops, the re-

* This is always possible, and with curves no more complicated than certain simple sextics. For example, assuming a finite side $A(0, 0)$, $B(2k, 0)$, a sextic of the form $y^2=A^2x^3(2k-x)^3$ will suffice. Since the maximum value of the positive determination of y is Ak^3, the size of the loop is easily controlled. If the vertices are $A(0, 0)$, $B(\infty, 0)$, then we could use a curve of the form $y^2(x^4+1)=4A^2x^3$ with maximum value of $+y$ equal to $3^{3/8}A$. Or again, if the side extended from $A(-\infty, 0)$ to $B(\infty, 0)$, we might use $y^2(x^2+1)=A^2$ with the maximum value of $+y$ equal to A. This treatment could be made to apply to any side by translating and rotating. Also non-algebraic curves could be used for this purpose such as, for example, $y^2=A^2e^{-x^2}$ for the last case above.

† On one side of $u_i=0$ this reduces further to $3\psi=0$ and on the other side to $\psi=0$ since u_i is the only factor of ψ which changes sign.

sulting system is $f_i=0$, $f_j=0$, $f_k=0$, $u_i+u_j+u_k=0$ and this is satisfied by and only by the triple vertex. The situation is similar for vertices of higher order. Outside the loops the reduction yields $\sum f_i=0$ which is impossible since every f_i is positive. Equation (6) therefore truly represents the polygonal points only.

It is of interest to note what changes occur in the geometry when certain modifications are made in these equations for a generalized polygon. Below we write a few of them with statements about their graphs, without making any attempt to go into detail; they can be readily checked.

The generalized polygon could be represented by the equation $\psi+\phi-(\psi-\phi)B(-\phi)=0$, since outside the loops the equation is meaningless. The equation

$$\sum \{u_i - u_iA(f_i) + f_i + f_iA(\phi)\} = 0$$

represents the sides of a polygon, the one vertex (if there be such) common to all sides, and those points that $f_i=0$ may have in common with $u_i+\sum f_i=0$. If we set, in equation (1), $\theta_i=L_{i-1}L_i$, $(i=1, 2, \cdots, n+1)$, with $L_0\equiv 1$, then equation (1) will represent, in addition to the polygon, all auxiliary lines $L_i=0$. Or again, if $\theta_i=1$, (1) will plot the complete polygon minus certain of the vertices (in general those true vertices not lying on another line-segment). If $\theta_i=1/L_{i-1}L_i$, then on $L_i=0$ the equation $\psi_i+\psi_{i+1}=0$ would have no meaning. Therefore in this case the graph would consist of the polygon minus the vertices.

In a similar manner other modifications will give new results. If (4) be changed to read $\psi+\phi+(\psi-\phi)A(\phi)=0$, we have the equation of what might be called the complementary polygon; that is, those portions outside of the loops of those lines coinciding with the sides of the given polygon. Changing (5) to read $|\psi|-\psi+\phi+|\phi|=0$, we get the polygon as before plus all points within and on the boundary of each of the loops which lie on one certain side of $u_i=0$. The interior points on the other side of $u_i=0$ are not represented nor are the points of the loop. The equation $\sum\{u_i-u_iA(f_i)\}=0$ represents the polygon plus all points exterior to the loops. Again, $\sum\{u_i+u_iA(\phi)\}=0$ plots all points within and on the loops plus the line $\sum u_i=0$. Also, $\sum\{u_i-u_iA(u_i)\}=0$ represents any convex area (the $++\cdots+$ region) formed by the lines $u_i=0$, including the boundary. The equation $\sum\{u_i-(\operatorname{sgn} R)u_iA(u_i)\}=0$, as well as equation (3), plots a region R with any sign-sets; and, by multiplying the left-hand members of such equations together, the equation of any rectilinear area, convex or not, can be formed. The equation $\sum\{u_i-u_iA(\phi)\}=0$ plots, exclusive of boundaries, all those regions in which an even number of the u's is negative and in addition those points of the line $\sum u_i=0$ lying within every region in which an odd number of the u's is negative.

THE MATHEMATICS OF EXTERIOR BALLISTIC COMPUTATIONS*

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1. Introduction. The mathematical theory upon which ballistic computations are based is subject to change depending upon two very different conditions. One is the kind of weapon and the consequent type of trajectory which is common at the time in actual gunnery, and the other is the available method of a strictly computational character. As a result of changes in these things it has happened that there are two theories which have formed bases for ballistic computations, and which have in turn become obsolete, and then returned to actual or prospective usefulness. One of these is the Siacci method for trajectories; the other is the adjoint system for differential variations.

2. The Siacci method. Previous to 1914, the common type of trajectory was one which nowhere departed very far from the horizontal. For a trajectory of this sort the Siacci method† constitutes an excellent approximation to the solution of the differential equations which formulate the commonly used physical hypotheses. These include the assumption that the force exerted by the air on the projectile consists only of the *drag*, that is, a force acting in a direction exactly opposite to that of the projectile velocity referred to the air. Thus if there is no wind the problem becomes two-dimensional; and the differential equations of motion for the projectile may be written

$$x'' = -Ex', \qquad y'' = -Ey' - g, \tag{1}$$

where primes denote time derivatives, and g is the acceleration of gravity. It is further assumed that we may write

$$E = \frac{GH}{C}, \tag{2}$$

where G is an empirical function of the velocity v and is called the *drag function*, H is the ratio of the air density to its standard value at sea level, and C is a constant for the trajectory and is called the *ballistic coefficient*. For different shapes of projectile there may be different drag functions. If such a drag function is used as corresponds reasonably well to the shape of the projectile under consideration, then equations (1) and (2) represent the motion of the projectile sufficiently well for most purposes connected with the computation of firing tables, and the value of C will show little variation between trajectories for the same projectile. Equations (1) may be written more explicitly

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* Presented at the Hanover meeting of the Mathematical Association of America on September 9, 1940.

† F. Siacci, Balistique Extérieur, Paris, 1892.

$$\frac{dx}{dt} = v_x, \qquad \frac{dv_x}{dt} = -Ev_x,$$

$$\frac{dy}{dt} = v_y, \qquad \frac{dv_y}{dt} = -Ev_y - g.$$

If in these we introduce the new variable $u = v_x \sec \theta_0$, called the *pseudo-velocity*, where $\tan \theta = v_y/v_x$ and θ_0 is the initial value of θ, the equations of motion become

$$\frac{dt}{du} = \frac{-1}{Eu}, \qquad \frac{dm}{du} = \frac{g \sec \theta_0}{Eu^2}, \tag{3}$$

$$\frac{dx}{du} = \frac{-\cos \theta_0}{E}, \qquad \frac{dy}{du} = \frac{-m \cos \theta_0}{E},$$

where $m = \tan \theta = v_y/v_x$.

The essential feature of the Siacci method in its original form is the use of two approximations which are valid if the trajectory remains in all parts nearly horizontal. From the definition of u it follows that $v = u \cos \theta_0 \sec \theta$. Since θ and θ_0 are both small and since G is always an increasing function of its argument, the error made is small if in (2) we replace $G(v) = G(u \cos \theta_0 \sec \theta)$ by $G(u) \cos \theta_0 \sec \theta$. This is the first Siacci approximation. The second consists in omitting both $\sec \theta$ and H as factors of E. This also is a valid approximation since each varies slightly from unity, $\sec \theta$ upward and H downward. By these approximations we get $E = G(u) \cos \theta_0 / C$, and the equations of motion become

$$\frac{dt}{du} = \frac{-C \sec \theta_0}{uG(u)}, \qquad \frac{dm}{du} = \frac{gC \sec^2 \theta_0}{u^2 G(u)}, \tag{4}$$

$$\frac{dx}{du} = \frac{-C}{G(u)}, \qquad \frac{dy}{du} = \frac{-Cm}{G(u)}.$$

All of these have separable variables except the equation for y. The other three can be immediately solved in terms of quadratures, and then the equation for y can also be solved in terms of a quadrature involving that already used for m. For example, if we let

$$S(u) = \int_u^U \frac{du}{G(u)},$$

where U is a constant larger than any velocity likely to be used, then

$$x = C[S(u) - S(v_0)].$$

If therefore $S(u)$ is computed and tabulated for a given drag function $G(u)$, x is expressible in terms of u merely by an entry in the table. In like manner each of the variables t, m, and y may be found in terms of u from a tabulated function of

u and hence in terms of x. This gives a practical method for finding trajectory results from four readily computed tables corresponding to any particular drag function.

All this depends on the condition that the trajectory departs only slightly from the horizontal. The increasing use of high angle fire both for anti-aircraft and ground impact emphasized the need of a method not subject to this restriction. This was furnished by the method of numerical integration, which is perfectly general and gives any solution of the equations (1) and (2) with any desired degree of accuracy. In 1918 numerical integration was adopted as the standard trajectory method; but the very laborious computation involved in this method made it necessary to supplement it in certain directions. In one of these the Siacci method continued in use as an auxiliary until about 1935. It then was dropped completely.

3. Modifications of the Siacci method. Shortly after this, however, a generalization of the Siacci method to trajectories that are nearly flat but not near the horizontal was developed by R. H. Kent* of the Aberdeen Proving Ground. Trajectories of this sort are of the kind needed for guns mounted on aircraft. In this method the first Siacci approximation is made but instead of the second, the factor $\sec\theta$ is canceled against $\cos\theta_0$, since they are nearly reciprocal, and H is treated as a constant. If this is combined with C by writing $C_H = C/H$, we get instead of equations (4) the following:

$$\frac{dt}{du} = \frac{-C_H}{uG(u)}, \qquad \frac{dm}{du} = \frac{gC_H \sec\theta_0}{u^2G(u)},$$

$$\frac{dx}{du} = \frac{-C_H\cos\theta_0}{G(u)}, \qquad \frac{dy}{du} = \frac{-C_H m\cos\theta_0}{G(u)}.$$

These have solutions expressible in terms of the Siacci functions in exactly the same manner as in (4). If the trajectory involves enough variation in altitude to require a consideration of variable density, we may retain H and express it in the usual exponential form $H = e^{-hy}$. From the flatness of the trajectory this is closely approximated by e^{-hm_0x}, which gives an equation for x in the form

$$e^{-hm_0x}dx = C_H\cos\theta_0\frac{du}{G(u)},$$

of which the integral is

$$\frac{1 - e^{-hm_0x}}{hm_0} = C_H\cos\theta_0\left[S(u) - S(v_0)\right].$$

From this point on, the Siacci functions are not applicable and the quadratures must be performed in each case; but this is much less laborious than numerical integration of the differential equations. A similar method can be applied if

* *Cf.* also, K. Popoff, Das Hauptproblem der aüsseren Ballistik, Leipzig, 1932.

there is an initial yaw which is damped out. The increased resistance due to this can also be expressed as an exponential in x or in t.

4. Differential variations. Just as the Siacci method for trajectories was threatened with extinction by the advent of numerical integration, so the use of the adjoint system for differential variations was abandoned with the advent of mechanical integration. When numerical integration is used, trajectories are computed only for standard conditions. These include:

a) standard muzzle velocity,
b) standard air density structure,
c) standard air temperature structure,
d) standard projectile weight,
e) no wind,
f) no rotation of the earth,
g) a particular selection of a few angles of departure.

The variation in range due to replacing one of these by a corresponding non-standard condition is called a differential variation. The *adjoint system** furnishes a method by which one single piece of numerical integration yields the differential variations from a given standard trajectory due to all kinds of non-standard conditions.

The general theory of this is as follows. Let x and y be the coördinates of the projectile at any instant t on a standard trajectory, and let $x+\xi$ and $y+\eta$ be its coördinates for the same value of t on a disturbed or non-standard trajectory. The differential equations of the disturbed trajectory will be

$$x'' + \xi'' = -(E + \Delta E)(x' + \xi') + \alpha_x,$$
$$y'' + \eta'' = -(E + \Delta E)(y' + \eta') - g + \alpha_y,$$

where ΔE is the change in E due to the disturbing cause as well as to the changes in the coördinates and components of velocity, and α_x and α_y are any new accelerations due to the disturbing cause, and where the initial conditions may also be altered. If the equations for the normal trajectory are subtracted from these and if all the variations are regarded as small quantities in the sense that the product of any two small quantities may be omitted, these equations become

$$\xi'' = -E\xi' - x'\Delta E + \alpha_x,$$
$$\eta'' = -E\eta' - y'\Delta E + \alpha_y.$$

If now ΔE is evaluated, it turns out to be a linear expression in ξ', η', and η.

5. Adjoint system. If then we include the identical relations $d\xi/dt = \xi'$ and $d\eta/dt = \eta'$, the equations for the variations may be seen to constitute a special case under the general form,

* *Cf.*, G. A. Bliss, Functions of lines in ballistics, Transactions of the American Mathematical Society, vol. 21, 1920, pp. 93–106.

$$
(5)\qquad
\begin{aligned}
\frac{d\xi}{dt} &= a_1\xi + b_1\eta + c_1\xi' + d_1\eta' + e_1,\\
\frac{d\eta}{dt} &= a_2\xi + b_2\eta + c_2\xi' + d_2\eta' + e_2,\\
\frac{d\xi'}{dt} &= a_3\xi + b_3\eta + c_3\xi' + d_3\eta' + e_3,\\
\frac{d\eta'}{dt} &= a_4\xi + b_4\eta + c_4\xi' + d_4\eta' + e_4.
\end{aligned}
$$

In the case considered many of the coefficients are zero. This system of linear first order equations has an adjoint system in four new variables λ, μ, ν, and ρ,

$$
(6)\qquad
\begin{aligned}
\frac{d\lambda}{dt} &= -a_1\lambda - a_2\mu - a_3\nu - a_4\rho,\\
\frac{d\mu}{dt} &= -b_1\lambda - b_2\mu - b_3\nu - b_4\rho,\\
\frac{d\nu}{dt} &= -c_1\lambda - c_2\mu - c_3\nu - c_4\rho,\\
\frac{d\rho}{dt} &= -d_1\lambda - d_2\mu - d_3\nu - d_4\rho.
\end{aligned}
$$

A well known theorem, which may be easily verified by direct substitution, states that any solution of the adjoint and any solution of the original system satisfy the relation,

$$
\frac{d}{dt}(\lambda\xi + \mu\eta + \nu\xi' + \rho\eta') = e_1\lambda + e_2\mu + e_3\nu + e_4\rho,
$$

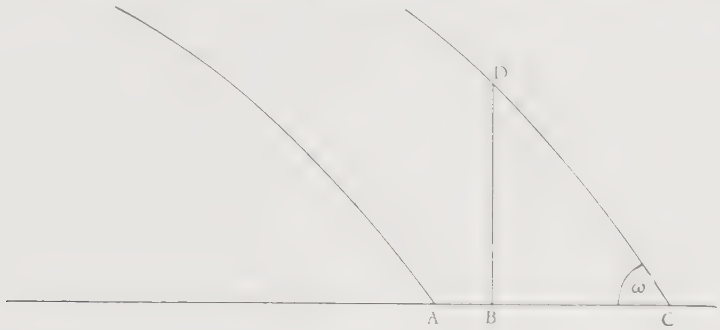
or

$$
(7)\qquad (\lambda\xi + \mu\eta + \nu\xi' + \rho\eta')\Big]_0^T = \int_0^T (e_1\lambda + e_2\mu + e_3\nu + e_4\rho)dt,
$$

where T is the time of flight to the point of fall. If now we select that particular solution of the adjoint system which has $\lambda=1$, $\mu=\cot\omega$, and $\nu=\rho=0$, when $t=T$, where ω is the angle of fall, the left side has a value at the upper limit equal to ΔX, the range variation on the level. In the figure, $\xi=AB$, $\eta=BD$, and $\Delta X=AC$. Equation (7) then becomes

$$
(8)\qquad \Delta X = \nu_0\xi_0' + \rho_0\eta_0' + \int_0^T (e_3\nu + e_4\rho)dt,
$$

where ξ_0' and η_0' denote the variations in the components of the muzzle velocity, and it is assumed that there is no variation in the position of the gun, so that $\xi_0=\eta_0=0$. The quantities e_1 and e_2 are always zero in this problem. Any variation is given by this formula for ΔX. For variations of the initial conditions, the integral vanishes; for non-standard forces the two outside terms vanish. In



any case, values of ν and ρ are required, which are a part of the solution of the adjoint system. If the trajectory is worked by numerical integration, the adjoint system may be solved by this method also; but the work proceeds backward along the trajectory from the point of fall to the muzzle, since the known conditions are at the point of fall.

6. **Relation to mechanical integration.** With the use of mechanical integration this method is not practical. For mechanical integration provides for the rapid finding of numerous solutions of one system of differential equations, but not for rapid work when there is a change in the differential equations themselves. Now the coefficients in the adjoint depend on the particular solution of the trajectory equations. Apparently then the use of the adjoint would have no value in connection with mechanical integration, where the variations are obtained by running independent trajectories and noting the actual differences.

Recently, however, an application* for it has been found which may prove useful even with mechanical integration. In any case, the adjoint admits of considerable simplification. The first equation in it becomes in the ballistic problem $d\lambda/dt=0$, so that $\lambda\equiv 1$. This reduces the system to the third order. Moreover, it is easy to show that the original system in the homogeneous case (where $e_3=e_4=0$) has a solution $\xi=x'$, $\eta=y'$. If now we have found two solutions of the homogeneous case by mechanical integration (that is, as differences) we get from these three solutions an algebraic solution of the adjoint by the identical relation (7) which here becomes $\xi+\mu\eta+\nu\xi'+\rho\eta'=\text{constant}$. If the two mechanical solutions are for muzzle velocity and angle, and are denoted by (ξ_v, η_v) and $(\xi_\theta, \eta_\theta)$, we have the following three equations:

$$
\begin{aligned}
x' + y'\mu - Ex'\nu - (Ey' + g)\rho &= 0, \\
\xi_v + \eta_v\mu + \xi_v'\nu + \eta_v'\rho &= \Delta_v X, \\
\xi_\theta + \eta_\theta\mu + \xi_\theta'\nu + \eta_\theta'\rho &= \Delta_\theta X,
\end{aligned} \tag{9}
$$

* Suggested by Dunham Jackson.

where the three constants on the right side are determined by considering the point of fall. These equations determine the particular solution of the adjoint completely; but it so happens that for two important applications only the muzzle values μ_0 and ν_0 are needed. This is because in these cases the indicated quadrature in (8) can be performed explicitly. In the case of a uniform range wind of velocity w_x, the integrand becomes $w_x(\nu'+1)$ and $\Delta X = \Delta_w X = w_x(T-\nu_0)$. In the case of uniform relative increase in density $\delta_0 H/H$, the integrand becomes $\mu'\delta_0 H/hH$, and

$$\Delta X = \Delta_H X = \frac{\delta_0 H}{hH}(\cot\omega - \mu_0).$$

The values of μ_0 and ν_0 are obtained by solving (9) with the particular muzzle conditons, $\xi_v = \eta_v = \xi_\theta = \eta_\theta = 0$, $\xi_v' = \Delta v_0 \cos\theta_0$, $\eta_v' = \Delta v_0 \sin\theta_0$, $\xi_\theta' = -v_0\Delta\theta_0 \sin\theta_0$, $\eta_\theta' = v_0\Delta\theta_0\cos\theta_0$. The use of these gives

$$\Delta_w X = w_x\left(T - \frac{\Delta_v X}{\Delta v_0}\cos\theta_0 + \frac{\Delta_\theta X}{v_0\Delta\theta_0}\sin\theta_0\right)$$

and

$$\Delta_H X = \frac{\delta_0 H}{hH}\left[\cot\theta_0 + \cot\omega - \frac{\Delta_v X}{\Delta v_0}\left(E_0\csc\theta_0 + \frac{g}{v_0}\right) - \frac{\Delta_\theta X}{\Delta\theta_0}\,\frac{g\cot\theta_0}{v_0^2}\right]. \tag{10}$$

The first of these is a familiar result which is commonly obtained by referring the motion to a set of axes fixed in the air. It has been in common use for wind effects. The density effect, however, is less familiar. Having derived it in this analytical way, we may seek a physical interpretation. To do this we may note that on a trajectory,

$$\frac{dv}{dy} = -E\csc\theta - \frac{g}{v}, \qquad \frac{d\theta}{dy} = \frac{-g\cot\theta}{v^2},$$

and $dH/dy = -hH$. This makes (10) become in the limit,

$$\frac{\partial X}{\partial H}\frac{dH}{dy} + \frac{\partial X}{\partial v_0}\frac{dv_0}{dy} + \frac{\partial X}{\partial\theta_0}\frac{d\theta_0}{dy} = -\cot\theta_0 - \cot\omega.$$

The two sides of this equation may each be interpreted as the change in range due to cutting off the trajectory at unit distance above the origin, the right side being obtained geometrically and the left side by means of the resulting changes in density, velocity, and angle. This interpretation shows how the rather surprising result is possible, that the effect of change of density can be expressed in terms of those of velocity and angle. The practical result anticipated is that density runs as well as wind runs may be omitted in preparing a firing table and the data derived from the effect of variation in angle and muzzle velocity. The improvement, however, is not yet in a practical stage since there is so far no provision for second order effects.

MATHEMATICAL EDUCATION

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This department of the MONTHLY *affords a place for the discussion of the place of mathematics in education, and other matters emphasizing the educational interests of those who teach mathematics. Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

AT WHAT LEVEL OF RIGOR SHOULD ADVANCED CALCULUS FOR UNDERGRADUATES BE TAUGHT?

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The above title is for good reasons in the form of a question—one on which experts differ. The opinions advanced below were developed in connection with a single year's experiment with a course at Reed College, where the writer was given free rein and encouragement by an experienced colleague, Professor F. L. Griffin. A somewhat unbalanced teaching background, consisting almost exclusively of freshman and sophomore courses on the one hand and some graduate courses on the other, may account for the writer's perhaps exaggerated sense of a chasm that needs bridging.

If we use the metaphor of a chasm, we might clarify the issues by raising three questions: First, is there really a chasm between the spirit of freshman and sophomore mathematics and the spirit of graduate mathematics? Second, if so, is the bridge to be made in the advanced calculus course? And if we agree to this, the third question is, how can the bridge be made?

As to the existence of the chasm, it may suffice to note that there is little in the usual freshman and sophomore work that would have seemed strange to an eighteenth century mathematician with his rough and ready methods, whereas it is hard to conceive of a good graduate course untouched by the renascence of rigor so largely characteristic of nineteenth century mathematics.

Consideration of a chasm between sophomore mathematics and graduate mathematics may seem irrelevant, since a large majority of students of advanced calculus will never take graduate mathematics. This brings us to our second question, namely, should the bridge be made in the advanced calculus course? On the answer to this, experienced American teachers disagree: One of the most respected and influential has in his advanced calculus text no mention whatever of uniform convergence and merely a remark on uniform continuity to the effect that it is a concept belonging to more advanced mathematics. An example at the other extreme is supplied by the authors of an excellent recent text, in which they not only prove Fourier's theorem under Dirichlet's conditions, but also prove Féjer's theorem on the summability of Fourier series. The writer's opinion (not that of an old practitioner, but possibly of some interest because he vividly recollects his own difficulties as a patient) is that the bridge should be made in this course—specifically, that advanced calculus students should be subjected to a rigorous treatment of the limit concept (that is, by the ϵ, δ method) since this concept is so fundamental for analysis, and furthermore, that the no-

tion of uniformity of a limit process should be developed, at least with regard to uniform convergence of series, and possibly, uniform continuity of functions.

This point of view may be argued by considering in turn the interests of the various types of students who enroll for advanced calculus. Consider first the small fraction who will go on to graduate mathematics. The psychological advantage of repetition should soften the austerity of the ϵ, δ method when it is encountered again in graduate work. Also, depending on the skill of the teacher and the relative homogeneity of the background of the students, it may be possible to "baby" them into the idea to an extent not desirable at the graduate level. For these students, some of the bridging may be made in other junior and senior courses, as in algebra and geometry, where some introduction to a postulational approach may be feasible, but there would seem to be little room for ϵ, δ except in the advanced calculus course.

Suppose we consider next those mathematics majors who will not go to graduate school. For these there may be some cultural benefit in getting a taste of rigor, or, to return to the metaphor of the bridge, in making at least a few trips from the domain of eighteenth century analysis onto the bridge to get a glimpse of what lies on the other side. This consideration, that mathematics majors should not be released from college in almost complete ignorance of the trend of analysis in the last century, applies also, with less force, to all others for whom advanced calculus constitutes a last course in mathematics.

Finally, there is the group who take the work for possible tool use in other fields. In the class at Reed College the majority of students were physicists and chemists; at other institutions there might also be engineers. Should these students be bothered with ϵ-analysis? A direct justification can be made as follows: Among the tool uses they may find for their mathematics, the use of infinite series is one of the most likely. It is conceivable that a student might some time be disturbed about the validity of integrating or differentiating a series. The simplest criteria involve uniform convergence. While ordinary convergence might be discussed without going beyond the intuitive notion of limit acquired in freshman and sophomore years; uniform convergence cannot be explained without the ϵ-idea. Since several advanced calculus texts, ostensibly written for engineers and physicists, include a treatment of uniform convergence, this would seem to be a not very radical program.

Before considering the last question, namely, how the bridge is to be made in the advanced calculus course, we might note that divorcing the second question, "Should we?" from the third question, "How?" is entirely too academic; practically, the decision as to doing it at all, depends to a great extent on how it can be done.

It is immediately evident that if a sufficiently wide territory is to be covered to meet the varying needs of the students, not all topics can be treated with the same degree of rigor. There are certain topics in which extreme rigor is simply not conventional; for example, in setting up surface or volume integrals for certain geometrical or physical quantities, the method of infinitesimals, while not

very rigorous, is nevertheless recommended by its efficiency; or again, in the transformation theory relating line integrals, surface integrals, and volume integrals, one rarely considers the most general curves or surfaces for which the transformations are valid. This carries over into the small amount of complex variable theory included in the course, for much of which Cauchy's integral theorem is basic, and the usual proof of which rests on Green's theorem for line integrals around closed curves. Other examples in which rigor is not customary are to be found in the elementary methods of integrating differential equations; in separating the variables, for instance, one wonders how often an engineer or physicist might get into trouble by not worrying about possible divisions by zero. The instructor is faced almost every day by questions of judgment and taste; if he expands on this topic he must sacrifice on that. On some topics such as these, an attempt to improve on the usual standards of carefulness was not deemed advisable by the writer.

A device which made more palatable the doses of rigor administered to the students was a more or less steady alternation of material of theoretical interest and of practical interest. The course lent itself to a presentation something like a cross between a sugar-coated pill (pill-coated sugar, the students thought) and a multi-decker sandwich, the alternate layers being somewhat better integrated than the comparison suggests. After generalizing the students' notion of function, we plunged into the ϵ, δ definition of limit as a basis for work on limits, continuity, and derivatives. Although this should logically have been preceded by a study of the real number system, it seemed pedagogically desirable to postpone a sketch of the Cantor theory until later in the year when the ϵ, δ methods were more familiar. The science students were reassured when after about three weeks we began working very practical-sounding problems on partial differentiation from various branches of physics. Such problems led naturally to the next bitter layer, implicit function theory.

The work on integration began with a brief introduction to the theory of the Riemann integral and led into the study of line, surface, and volume integrals, with their obvious physical applications. The little vector analysis that had been previously introduced in connection with the directional derivative was now developed further so that the transformation theory for integrals could be brought into very compact form. After this, a couple of weeks were spent on ordinary differential equations. Besides reviewing the elementary methods the students had had in the sophomore course, the theory of the Wronskian was developed and applied to initial value problems and the method of variation of parameters. Series solutions were postponed until after the work on complex variables.

The students now feeling that, if ϵ and δ were not old friends, at least they inspired less discomfort after a few months' acquaintance, it seemed worth while to take a week to outline some of the Cantor theory of the real number system. While most of this was charged against the account headed "Cultural," we did emerge with a theorem basic for our next topic, infinite series, namely,

that *a necessary and sufficient condition that a sequence,* $a_1, a_2, a_3, \cdots$, *have a limit is that it be convergent in the sense that for every positive* ϵ *there exist an* N *such that* $|a_m - a_n| < \epsilon$, *for* $m, n > N$. Any "proof" of this fundamental theorem not based explicitly on some theory of the real number system must depend on a little chicanery. Before tackling infinite series we made the next extension of the number system beyond the real numbers, with an eye to killing two birds (real and complex) with the same stone in much of the work on series. To make the students feel more at home in the field of complex numbers, the theory of complex variables was carried to the stage of defining analyticity and the elementary functions of a complex variable. Those few who had had a shoulder-shrugging or indulgent attitude toward questions of generalizing number systems and functions, and the enumerability of number systems, became eager pupils again when we studied the practical tests for convergence of series. It took only a few minutes to develop a few horrible examples sufficient to convince the most hard-boiled of such students of the necessity of care when manipulating series. The treatment of power series was simplified by the previous work on complex variables and in turn facilitated the later development of that subject. We returned to this after a little work on Fourier series, and developed some of the theory based on Cauchy's integral theorem, including Cauchy's integral formula, Taylor's series, the structure of analytic functions in the neighborhood of zeros and poles, evaluation of improper integrals by contour integration, and Laurent's series.

The considerable amount of work on power series engendered a confident attitude on the part of the students when we attacked the solution of ordinary differential equations by means of series. The various difficulties encountered in general were illustrated by a detailed study of the Bessel equation. A few meetings were spent solving some partial differential equations chosen to give rise to Fourier series and Bessel functions. The main purpose here was to give some idea of the technique of separating variables; questionable procedures were frankly pointed out but not justified. The course was concluded with a smattering of differential geometry and of the calculus of variations, the work on the latter centering on the Euler equations, which are by far the most important conditions in the physical applications.

Several favorable factors enabled us to carry out this apparently over-ambitious program: the class met four hours a week and understood that eight hours of outside work would be required; we were fortunate in finding a suitable text, so that the time that would otherwise have been consumed in taking notes could be devoted to further explanation of theory and problems; and the calibre of the students was a stimulus to the instructor.

One of the most valuable aids at this bridge-crossing stage is of course a generous use of graphical illustrations; for instance, the graphical interpretation of limit, continuity, uniform continuity, and uniform convergence, all these in the case of real variables. With complex variables, while the pictures based on the graph of the function break down, the circle diagrams are of some help.

It is almost impossible to overdo the business of easing the students' way into the rigorous methods. It is hard always to keep in mind that certain manipulations, as familiar to us as the binomial theorem, say manipulations of inequalities, like applications of $|A+B| \leqq |A|+|B|$, seem to be unfamiliar to them in concept as well as symbolism. This perhaps suggests some revisions in freshman and sophomore work, but that is another story. Again, the usual argument that since a certain quantity can be made less than any positive ϵ, it can hence be made less than $\epsilon/2$ or ϵ/M, seems to them subtle, and even dubious. Another common mode of reasoning which apparently is not immediately crystal clear to them is of the following type: If one statement is valid for $|x-a| < \delta_1$, and another for $|x-a| < \delta_2$, then if $\delta_1 \neq \delta_2$, they are both valid for $|x-a| < \delta$, where δ is the smaller of δ_1 and δ_2. In all these cases the students' acceptance of the general arithmetical argument was hastened by specific numerical and geometrical examples.

The following example will give an idea of the kind of approach we used to make a definition concrete: To prove $\lim_{x\to a} x^2 = a^2$ we first took a definite value for a, namely, $a=2$. We set a numerical ϵ, namely, $\epsilon=0.1$, and produced a δ which worked for this ϵ. We then repeated with $\epsilon=0.001$, and finally with an arbitrary positive ϵ, but still with $a=2$. After this we considered a general a. The arithmetical procedure was supplemented by graphical pictures. This succeeded fairly well, in fact one of the students offered a comment on the case for a general a which amounted to an appreciation of the non-uniform continuity of the function x^2 on an infinite interval, and this was before we had mentioned uniformity.

Whenever faced with the problem of deciding whether rigorous treatments should be attempted, or else better omitted as being appropriate only to a higher course, the writer tried to adhere to a principle of formulating all definitions with the greatest care, and proving, whenever feasible, at least those theorems having a very wide scope of application. Thus, on infinite series, we made the proofs of practically all our theorems on manipulation of series in general; much of this we utilized in a careful treatment of power series; but when we came to Fourier series we skipped the proof of Dirichlet's theorem, although we discussed the significance of this and some other theorems on Fourier series, and worked many problems on finding expansions in Fourier series. The students, for instance, were very much interested in the Gibbs phenomenon; we noted it qualitatively on some graphs, but did not undertake the proof. Sometimes there was a group of theorems, all of sufficient importance to deserve careful statement, but such that a proof of just one of them might suggest the general mode of attack; for example, the set of theorems on the limit of a sum, difference, product, and quotient, or the four fundamental theorems on real functions continuous on a closed interval. Occasionally it seemed excusable merely to mention the existence of certain theorems without even carefully stating them, in hopes of egging on a superior student or as a salve to conscience toward the rest; for instance, the existence of conditions in the calculus of variations other than the

Euler equations, or the existence of existence theorems in the theory of differential equations.

Our dilemma may be neatly summarized by the apt phrase Professor E. B. Wilson uses in the preface to his advanced calculus text: *rigor vs. vigor.* There are some situations where rigor may bolster vigor instead of undermining it; thus, if a student has had a rigorous treatment of power series, giving him command of a precise knowledge of the freedom with which they may be manipulated in various ways, such a student should not be incapacitated in comparison with one who uses power series naïvely. Realizing that this example is not entirely typical, and that the striving for rigor may be overdone to the extent of a stupid duplication of parts of more advanced courses far over the students' heads, the writer still maintains that the attitude of the advanced calculus teacher should not be such a straining for vigor that rigor is completely neglected, but a searching for some synthesis in this difficult teaching problem.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to a l forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

ON THE CIRCLES OF PAPPUS*

V. THÉBAULT, Le Mans, France

1. Let A, B, C be three points on a straight line, C lying between A and B. (See accompanying figure.) On the same side of the line describe the semicircles (O), (O_1), and (O_2) with diameters $AB=2a$, $AC=2b$, and $BC=2c$, $(b>c)$. Consider the circle (ω_1) of radius ρ_1, tangent to the three semicircles; then the circles (ω_2), (ω_3), $\cdots$, (ω_n), of radii ρ_2, ρ_3, $\cdots$, ρ_n, all tangent to the semicircles (O) (O_1) and each one tangent to the preceding circle. Such circles are called *circles of Pappus*. (*Collected Mathematical Works*, Theorem XV, Book IV.) Let y_1, y_2, $\cdots$, y_n be the ordinates of the centers of the circles measured from the line AB.

THEOREM. *A necessary and sufficient condition that the line of centers of two of the circles* (ω_1), (ω_2), $\cdots$, (ω_n) *be parallel to the line* AB, *is that the length of the segment* AB *be equal to* k *times that of* BC, k *being an integer.*

In a recent note (this MONTHLY, vol. 47, 1940, pp. 19–24), M. G. Gaba has shown that this condition is sufficient. We shall prove that it is necessary.

A transformation of the figure by the inversion $(A, 4ab)$ enables one to arrive at the formulas†

* Translated from the French by T. Hailperin, Cornell University.

† A particular case of a more general relationship which we have derived; see L'Enseignement Mathématique, vol. 34, 1935, p. 316.

$$\rho_n = \frac{bc(b+c)}{b^2+n^2c^2+bc}, \quad y_n = 2n\rho_n. \tag{1}$$

A necessary and sufficient condition that the line of centers $\omega_p\omega_q$ of the two circles (ω_p) and (ω_q), $(1 \leqq p \leqq n,\ 1 \leqq q \leqq n)$, be parallel to AB is that $y_p = y_q$ or $p\cdot\rho_p = q\cdot\rho_q$; in virtue of (1), this leads to the equation

$$(b/c)^2 + (b/c) - pq = 0$$

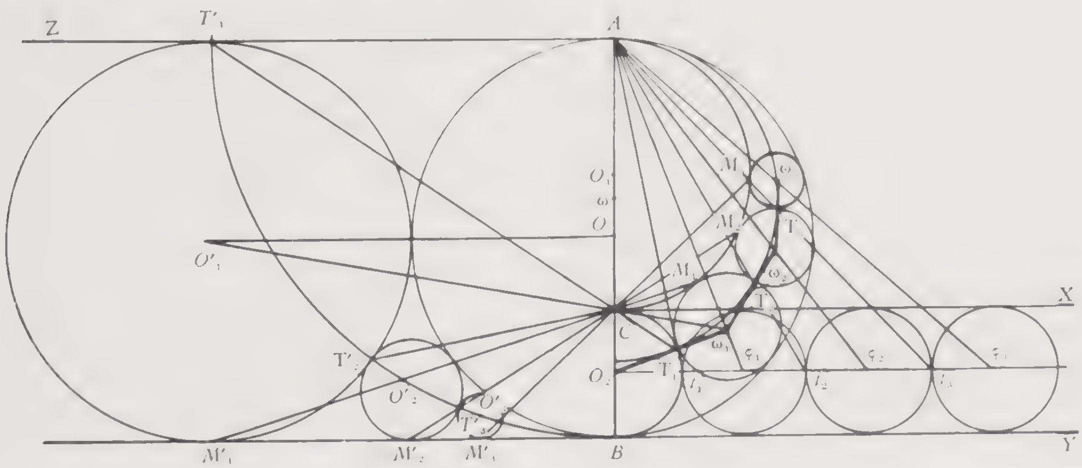
whose roots are not real unless $4pq+1 = x^2$. Since x is an odd integer, we must have

$$pq = \left(\frac{x+1}{2}\right)\left(\frac{x-1}{2}\right) = k(k+1),$$

and

$$b/c = k \quad \text{or} \quad -(k+1).$$

For each decomposition of the number $k(k+1)$ into a product of two factors, there corresponds a line $\omega_p\omega_q$ parallel to AB.



2. By the inversion $(A, 4ab)$, the circles (ω_1), (ω_2), $\cdots$ are transformed into circles (ϕ_1), (ϕ_2), $\cdots$ equal to (O_2), tangent to the lines CX, BY which are perpendicular to BC at C, B; and each of the circles is tangent to the succeeding one. The points of contact t_1, t_2, $\cdots$, of the circles (O_2) and (ϕ_1), (ϕ_1) and (ϕ_2), $\cdots$, are transforms of the points of contact T_1, T_2, $\cdots$ of (O_2) and (ω_1), (ω_1) and (ω_2), $\cdots$. The points T_1, T_2, $\cdots$ are on a circle (ω), the transform of the line $\delta \equiv (t_1, t_2, \cdots)$, which has its center on AB and a radius $\rho = 2ab/(a+b)$. This property holds no matter what the relation of b to c.

3. On transforming the original figure by the inversion $(C, -4bc)$, the circles (ω_1), $\cdots$, (ω_n) become the circles (O_1'), $\cdots$, (O_n'), each one tangent to its succeeding one and all tangent to the circle (O) and the line BY below AB. These

circles remain fixed when C varies between A and B, and the transforms $T_1', \cdots, T_n'$ of $T_1, \cdots, T_n$ are fixed points. It results from this that the points $T_1, \cdots, T_n$ describe arcs of fixed circles which are circumscribed about the triangles $ABT_1', \cdots, ABT_n'$, the arcs lying above AB.

In the same inversion, the points of contact $M_1, \cdots, M_n$ of the circles $(\omega_1), \cdots, (\omega_n)$ with the variable circle (O_1)—which points are transformed into points $M_1', \cdots, M_n'$ coincident with the points of contact of the circles $(O_1'), \cdots, (O_n')$ with BY—describe arcs of fixed circles circumscribed about the triangles $ABM_1', \cdots, ABM_n'$, the arcs lying above AB.

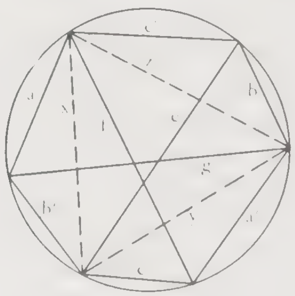
4. Under the same hypothesis that C varies between A and B, the center ω_n of the circle (ω_n) describes an arc of an ellipse passing through A and B, lying above AB, with focus at O and eccentricity $c=1/2n$. The directrix of this ellipse is parallel to AB at a distance of $2na$.*

ON THE CONVERSE OF A THEOREM BY FUHRMANN

H. F. Sandham and V. W. Graham, Dublin

Let the opposite sides of a convex hexagon inscribed in a circle be a, a'; b, b'; c, c'; and let the principal diagonals be e, f, g (so chosen that a, a', e have no common vertex, *etc.*). Then†

$$efg = aa'e + bb'f + cc'g + abc + a'b'c'. \tag{1}$$



This theorem of Fuhrmann follows at once from Ptolemy's theorem. Considering the various quadrilaterals inscribed in the circle, we have

$$-ez + c'y + bx = 0, \qquad cz - fy + a'x = 0, \qquad b'z + ay - gx = 0;$$

whence

$$\begin{vmatrix} -e & c' & b \\ c & -f & a' \\ b' & a & -g \end{vmatrix} = 0,$$

which reduces to the given condition.

* A particular case of a general property which we have given; *loc. cit.*, p. 319.

† See R. A. Johnson, Modern Geometry, p. 65.

It is remarkable that this simple condition is sufficient to insure that the six vertices of a hexagon lie on a circle. Writing the condition in the form

$$g(ef - cc') = aa'e + bb'f + abc + a'b'c',$$

we see that $ef - cc'$ is positive. Similarly, $fg - aa'$ and $eg - bb'$ are positive. Now by Ptolemy's theorem we have

$$-ez + c'y + bx \geqq 0, \tag{2}$$

$$cz - fy + a'x \geqq 0, \tag{3}$$

$$b'z + ay - gx \geqq 0. \tag{4}$$

From (2), we have $(c'y+bx)/e \geqq z$; therefore, from (3), $(bc+a'e)x/(ef-cc') \geqq y$, since $ef-cc'$ is positive. Combining this with (4) gives us

$$aa'e + bb'f + cc'g + abc + a'b'c' \geqq efg. \tag{5}$$

The equality sign holds in (5) only if it holds in each of (2), (3), and (4). Hence if the condition (1) is satisfied we have three cyclic quadrilaterals such that each has three points in common with one of the other two. They are therefore all inscribed in the same circle, or the six vertices of the hexagon are concyclic.

Note by the Editor. Professor Johnson has pointed out that this proof of the converse theorem needs no assumption of the convexity of the hexagon, for the inequalities (2), (3), and (4) are true even if the quadrilaterals in question are not convex. However, the proof establishes that the equality sign holds in each case, and so the quadrilaterals are necessarily convex; it is then easy to show that the hexagon is also convex. Thus the complete result is that if

$$efg = aa'e + bb'f + cc'g + abc + a'b'c',$$

the hexagon is convex and its vertices are on a circle.

R. J. W.

A FORMULA FOR REPEATED INTEGRATION BY PARTS

F. L. MANNING, Ursinus College

Many problems in elementary calculus using integration by parts require that the formula be applied several times. Typical of this kind of problem are $\int x^n \sin x\, dx$, and $\int x^n e^x dx$. Integrals of this type are easily evaluated by the following general formula:

$$\begin{aligned}\int f(x)g(x)dx &= f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - \cdots \\ &\quad + (-1)^n f^{(n)}(x)g_{n+1}(x) + (-1)^{n+1}\int f^{(n+1)}(x)g_{n+1}(x)dx.\end{aligned} \tag{1}$$

Here $f'(x), f''(x), \cdots$ are the successive derivatives of $f(x)$, and $g_1(x), g_2(x), \cdots$ are the successive integrals of $g(x)$; that is,

$$f^{(i+1)}(x) = \frac{d}{dx} f^{(i)}(x), \qquad \text{and} \qquad g_{i+1}(x) = \int g_i(x)dx.$$

If $f(x)$ is a polynomial of degree n, then $f^{(n+1)}(x) = 0$, and the last term of (1) is a constant. Hence if $g(x)$ is of such a form that its successive integrals are easily found, the value of $\int f(x)g(x)dx$ may be written down in one step.

Formula (1) is obtained by successive integration by parts. Thus

$$\begin{aligned}\int f(x)g(x)dx &= f(x)g_1(x) - \int f'(x)g_1(x)dx \\ &= f(x)g_1(x) - f'(x)g_2(x) + \int f''(x)g_2(x)dx \\ &= \cdots .\end{aligned}$$

This formula is not given in elementary texts but might well be introduced because it is easy to remember, works well with many text-book problems listed under this topic, and seems to help the students avoid some of the algebraic complications that bother underclassmen when using the ordinary formula.

PROOFS OF VECTOR PRODUCT EXPANSIONS BY DETERMINANTS

J. S. Frame, Brown University

The proofs of certain vector product expansions can be greatly simplified by the use of an elementary theorem on determinants which is embodied in the formula

$$(1) \qquad |\mathbf{ABC}| \cdot |\mathbf{DEF}| = \begin{vmatrix} \mathbf{A}\cdot\mathbf{D} & \mathbf{A}\cdot\mathbf{E} & \mathbf{A}\cdot\mathbf{F} \\ \mathbf{B}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{E} & \mathbf{B}\cdot\mathbf{F} \\ \mathbf{C}\cdot\mathbf{D} & \mathbf{C}\cdot\mathbf{E} & \mathbf{C}\cdot\mathbf{F} \end{vmatrix},$$

where we define the scalar product of two vectors and the determinant of three vectors by the usual formulas

$$(2) \qquad \mathbf{A}\cdot\mathbf{B} = a_1b_1 + a_2b_2 + a_3b_3,$$

$$(3) \qquad |\mathbf{ABC}| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

If $|\mathbf{A}|$ and $|\mathbf{B}|$ are the lengths of the two vectors $\mathbf{A}$ and $\mathbf{B}$, and $(\mathbf{A}, \mathbf{B})$ is the angle between them, then by applying the law of cosines to a triangle whose sides are $|\mathbf{A}|$, $|\mathbf{B}|$, and $|\mathbf{A}-\mathbf{B}|$, and expanding the product $(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B})$, we obtain the formula

$$(4) \qquad \mathbf{A}\cdot\mathbf{B} = |\mathbf{A}| \cdot |\mathbf{B}| \cos(\mathbf{A}, \mathbf{B}),$$

which will be required below.

We define the components of the vector product $\mathbf{P}$ of two vectors $\mathbf{A}$ and $\mathbf{B}$ in the usual manner, namely,

$$\text{(5)}\quad \mathbf{P} = \mathbf{A} \times \mathbf{B}, \quad \text{if} \quad p_1 = a_2b_3 - a_3b_2, \quad p_2 = a_3b_1 - a_1b_3, \quad p_3 = a_1b_2 - a_2b_1.$$

Then if $\mathbf{V}$ is an arbitrary vector, we may write the equations

$$\text{(6)}\qquad \mathbf{V}\cdot\mathbf{P} = \mathbf{V}\cdot\mathbf{A}\times\mathbf{B} = |\mathbf{VAB}|,$$

$$\text{(7)}\qquad \mathbf{A}\cdot\mathbf{P} = \mathbf{B}\cdot\mathbf{P} = 0,$$

$$\text{(8)}\qquad \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{D} = \mathbf{P}\cdot\mathbf{C}\times\mathbf{D} = |\mathbf{PCD}| = \mathbf{P}\times\mathbf{C}\cdot\mathbf{D} = (\mathbf{A}\times\mathbf{B})\times\mathbf{C}\cdot\mathbf{D},$$

which follow immediately from the definitions (2), (3), (5) and from the rules for expanding determinants. Applying (6), (8), (1), and (7), we then have

$$\text{(9)}\qquad \begin{aligned}(\mathbf{V}\cdot\mathbf{P})(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{D}) &= |\mathbf{VAB}|\cdot|\mathbf{PCD}| \\ &= \begin{vmatrix} \mathbf{V}\cdot\mathbf{P} & \mathbf{V}\cdot\mathbf{C} & \mathbf{V}\cdot\mathbf{D} \\ \mathbf{A}\cdot\mathbf{P} & \mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D} \\ \mathbf{B}\cdot\mathbf{P} & \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D} \end{vmatrix} = \begin{vmatrix} \mathbf{V}\cdot\mathbf{P} & \mathbf{V}\cdot\mathbf{C} & \mathbf{V}\cdot\mathbf{D} \\ 0 & \mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D} \\ 0 & \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D} \end{vmatrix}.\end{aligned}$$

Since $\mathbf{V}$ may be chosen so that $(\mathbf{V}\cdot\mathbf{P})\neq 0$, except when $\mathbf{P}=\mathbf{A}\times\mathbf{B}=0$ and both factors on each side of equation (9) vanish, we may cancel the factor $\mathbf{V}\cdot\mathbf{P}$ from both sides of (9) and obtain the identity

$$\text{(10)}\qquad \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}\times\mathbf{D} = \begin{vmatrix} \mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D} \\ \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D} \end{vmatrix}.$$

By (8) this identity may be written in the form

$$\text{(11)}\qquad (\mathbf{A}\times\mathbf{B})\times\mathbf{C}\cdot\mathbf{D} = [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}]\cdot\mathbf{D},$$

which holds for all vectors $\mathbf{D}$. Hence we obtain the identity

$$\text{(12)}\qquad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}.$$

A special case of (10), combined with (4), gives us the formulas

$$\text{(13)}\qquad |\mathbf{A}\times\mathbf{B}|^2 = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2 = |\mathbf{A}|^2\cdot|\mathbf{B}|^2[1-\cos^2(\mathbf{A},\mathbf{B})],$$

$$\text{(14)}\qquad |\mathbf{A}\times\mathbf{B}| = |\mathbf{A}|\cdot|\mathbf{B}|\sin(\mathbf{A},\mathbf{B}).$$

The identities (10), (12), and (14), which we have just proved, embody the chief properties of the vector product which are not trivial consequences of the definition (5).

Formula (1) may also be used to show that the determinant of three mutually perpendicular vectors is equal to plus or minus the volume of the rectangular parallelopiped built on these vectors as edges; for if $\mathbf{A}\cdot\mathbf{B}=\mathbf{B}\cdot\mathbf{C}=\mathbf{C}\cdot\mathbf{A}=0$, and we set $\mathbf{D}=\mathbf{A}$, $\mathbf{E}=\mathbf{B}$, $\mathbf{F}=\mathbf{C}$ in (1), then we have

$$\text{(15)}\qquad \begin{aligned} |\mathbf{ABC}|^2 &= (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{C}), \\ \pm|\mathbf{ABC}| &= |\mathbf{A}|\cdot|\mathbf{B}|\cdot|\mathbf{C}| = \text{volume of parallelopiped of } \mathbf{A}, \mathbf{B}, \mathbf{C}.\end{aligned}$$

Since the addition of a multiple of **A** to the vector **B**, and the addition of multiples of **A** and **B** to the vector **C**—so chosen as to reduce an arbitrary trihedral of vectors **A**, **B**, **C** to a rectangular trihedral—does not change either the determinant or the volume of the parallelopiped, it follows that these two quantities are equal numerically for each set of three vectors **A**, **B**, and **C**.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department at the Mathematical Association of America, 531 West 116th St., New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

A Detailed Proof of the Chi-Square Test of Goodness of Fit. Harvard Phi Beta Kappa Prize Essay for 1939. By E. R. Greenhood, Jr. Cambridge, Harvard University Press, 1940. 16+61 pages.

Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments. New York, Federal Works Agency, Work Projects Administration for the City of New York, 1939. 405 pages.

Mathematics of Accounting and Finance. By C. H. Langer and T. B. G. Gill. Chicago, Walton Publishing Company, 1940. Part I, 26+114+8 pages. Part II, 19+pp. 497 to 974+64 pages of tables +pp. 115–284+9 pages.

Handbook of Mathematical Tables and Formulas. Second edition. Compiled by R. S. Burington. Sandusky, Ohio, Handbook Publishers, Inc., 1940. 275 pages.

Plane and Spherical Trigonometry. Second edition. By L. M. Kells, W. F. Kern, and J. R. Bland. New York and London, McGraw-Hill Book Company, Inc., 1940. 15+401 pages. $2.00.

A Study of the Number Concept of Secondary School Mathematics. By H. F. Fehr. (Dissertation, Teachers College, Columbia University, 1940.) Upper Montclair, New Jersey, State Teachers College, 1940. 202 pages. $1.60.

REVIEWS

Integration of Ordinary Differential Equations. By E. L. Ince. (University Mathematical Texts.) Edinburgh and London, Oliver and Boyd, 1939. 8+148 pages. 4/6s.

This small text contains a somewhat more extended list of formal methods of solving ordinary differential equations than the average book of its type. At the same time, these methods are clearly presented and well illustrated, and at many points the reader is given more of a glimpse into the interrelations and background of the methods than in other books. Since the object of the text is to enable one to write down formally the solution of a differential equation, existence theorems and other questions of rigor are omitted. In spite of this, the

reader gets the general impression that an effort has been made to say exactly what is meant. The author is not afraid of absolute values, and, in particular, he uses logarithms of absolute values in integration formulas to avoid imaginary constants. The book is purely mathematical with no physical applications either in the text or the problems. However, the geometric meaning of a differential equation is considered at some length, and numerous geometric problems are given.

The work is divided into six chapters under the following titles: I, Equations of the First Order and Degree; II, Integral Curves; III, Equations of Higher Degree; IV, Equations of the Second and Higher Orders; V, Linear Equations; VI, Solution in Series. These chapters cover rather thoroughly the methods usually given under these heads. The following additional features should be noted. The Riccati equation is included in Chapter I, the general solution being obtained in terms of one particular solution. The form of the solutions in the neighborhood of a singular point is discussed in Chapter II, and the linear fractional equation is studied in detail. A three-dimensional interpretation of the differential equation is given in Chapter III, and the p-discriminant, singular solutions (envelopes), cuspidal loci and tac-loci arise naturally. (It is assumed that the student is already familiar with the envelope of a family of curves.) In Chapter V operational theory is developed so far that the linear equation with first degree coefficients can be solved in terms of definite integrals. The last chapter introduces the student to the indicial equation and the hypergeometric, Legendre, and Bessel functions.

The 300 problems are all put at the end of the book so that the student will have to identify the type as well as solve the problem. This, of course, necessitates a careful choice by the teacher of certain problems to be done when each individual method is studied.

R. H. Cameron

Differential and Integral Calculus. By R. R. Middlemiss. New York, McGraw-Hill Book Co., 1940. 10+416 pages. \$2.50.

This text covers the traditional material of a five-hour, two-semester, first course in calculus. The indefinite integral is introduced about a third of the way through. Duhamel's principle is stated and is used in applications to geometry and mechanics. While the treatment is satisfactory for the most part, there are some passages of exceptionally good exposition and some that are careless and inadequate.

No concise definition of function is given; the first paragraph, if so intended, applies only to single-valued functions defined for all values of the independent variable. On page 5, the increment of the independent variable is defined as "small"; this is unfortunate, and is often repeated.

The discussion of maxima and minima is unsatisfactory. The statement on page 66, that "ordinarily" the second derivative is positive at a minimum point and negative at a maximum point, is misleading since the author evidently con-

siders the example $y=x^4$ sufficiently "ordinary" to use it, in another connection, on the same page.

It is inconsistent to devote seven pages to a review of trigonometry, including definitions of the trigonometric functions on page 69, while assuming on page 273 that the student will recall from analytic geometry that the coefficients of $Ax+By+Cz+D=0$ are direction numbers of the normal to the plane represented by the equation. On page 114, the vector concept and the parallelogram law are used without explanation.

On the other hand, Chapters II and XII, Limit of a Function, are very good. The latter contains an exceptionally clear treatment of L'Hospital's theorem. There is a good discussion of order of infinitesimals in Chapter XIII, and throughout the text the applications to geometry and mechanics are well handled.

In addition to familiar exercises, a number of problems, designed to make the student think about fundamental concepts, are offered; this is a step in the right direction.

Chapters XXVI and XXVII, Infinite Series and Expansion of Functions, are excellent. The traditional proof of Taylor's theorem often puzzles the beginner; he follows the logic, but the arbitrary fashion in which the equation defining the remainder is usually introduced smacks of pulling a rabbit from a hat. The exceptionally good presentation of this topic in the present text makes the proof of Taylor's theorem seem natural and plausible.

The page appearance is good; there are wide margins and, except for fractional exponents, the type is clear. No misprints were found. The isometric drawings are to be commended, but a few of the diagrams leave something to be desired. There is an index, but its usefulness is impaired by not being in the customary place; a set of answers to problems comes after the index.

E. J. PURCELL

A Detailed Proof of the Chi-Square Test of Goodness of Fit. Harvard Phi Beta Kappa Prize Essay for 1939. By E. R. Greenhood, Jr. Cambridge, Harvard University Press, 1940. 13+61 pages.

Each year the Harvard Chapter of the Phi Beta Kappa Society publishes an honor's thesis. This is the sixth of these publications and, incidentally, the first one to deal with a topic from mathematics. It is an exceedingly entertaining little monograph, and for this reason, and others, is highly recommended to all teachers of mathematics, even to those whose courses do not include one in statistics.

Too often when one "gets" a principle or theorem in mathematics the initial strangeness is lost forever, the thing acquires a new aspect altogether, and when one has used and taught the principle repeatedly it becomes difficult to sympathize with the groping of the untutored. What seems so clear to us should be equally evident to all. It is useful, therefore, to try to see our subject again as

something new and strange, as it looked before we hit upon the key to its inner being.

Mr. Greenhood has by no means forgotten his own difficulties in mastering "chi-square" and its distribution, and he strives to elucidate in a fashion that will be adequate for anyone interested who has "had as a background a course or two of calculus plus an introductory course in probability, but no more." He attempts to anticipate and to obviate all the perplexities and pitfalls that might arise for such a reader. He describes his own intellectual peregrinations through a new domain. Indeed, the author does more; for having found in the literature no single derivation of the distribution function which he, himself, could easily understand in its entirety (though he makes acknowledgment to some which are close and from which he borrows), he has constructed a derivation of his own. This proof is clear to him, and must therefore be simple enough to be "definitely within ordinary comprehension"! Finally, he adds to this a discussion of the uses and limitations of the chi-square test, with critical comments on several related papers.

Mr. Greenhood is to be congratulated for his achievement. He has made an extensive and thoughtful excursion into the literature of his subject, and he has emerged with a well-rounded, coherent account of his adventures and a guide book for those who wish to follow. It would be well to have other books like it.

A. S. HOUSEHOLDER

The Mathematics of Business. By H. E. Stelson. Boston, Houghton Mifflin Company, 1940. 14+464 pages. $2.50.

The book deals largely with the applications of percentage, simple and compound interest, and insurance to a variety of problems in domestic economy and financial transactions. For the young student, it might well serve not only as a text on arithmetic, but also as a lucid introduction to the terminology and actual practices in some branches of finance, real estate, and governmental agencies.

The greater part of the contents may be understood by a student with no training in algebra. There are chapters on logarithms, powers and roots, as well as on arithmetic and geometric progressions. In a chapter on statistics, the author deals briefly with the arithmetic mean, the median and mode, and methods of graphical representation of data. The formulas of compound interest, the sum and present value of an annuity, are fully derived, and the usual problems in percentage, bank discount, trade discount, and related topics are amply covered. There is a most pleasing and timely chapter on installment buying, and on the various small loan agencies. The chapters on insurance deal in an elementary way with the principles of ordinary life, term, and endowment policies; group insurance, Federal and private old age insurance, annuities, automobile, and fire insurance. There is an abundance of illustrative problems throughout the text, with many exercises. Answers are provided to all odd-numbered problems. The book contains tables of logarithms, compound interest, annuities, and a table

of powers and roots. In addition, there are many illustrative diagrams and short tables in the main body of the book.

The text suffers here and there from inadequate treatment of topics which depend, in essence, on a broader mathematical background. In chapter one, for instance, the important topic of approximate numbers is treated in a paragraph. A *rule* (made plausible by an illustration) is given for the number of digits to be retained in the product of two numbers. Only three pages are devoted to the theory of probability, in relation to life insurance, and the exercises are made to depend on "hints" giving the assumed theory. The chapter on finding the square root of a number may be pedagogically intelligible, but integers are badly confused with "numbers" throughout; the entire section deserves revision. In the same chapter, the method of (linear) interpolation is explained, and the student is cautioned that "the process of interpolation will not in general give satisfactory results . . . for more than one extra digit" (beyond the tabulated argument, presumably). That, of course, depends on the table, and does not happen to be true for the two examples of direct and inverse interpolation given by the author.

For the most part, however, the book is clearly written and the style is agreeable on the whole, especially if one is in sympathy with the author's premise, page 63, that "a thorough knowledge of investments is a large factor in good citizenship." The following typographical or editorial errors were noted: page 256, in the line before the last, read B/n for B/r; page 68, Exercise 1 is probably misstated.

GERTRUDE BLANCH

The Physical Sciences. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, Inc., 1940. 17+754 pages. $5.00; to schools, $3.75.

This book, which contains a minimum of mathematics, is a suitable text for a survey course covering the fields of physics, chemistry, meteorology, geology, and astronomy. With a few exceptions the text is thoroughly up-to-date. The historical concept of magnetic poles is used to discuss magnetic forces rather than the modern notion that magnetic forces are primarily forces between moving charges. The Lewis-Langmuir octet theory of atomic structure is prominently described to the exclusion of the modern statistical "electron cloud" picture. The definition given for the center of gravity of an object is misleading, since no mention is made of the assumption of a system of parallel gravitational forces. The format, typography, and illustrations of the text are excellent. Exercises and reading references are given for each chapter.

W. E. BLEICK

Introductory Business Mathematics. By J. S. Georges and W. H. Conley. New York, Henry Holt and Co., 1940. 10+326 pages. $2.40.

This book is an effort to provide practical mathematical training for students in business departments of junior colleges who seek employment in stores, offices, and industrial institutions on completion of not more than four semesters attendance. The text is divided into three sections, the first dealing with arith-

metic and with arithmetic methods, the second with algebraic methods, and the third with graphical methods.

Included in the first section are such topics as sales records, pay rolls, bank statements, interest, and insurance. Systems of linear equations, quadratic equations, and the concept of a function are discussed in the second section. There is also a chapter on logarithms and the use of the slide rule which is particularly well done. Frequency distributions, histograms, broken line and functional graphs receive consideration in the final section. The concluding chapter, dealing with the construction and use of nomograms, is a feature of this book not generally found in texts of this type.

Definitions are clearly indicated, numerous proofs and derivations are given, and each theorem or rule is illustrated with one or more examples. There are 240 such illustrative examples in the book. Each new topic is followed with a series of questions to stimulate discussion, as well as a wide variety of problems. A selection of problems for a cumulative review follows each chapter. Tables for simple and compound interest and common logarithms are provided.

The reviewer detected several unimportant misprints, and an error in an illustrative example on page 41 which may be rather confusing to the less discerning student. The book, though elementary in nature and limited in scope, should prove a valuable addition to the texts in the field for which it is designed.

W. D. WRAY

Textes Mathématiques Babyloniens. Edited and translated by F. Thureau-Dangin. Leiden, E. J. Brill, 1938. 40+243 pages. 25 guilders.

This book is meant to be the first of a series entitled "Ex Oriente Lux" published by La Société Orientale Ex Oriente Lux. Thureau-Dangin is considered one of the world's greatest authorities on the Sumerian language. The present reviewer, who is not an Assyriologist, will give here only a factual survey of this scholarly work. As stated in the introduction, this arrangement of Babylonian mathematical texts has been prepared by the author with the hope that it will assist the mathematicians in ascertaining the debt which their science, especially algebra, owes to the Babylonians. Instead of presenting the material in its original cuneiform text the author has transcribed the ideograms into their phonetic equivalents.

The texts are preceded by an introduction of forty pages in which the author outlines the general characteristics of Babylonian mathematics. He includes some explanation of the sexagesimal system, brief tables of weights and measures, and a detailed analysis for several problems from the text proper. This analysis shows that the Babylonians had general solutions for first and second degree problems. There existed tables for the sum of the cube and square of numbers which aided the solution of certain third degree problems. Some third degree problems were solved by trial. Thureau-Dangin presents his analysis with the classical algebraic notation, but he is careful to note that the Babylonians had no such tool to assist them in the manipulation of algebraic ideas.

Following the introduction are five chapters in which the actual texts are

grouped according to the archeological collections from which they originate. Chapter one contains texts from the British museum; two from the museum at the Louvre; three from the University of Strasbourg; four from the Berlin museum; and five from Yale University. The texts within each of these groups are identified with three distinct periods of Babylonian history: (1) the First Dynasty of Babylon, about 2200–1900 B.C.; (2) the Middle Babylonian Period, about 2000–1930 B.C.; (3) the Seleucid Era, 312–65 B.C.

The texts all consist of a collection of problems. Each problem is given first in its phonetic form and then followed by a French translation. The problems are all of the same form. The statement of the problem is given by the scribe who is writing in the first person. The explanation of the solution is usually expressed in the imperative, but is sometimes given in the second person. Many of the problems deal with practical land measurements. All the examples are stated in geometrical terminology.

In an appendix the author lists a number of texts to which he has referred in preceding footnotes. Some of these texts are transcribed and given in the same manner as those in the first five chapters of the book. However, most of these texts are merely identified by proper references and not actually reproduced. At the end of the book is a vocabulary which contains: (1) the meanings of mathematical terms used by the Babylonians; (2) the phonetic names of ordinal and cardinal numbers; and (3) a list of ideograms and their phonetic equivalents.

ROBERTA F. JOHNSON

Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments. New York, Federal Works Agency, Work Projects Administration for the City of New York, 1939. 405 pages.

Among the projects supported by the Federal Works Agency was the compilation of numerous mathematical tables. A committee of well known mathematicians was first selected to choose the particular tables to be calculated, an administrative supervisor appointed, and a technical director selected to direct the actual work undertaken by a staff of six computers. The whole undertaking was sponsored by the Bureau of Standards.

The present volume includes the circular and hyperbolic sines and cosines, to nine places of decimals, of the argument in radian measure, from 0 to two radians, at intervals of 1/10000. A short introduction explains the method of compilation and the controls employed for attaining a high degree of accuracy.

The tables are printed in large type, fifty entries per page on a 6 by 9 block, with generous spacing, making them particularly easy to use.

A few supplementary tables follow, including one with argument at intervals of 1/10 radian from 0 to ten radians, one converting radians to degrees, for various small intervals, to six places of decimals.

The method of reproduction of the typewritten text is, except for the title page, a photo-offset process that provides a clear and legible page.

VIRGIL SNYDER

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N.J.

ANNUAL REPORTS, 1939–1940

NATIONAL HONORARY MATHEMATICS FRATERNITIES

Each year we receive a number of requests regarding membership in and activities carried on by national honorary mathematics fraternities. For that reason, we requested Professor W. E. Milne, Director-General of *Pi Mu Epsilon* and Professor C. V. Newsom, President Pythagoras of *Kappa Mu Epsilon* to submit a statement of the aims of the respective national organizations and a report of the new chapters becoming affiliated with them during the past year.

PI MU EPSILON

Pi Mu Epsilon is a national academic fraternity devoted to the promotion of scholarship and interest in mathematics among students of university grade. The individual chapters endeavor to accomplish this purpose through the election of student members on a basis of scholarship, and also by meetings, addresses, and papers on suitable topics, together with other activities intended to arouse and maintain a real interest in the field of mathematics. The very excellent programs which have been published from time to time in this MONTHLY give testimony of the success with which many chapters are fulfilling their purpose.

The national fraternity exists primarily for two reasons: (1) To assist and strengthen the local chapters by giving them the backing, prestige, and unity of policy that results from membership in a nation-wide organization; and (2) To consider applications for new chapters, and, upon favorable vote of a majority of chapters, to admit qualified groups as new chapters of the fraternity.

The following national officers were elected for a term of three years beginning April 1, 1939:

Director-General: Professor W. E. Milne, Oregon State College, Corvallis, Oregon
Vice-Director-General: Professor Lincoln La Paz, Ohio State University, Columbus, Ohio
Secretary-General: Professor J. S. Gold, Bucknell University, Lewisburg, Pennsylvania
Councillors-General:
Professor H. H. Downing, University of Kentucky, Lexington, Kentucky
Professor W. W. Elliott, Duke University, Durham, North Carolina
Professor G. C. Evans, University of California, Berkeley, California
Professor R. A. Johnson, Brooklyn College, Brooklyn, New York.

The name *Pi Mu Epsilon* is taken from the initial letters of the three principal words in the Greek motto:

Τὴν παίδευσιν καὶ τὰ μαθηματικὰ ἐπισπεύδειν

—"To promote scholarship and mathematics"—which well summarizes the aims and ideals of the fraternity.

W. E. MILNE, *Director-General*

Three new chapters of *Pi Mu Epsilon* have been established during the past two academic years. This makes the total number of chapters thirty-nine, all of which are active, and a total membership of over 6,740.

The *Wisconsin Beta Chapter* of *Pi Mu Epsilon* was installed at the *University of Wisconsin* on April 12, 1939, by the Director-General W. E. Milne who was also the principal speaker at the banquet which was held in conjunction with the installation ceremony. Thirty-two members were initiated. Officers are: Director, R. D. Wagner; Secretary, Jack Kelly; Treasurer, Fred Gruenberger; Permanent Secretary, L. R. Wilcox.

The *Louisiana Alpha Chapter* of *Pi Mu Epsilon* was installed at *Louisiana State University* on December 9, 1939, by the Secretary-General J. S. Gold. Nineteen members were initiated. Professor S. T. Sanders, head of the Department of Mathematics, was the principal speaker at the installation banquet. Officers are: Director, Carolyn Rosenthal; Secretary-Treasurer, J. C. Stewart; Permanent Secretary, Marelena White.

The *Michigan Alpha Chapter* of *Pi Mu Epsilon* was installed at *Michigan State College* on June 1, 1940, by Professor Wayne Dancer of the University of Toledo who also addressed the installation banquet. Representatives from the Chapters of Ohio State University, University of Kentucky, and Iowa State College were present. Twenty-four members were initiated. The Permanent Secretary is Professor J. W. Zimmer of the Department of Mathematics.

J. S. Gold, *Secretary-General*

KAPPA MU EPSILON

Kappa Mu Epsilon is a mathematics fraternity for accredited colleges and universities. Chapters are placed in petitioning institutions only after it has been ascertained that there is an active undergraduate mathematics club, and that there are ample facilities for carrying on a well-rounded undergraduate program in mathematics. The fraternity provides a medium for the recognition of high scholastic attainments in the field of mathematics, and has for its purpose the bringing of students and faculty together in a spirit of common endeavor.

The following officers served their first year in office during the year 1939–40:

President Pythagoras: Professor C. V. Newsom, University of New Mexico, Albuquerque, New Mexico

Vice-President Euclid: Professor E. H. Taylor, Eastern Illinois State Teachers College, Charleston, Illinois

Secretary Diophantus: Miss E. Marie Hove, Nebraska State Teachers College, Wayne, Nebraska

Treasurer Newton: Professor H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Historian Hypatia: Dr. Kathryn Wyant, Forreston, Illinois

Past President Zeno: Professor J. A. G. Shirk, Kansas State Teachers College, Pittsburg, Kansas.

Kappa Mu Epsilon now has twenty-four chapters, the activities of which have been reported from time to time in this Monthly. Of the twenty-four chapters, six were installed during the spring of 1940. They are as follows:

South Carolina Alpha Chapter, Coker College, Hartsville

Installed April 5, 1940........................8 charter members
Installing officer..........................Miss Orpha Ann Culmer

President Leibniz . . . . Frances Humphries
Vice-President Pascal . . . . Edith Mitchell
Secretary Thales . . . . Eunice Mitchell
Treasurer Gauss . . . . Elsie Neighbors
Secretary Descartes . . . . Caroline M. Reaves

Texas Alpha Chapter, Texas Technological College, Lubbock

Installed May 10, 1940 . . . . 25 charter members
Installing officer . . . . Professor C. V. Newsom
President Lobatchewsky . . . . Joe Foote
Vice-President Agnesi . . . . Aliene May
Recording Secretary Noether . . . . Marie McCrummen
Treasurer Cayley . . . . Rance Jones
Secretary Descartes . . . . Mrs. Opal L. Miller
Faculty Sponsor . . . . Professor R. K. Wakerling

Texas Beta Chapter, Southern Methodist University, Dallas

Installed May 15, 1940 . . . . 23 charter members
Installing officer . . . . Professor C. V. Newsom
President Galois . . . . Julia Smith
Vice-President Abel . . . . Paul D. Minton
Secretary-Treasurer Pascal . . . . Marguerite Summers
Secretary Descartes . . . . Paul K. Rees
Faculty Sponsor . . . . Professor K. L. Palmquist

Kansas Gamma Chapter, Mount St. Scholastica College, Atchison

Installed May 26, 1940 . . . . 12 charter members
Installing officer . . . . Miss E. Marie Hove
President Nicomedes . . . . Sarah Alice Woodhouse
Vice-President Tartaglia . . . . Bobbe Powers
Secretary Galileo . . . . Margaret Mary Kennedy
Treasurer Napier . . . . Mary Hughes
Secretary Descartes . . . . Sister Helen Sullivan

Iowa Beta Chapter, Drake University, Des Moines

Installed May 27, 1940 . . . . 23 charter members
Installing officer . . . . Professor H. Van Engen
President . . . . Floyd Beasley
Vice-President . . . . R. Bernard Smith
Secretary . . . . Dixie Lippincott
Treasurer . . . . Kenneth Austin
Secretary Descartes . . . . Professor Floyd Woodyard

New Jersey Alpha Chapter, Upsala College, East Orange

Installed June 3, 1940 . . . . 11 charter members
Guest installing officer . . . . Professor E. H. C. Hildebrandt
President Thales . . . . Bernard Morrow
Vice-President Apollonius . . . . Marian Fialk
Secretary Abel . . . . Marjorie Dargue

Treasurer Fibonacci . . . . Edith Olson
Historian Gauss . . . . Anna Zmurkewitz
Secretary Descartes . . . . Professor Martin Nordgaard

C. V. NEWSOM, *President Pythagoras*

CLUB REPORTS, 1939–1940

Graduate and Research Clubs. Although most of the clubs reporting to this department limit their membership to undergraduate students, there are research groups consisting of graduate students and faculty members meeting a number of times during the year to consider papers on original research. Since announcements of these meetings and the papers submitted are not made in any other mathematical journals, it may prove of interest to list such reports in these pages. Reports received to date include those from the *Oliver Mathematical Club* at *Cornell University* and the *Mathematics Round Table* of the *University of Illinois.* Reports of other such groups for meetings held during 1939–40 as well as the calendar for the present college year will be welcomed.

Oliver Mathematical Club, Cornell University

p-Adic numbers, by B. W. Jones; Vitali's covering theorem, by J. F. Randolph; A statistical method in the theory of functions, by M. Kac; Statistical independence and some of its applications, by M. Kac; A theorem on sequences, by R. J. Walker; The mean of a sample, by J. H. Curtiss; The elements of transformation topology, by G. E. Schweigert; Regular ternary quadratic forms, by G. B. Thomas; Tauberian theorems for double series, by R. P. Agnew; Moving pictures and slides explaining the "Isograph" for solving an algebraic equation; The rootograph, an instrument to determine the roots of polynomials up to the thirtieth degree, by M. G. Malti; The separation of the projective plane by n straight lines, by W. B. Carver; Direct methods in the calculus of variations for the solution of integral equations, by M. Golomb; Measure for measure, by S. Sherman; Metrics for the space of permutations of positive integers, by W. A. Hurwitz; The limit of orbits, by G. E. Schweigert; The harmonic analysis of number theoretic functions, by M. Kac; Tauberian conditions, by R. P. Agnew; Cantor set, by J. F. Randolph; Uniqueness problems for rational numbers, by F. Herzog; The solution of differential equations by Laplace transformations, by M. Golomb; The transfinite diameter of a point set, by J. H. Curtiss; Related genera of quadratic forms, by B. W. Jones; Lorentz geometry, by J. W. Givens; Mathematical problems in cosmic radiation, by H. A. Bethe. Officers were: President, B. W. Jones; Secretary, J. W. Givens.

Mathematics Round Table, University of Illinois

The postulation of manifolds in hypersurfaces, by R. A. Miller; Boundary value problems of ordinary differential equations, by K. L. Nielsen; The generation of Pearson's system of frequency curves, by E. E. Blanche; Hypergroups, by R. S. Pate; Elementary valuation theory, by P. W. Carruth; Groups with operators, by P. E. Lewis; Differential equations with several parameters, by C. W. Moran; Elementary divisor theory in non-commutative rings, by Y. L. Luke; Related conics, by C. F. Strobel. Officers were: first semester: President, R. A. Miller; Secretary, K. V. Knight; and second semester: President, D. W. Starr; Secretary, Lois Kiefer.

REPRINTS OF MATHEMATICAL MAGAZINE ARTICLES

Are there any readers of this department who have reprints of their articles or dissertations in their files or stored away in their attics, who would like to make these available for further study to members of our mathematics clubs? Several people have spoken to us of their willingness to make available to others such materials as they have on hand, and a number of club representatives have stated that they are trying to build up a club library of books and articles. One club president made inquiry for papers which give rise to further questions and problems and which could be considered by advanced undergraduate and beginning graduate students. We believe that clubs will be willing to pay the postage charges and any other nominal costs involved. May we suggest that all those interested write us as soon as possible, listing the title, author, page reference in original publica-

tion, cost of mailing, and other charges. This information will then be published in a later issue of the MONTHLY in this department, or sent in our spring letter to clubs.

SEASON'S GREETINGS

The following way of expressing these greetings has been used by some of our readers in writing to their mathematically minded friends, and we should like to pass these along as an expression of our own best wishes, to be read late in December:

$$\int (x)dx \quad \frac{\sum x}{N} \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \left(\frac{\sum xy}{\sigma_x\sigma_y}\right)^2 \quad f(x)$$

$$X \quad \frac{\sum x}{N} \quad P\left(1+\frac{j}{m}\right)^{mn} \quad \sqrt{\frac{\sum x^2}{N}}$$

$$+$$

$$\frac{2ab}{a+b} \quad \int f(x)dx \quad 8 \text{ arc tan } 1 \quad f(x)$$

$$\sum_{n=c}^{n=95} D_n \quad \int wh\ dv$$

$$f(x) \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad \frac{h}{2}(b_1+b_2) \quad \frac{C}{2\pi}.$$

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 441. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given ED, construct an isosceles triangle ABC, with apex C, so that E lies on the altitude CD, and two perpendicular transversals drawn through E divide the area of the triangle into four equal parts.

E 442. *Proposed by V. Thébault, Le Mans, France.*

In the scale of B there is a perfect square of sixteen digits of the form $abcdefghabcdefgh$. What is the smallest possible value of B?

E 443. *Proposed by N. A. Court, University of Oklahoma.*

(a) Two triangles, one inscribed in the other, are in perspective. Prove that on a parallel to the axis, the center of perspective trisects the intercept between any pair of corresponding sides. (*Educational Times Reprints*, vol. 3, 1903, p. 57, Question 15259.)

(b) Two tetrahedra, one inscribed in the other, are in perspective. Prove that on a line parallel to the plane of perspective, the center of perspective quadrisects the intercept between any pair of corresponding faces.

E 444. *Proposed by Harry Goheen, Reed College, Portland, Oregon.*

Prove that there is no prime p such that $p^n+1=2^m$ if $n>1$, and that there is no prime p such that $p^n-1=2^m$ if $n>2$.

E 445. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that if $0<m<n+2$, then

$$\sum_{r=0}^{m-1}(-1)^r\binom{n}{r}^{-1} = \frac{n+1}{n+2}\left\{1-(-1)^m\binom{n+1}{m}^{-1}\right\}.$$

SOLUTIONS

E 405 [1940, 48]. *Proposed by James Travers, Harrow, England.*

Construct points P and Q on the respective sides AB and BC of a given triangle ABC so that $AP=PQ=QC$.

I. *Solution by William Douglas, Courtenay, B. C.*

Construction. On CB lay off $CL=AB$. With center L and radius AB, draw a circle cutting the parallel through B to AC in H. Then CH meets AB in the required point P, and the parallel to LH through P meets BC in Q.

Proof. Let the parallel to AB through H meet AC in M, and BC in N. Then $MH=AB=HL=LC$. But quadrangles $MHLC$ and $APQC$ are homothetic. Hence $AP=PQ=QC$.

II. *Remark by W. B. Clarke, San José, Calif.*

Since it is not always possible to locate P between A and B, and Q between B and C, we may as well waive this implied restriction. The circle cuts the parallel through B to AC in two points H, giving two solutions unless $\angle B=60°$, in which case one of the lines CH is parallel to AB. Moreover, L may be replaced by a point L', distant AB from C on BC produced, giving a second circle, which meets BH if $BL'\sin C\leqq AB$, *i.e.*, if $\sin A+\sin C\leqq 1$. When $\angle B=120°$, one of the new lines CH is parallel to AB. To sum up, there are two, three, or four solutions, according as $\sin A+\sin C-1$ is positive, zero, or negative, except when $\angle B=60°$ or $120°$, in which cases the number of solutions is one less. There are only two solutions when $B<60°$ or $60°<B\leqq 90°$; four when $B>120°$.

III. *Bibliographical Note by N. A. Court, University of Oklahoma.*

The problem was stated and solved by I. Aleksandrov, *Geometricheskye Zadachy na Postroyenye*, Russia, 1882, p. 78. A French translation by D. Aitoff, *Problèmes de géométrie élémentaire*, appeared in 1899; there the problem may be found on p. 72. In the *Proceedings of the Edinburgh Mathematical Society*, vol. 2, 1884, the problem is credited to James Edward, and his solution of it published

(p. 5), as well as a second solution by J. S. Makay (p. 27). The problem was proposed by F. Morley in the *Educational Times* as Question 9367, and some ten years later was solved algebraically by I. Arnold in *Reprints*, vol. 69, 1898, p. 93. The problem appears again as Question 16253 in the same collection, vol. 13, 1908, p. 81; four solutions are offered. A solution is included by N. A. Court in his *College Geometry*, Richmond, Va., 1925, p. 44. It was taken up in 1929 by *School Science and Mathematics*, Chicago, as Question 1062; two solutions may be found on p. 644. It also occurred as Question 139 of the *National Mathematics Magazine*, where a solution was published in Oct., 1937. J. Rosenbaum generalized the problem, replacing the three equal segments by three segments having given ratios. The generalized problem and its solution were published in this MONTHLY, vol. 31, 1924, p. 311.

Also solved by W. E. Buker, V. W. Graham, Ida R. Kaplan, E. C. Kennedy, Elmer Latshaw, D. L. MacKay, Hazel E. Schoonmaker, E. P. Starke, J. E. Trevor, C. W. Trigg, and the proposer.

E 406 [1940, 110]. *Proposed by David Segal, Kosow Huculski, Poland.*

Prove that

$$\sum_{k=0}^{[n/2]} \binom{2n+1}{4k} = 2^{n-1}(2^n \pm 1).$$

Solution by Esther Szekeres, Shanghai, China.

Using De Moivre's theorem, we have in succession,

$$(1 + i)^m = 2^{m/2}\{\cos (m\pi/4) + i \sin (m\pi/4)\},$$

$$\sum_{k=0}^{[m/4]} \left\{\binom{m}{4k} - \binom{m}{4k+2}\right\} = 2^{m/2} \cos \frac{m\pi}{4},$$

$$\sum_{k=0}^{[m/4]} \left\{\binom{m}{4k+1} - \binom{m}{4k+3}\right\} = 2^{m/2} \sin \frac{m\pi}{4}.$$

On the other hand,

$$\sum_{k=0}^{[m/4]} \left\{\binom{m}{4k} + \binom{m}{4k+2}\right\} = 2^{m-1},$$

$$\sum_{k=0}^{[m/4]} \left\{\binom{m}{4k+1} + \binom{m}{4k+3}\right\} = 2^{m-1}.$$

By adding or subtracting pairs of these identities, we obtain $\sum_{k=0}^{[m/4]}\binom{m}{4k+j}$, where m is any positive integer and $j=0, 1, 2, 3$. In particular,

$$\sum_{k=0}^{[n/2]} \binom{2n+1}{4k} = 2^{2n-1} + 2^{n-1/2} \cos \left(\frac{n\pi}{2} + \frac{\pi}{4}\right),$$

as required.

Also solved by D. X. Gordon, V. W. Graham, Emma Lehmer, C. E. McCauley, C. W. Moran, E. P. Starke, and R. M. Walter.

E 407 [1940, 110 and 396]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Let A', B', C' be the feet of the altitudes of a triangle ABC, and H the orthocenter. Let the parallels through H to $B'C'$, $C'A'$, $A'B'$ meet BC, CA, AB in D, E, F, respectively; and let the parallels through H to BC, CA, AB meet $B'C'$, $C'A'$, $A'B'$ in D', E', F'. Prove that the points D, E, F and D', E', F' lie respectively on two parallel lines, perpendicular to the Euler line.

Solution by V. W. Graham, Dublin, Ireland.

Produce $B'C'$, $C'A'$, $A'B'$ to meet BC, CA, AB in P, Q, R, respectively. Then PQR is the orthic axis of the triangle, and is perpendicular to the Euler line. (See R. A. Johnson, *Modern Geometry*, p. 199, or the *Editorial Note* below.)

Since HE is parallel to QC', and HD to PC', we have

$$QE/EC = C'H/HC = PD/DC;$$

therefore DE is parallel to PQR. Similarly DF is parallel to PQR. Hence DEF is perpendicular to the Euler line. Similarly, so is $D'E'F'$.

Also solved by W. B. Clarke and Emanuel Mehr.

Editorial Note. In the notation of the *Editorial Note* to E 383 [1940, 242], the orthic axis and the Euler line have direction cosines proportional to

$$\left(\frac{m}{n}-\frac{n}{m},\ \frac{n}{l}-\frac{l}{n},\ \frac{l}{m}-\frac{m}{l}\right) \quad\text{and}\quad \left(l+\frac{1}{l},\ m+\frac{1}{m},\ n+\frac{1}{n}\right).$$

They are perpendicular, since $(m^2-n^2)(l^2+1)+(n^2-l^2)(m^2+1)+(l^2-m^2)(n^2+1)=0$.

E 408 [1940, 110]. *Proposed by W. C. Rufus, Observatory of the University of Michigan.*

From front to rear of an advancing army detachment was ten miles. A rear-guard messenger, dispatched to a guard-house directly behind his position in the line of march, returned without loss of time and then proceeded immediately to the van-guard and again returned. He noted that he had overtaken his guard ten miles from the starting-point, and that the time spent on each errand had been the same. How far was the guard-house from the starting-point, and how far did the messenger travel altogether?

Solution by E. K. Paxton, Washington and Lee University.

Let x miles be the distance of the guard-house from the starting-point. Since the time consumed on each errand was the same, the messenger had advanced 10 miles on the completion of each errand. Therefore he went rearward x miles and forward $x+10$ miles each time, a total distance of $4x+20$ miles. Since the detachment was 10 miles long, the rear-guard advanced x miles while the messenger advanced $x+10$ miles and reached the van-guard. Hence the relative rate of messenger to army is $(2x+10)/10$ for the first errand, and $(x+10)/x$ for part

of the second. Equating these expressions, we obtain $x=5\sqrt{2}$. Thus the guard-house was 7.071 · · · miles from the starting-point, and the messenger travelled 48.284 · · · miles.

Also solved thus by Louis Bauer, W. E. Buker, Edwin Comfort, Wm. Douglas, J. I. Nuseireh, E. P. Starke, C. W. Trigg, and the proposer. Interpreting the words "had overtaken" as referring to the end of the second errand, the following people obtained the alternative solution $x=5(\sqrt{5}+1)/2=8.09 \cdots$: Michael Aissen, Eli Borok, E. Fletcher, R. E. Greenwood, and B. C. Zimmerman. Other values of x submitted were 2.5, 5.59, 8.75.

E 410 [1940, 110]. *Proposed by C. H. Hardingham, Harpenden, England.*

What are the smallest positive integers a, b, c which are the sides of a triangle whose medians also are integers?

Partial Solution by W. E. Buker, Pittsburgh Public Schools.

The problem of finding triangles whose sides and medians are integers is an old one. (See L. E. Dickson, *History of the Theory of Numbers*, vol. 2, pp. 202–205.) Particular solutions were obtained by Euler and rediscovered many times, the simplest being 174, 170, 136 for the sides and 127, 131, 158 for the medians. (A recent account is to be found in Alliston, *Mathematical Snack Bar*, pp. 24–25.) While these investigations do not seem to prove that the above solution is the smallest one, I suggest that otherwise the problem is scarcely elementary.

Also solved by E. P. Starke and the proposer.

Editorial Note. The squares of the medians are easily seen to be

$$-a'^2+2b'^2+2c'^2, \qquad 2a'^2-b'^2+2c'^2, \qquad 2a'^2+2b'^2-c'^2,$$

where $a'=a/2$, $b'=b/2$, $c'=c/2$. Euler (*Opera postuma*, vol. 1, 1862, pp. 102–103) observed that the values

$$a'=(m+n)p-(m-n)q, \quad b'=(m-n)p+(m+n)q, \quad c'=2\left|mp-nq\right|$$

make the third median $2(np+mq)$, and that the other medians are integers too if $p=(m^2+n^2)(9m^2-n^2)$ and $q=2mn(9m^2+n^2)$, m and n being integers subject to certain inequalities. Discarding any common factor of p and q, we obtain the following simple cases. (But other expressions for p and q might provide hitherto unknown solutions.)

m	n	p	q	Half sides	Medians
1	2	25	52	127, 131, 158	261, 255, 204
2	1	175	148	377, 619, 404	975, 477, 942
3	1	200	123	277, 446, 477	881, 640, 569
2	3	13	20	85, 87, 68	131, 127, 158
5	3	68	65	207, 328, 145	463, 142, 529

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3971. *Proposed by J. R. Musselman, Western Reserve University.*

If O be the circumcenter of the triangle $A_1A_2A_3$ and M_i be the other points of intersection of the circle with the lines OA_i, show that the three circles passing through O, and on M_i with centers on A_jA_k respectively, meet at the point of Feuerbach for the tangential triangle of $A_1A_2A_3$.

3972. *Proposed by N. A. Court, University of Oklahoma.*

With the traces of a plane on the edges of a tetrahedron as centers, spheres are drawn orthogonal to the circumsphere of the tetrahedron. Show that the twelve points of intersection of the six spheres with the respective edges form a desmic system.

3973. *Proposed by V. Thébault, Le Mans, France.*

The symmetrics of the Apollonian circles of a triangle with respect to the corresponding midpoints of the sides are orthogonal to the circumcircle of the anticomplementary triangle. N. A. Court has given other properties of these circles in this MONTHLY, 1926, p. 373.

3974. *Proposed by V. Thébault, Le Mans, France.*

If the straight lines joining the vertices A, B, C, D of a tetrahedron to the points A_1, B_1, C_1, D_1 in the planes of the corresponding opposite faces are concurrent, and also the straight lines joining the same vertices to the isogonal conjugates A_2, B_2, C_2, D_2 of A_1, B_1, C_1, D_1 with respect to the corresponding face triangles are concurrent, the tetrahedron is isodynamic, and conversely. For isodynamic tetrahedrons see N. A. Court, *Modern Pure Solid Geometry*, p. 276.

SOLUTIONS

3874 [1938, 323]. *Proposed by V. Thébault, Le Mans, France.*

If a conic is inscribed in a triangle so that one of its principal axes passes through the circumcenter of the triangle, the auxiliary circle of the conic corresponding to the axis considered is tangent to the nine-point circle of the triangle, and conversely.

Note. This theorem completes a proposition of M. Weill, *Nouvelles Annales de Mathématique*, 1880, p. 253.

Editorial Note. Goormaghtigh's solution of 3857 [1940, 183] furnishes a solution of this problem. The proposer gave the following: Let $LL'=2a$ be the axis of the real foci of the inscribed conic, and $MM'=2b$ be the other axis, the two axes intersecting at right angles at the center ω of the conic. (1) Weill's theorem: *The axis LL' passes through O, the circumcenter of the considered triangle ABC.* In this case we have from a theorem of Laguerre, $\pm N\omega=(R/2)\pm a$, where N is the center of the nine-point circle and R is the radius of the circumcircle. (2) The proposer's theorem: *The axis MM' passes through O.* We obtain in the same way, $\pm N\omega=(R/2)\pm b$. These two theorems generalize Feuerbach's theorem on the nine-point circle and the inscribed circle. He remarked also that this important theorem would appear in a supplement of *Mathesis.*

The theorem of this problem results from the considerations in the *Note* to the solutions of 3882 and 3883 [1940, 404]. If the isogonal conjugate points P, P' are not collinear with O, and Γ' with the center S' is the isogonal conjugate of OP, and Γ with the center S is the isogonal conjugate of OP', then S and S' are distinct points on the nine-point circle (N), SS' is the common chord of (N) and the common pedal circle of P and P', and S, S' are the orthopoles of OP', OP, respectively. Hence, if P, P' are collinear with O, the common pedal circle of P and P' is tangent to (N) at the orthopole of the diameter d fixed by P, P'. This common pedal circle is the auxiliary circle of the inscribed conic for the axis of the foci P, P'. If d rotates about O from a position in which P, P' are real to one where they are conjugate imaginaries, as will be shown later, P and P' move on a cubic, and d must pass through a special position where P and P' coincide in an incenter or excenter. In this passage through the special position, the focal axis turns through a right angle, the auxiliary circle remains real, and tangent to (N) at the orthopole of d.

The locus γ of P and P' is easily determined. Suppose that d does not pass through an incenter or excenter, and that it cuts its isogonal conjugate Γ in P and P'. Then, since P lies on both d and Γ, its isogonal conjugate must be on both Γ and d, and it must be P'. The locus must pass through O and H, the orthocenter, and it then easily follows that every diameter d contains three and only three points of the locus. If AO cuts BC in A', then A, A' are in this case isogonal conjugates on γ. Thus γ passes through A, B, C, A', B', C', O, H, the incenter, and three excenters. Obviously, the tangents to γ at the incenter and the three excenters pass through O, and OH is tangent at O. The tangent at a vertex, say A, is the altitude of the triangle for A. For, if P is a point on γ near A and P' is its isogonal conjugate, then AP and AP' are isogonal with respect to the angle A. As P approaches A, P' approaches A'; the isogonal conjugate of AA' with respect to angle A is the altitude at A, and hence this altitude is tangent at A. There are three asymptotic directions which are easily determined. Let $\overline{A}$ be the bisector of that arc of (O) from B to C which contains A, let the angle between $O\overline{A}$ and d be θ, and let $\angle \bar{A}OA=\alpha$. Then it is easily seen that the asymptotic directions of Γ, the isogonal conjugate of d, make with $O\overline{A}$ the angles $(\pi+\alpha-\theta)/2$, $(2\pi+\alpha-\theta)/2$. Then the asymptotic directions of γ are ob-

tained by setting these angles equal to θ and $\pi+\theta$. We thus find for γ the asymptotic directions $\alpha/3$, $(\pi+\alpha)/3$, $(2\pi+\alpha)/3$. Thus there are three diameters to each of which corresponds an inscribed parabola, and the corresponding auxiliary circles tangent to (N) are straight lines. The three foci, which lie on both (O) and γ, form an equilateral triangle. The degenerate cases for the inscribed conics are in this instance the three conics such as the one with foci A, A', and three others such as the one with foci B, C. In the first type the pedal circle, auxiliary circle for the axis of foci A, A', has AA' for diameter with the orthopole A_h, the foot of the altitude from A. This circle is internally tangent to (N) at A_h. For the other type with the real foci B, C, the imaginary foci are on OA_0, where A_0 is the midpoint of BC and the orthopole of OA_0. In this case the auxiliary circle for the axis along A_0O is a null circle, A_0. Moreover, if ABC is not isosceles, it follows that the three diameters such as OA_0 cut γ in only one real point, O. As for the diameter OH, the common pedal circle coincides with (N). If ABC is isosceles, say $AB=CA$, then obviously AA_0 is part of the degenerate γ, and the other part is a hyperbola with AA_0 as an axis. For this hyperbola A, O and H, A_0 are conjugate pairs with respect to it, and hence its center is at the centroid G.

3893 [1938, 631]. *Proposed by Norman Anning, University of Michigan.*

From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible triangles is the integer nearest to $n^2/12$.

I. *Solution by J. S. Frame, Brown University, Providence, R. I.*

Each of the distinct equilateral, isosceles, or scalene triangles having its vertices at the vertices of a regular n-gon is congruent respectively to one, three, or six triangles having a given vertex A. If the number of distinct triangles of these types are E, I, and S, respectively, then since each pair of vertices different from A determines with A a triangle, we have

$$(n-1)(n-2)/2 = E + 3I + 6S. \tag{1}$$

Counting the distinct triangles ABC whose angles B and C are equal, we find

$$I + E = (n-2+d)/2, \qquad E = 1 - c, \tag{2}$$

where c and d are the residues 0 or 1, such that

$$n^2 \equiv c \pmod{3}, \qquad n^2 \equiv d \pmod{4}. \tag{3}$$

Solving for $S+I+E$ from equations (1) and (2), we have

$$\begin{aligned} 12(S+I+E) &= (n-1)(n-2) + 3(n-2+d) + 4(1-c) \\ &= n^2 + 3d - 4c. \end{aligned} \tag{4}$$

Hence $S+I+E$ is the integer nearest to $n^2/12$.

II. *Solution by F. C. Auluck, Dyal Singh College, Lahore.*

Let the regular n-gon be $A_1A_2A_3 \cdots A_n$. All kinds of possible triangles can be formed with A_1 as one vertex. We divide all the representative triangles (that is the representatives of all the classes of similar triangles) into the following groups. In the first group place all those which have their smallest side equal to A_1A_2; in the second group all those which have their smallest side equal to A_1A_3; in the rth group all those which have their smallest side equal to A_1A_{r+1}. Each representative triangle belongs to one and only one group. Let y be the required number of different triangles. We enumerate the number of such triangles in each group.

If n is even,

$$y = (n-2)/2 + (n-4)/2 + (n-8)/2 + (n-10)/2 + \cdots$$
$$= \frac{1}{2}\left\{\frac{n^2}{4} - \frac{n}{2} - n\left[\frac{n}{6}\right] + 3\left[\frac{n}{6}\right]^2 + 3\left[\frac{n}{6}\right]\right\},$$

where $[n]$ is the greatest integer in n.

If n is odd,

$$y = (n-1)/2 + (n-5)/2 + (n-7)/2 + (n-11)/2 + \cdots,$$
$$= \frac{1}{2}\left\{\frac{n^2-1}{4} - (n-3) - (n-3)\left[\frac{n-3}{6}\right] + 3\left[\frac{n-3}{6}\right]^2 + 3\left[\frac{n-3}{6}\right]\right\}.$$

The values of y for the four cases, with respect to mod 6, $n \equiv 0, 3, \pm 1, \pm 2$ are respectively $n^2/12$, $n^2/12+1/4$, $n^2/12-1/12$, $n^2/12-1/3$. Hence the desired result follows.

Solved also by Harry Gershenson, Michael Goldberg, E. R. Heineman, and Chang Shou Lien.

Editorial Note. Chang used the same division into types as in I, and obtained $12(S+I+E)=n^2+2-3\delta(n)+4\epsilon(n)$, where $\delta(n)$ is 1 or 2 according as n is odd or even, and $\epsilon(n)$ is 1 or 0 according as n is, or is not, divisible by 3. The remaining solvers made use of the fact that the desired number is the number of essentially different sets of three positive integers which satisfy $x+y+z=n$. Gershenson stated that Dickson's *History of the Theory of Numbers*, vol. 2, pages 115, 126 gives the number as stated in the problem. Goldberg gave an interesting solution by considering the positive integral solutions as represented by points in the cartesian plane; and this gives a triangular array having $n-2$ points on each side. The distinct solutions are symmetric with respect to the three axes of the triangle. If s, a, c are respectively the total number of points in the triangle, the number on an axis, and the number at the centroid, he found that the desired number is $p=(s+3a+2c)/6$, and determined the results for the four cases as above.

In solution I it is easily seen that

$$3c = 2 - (\omega^n + \omega^{2n}), \qquad 2d = 1 - (-1)^n,$$

where ω is an imaginary cube root of unity. In Goldberg's solution

$$3c = 1 + \omega^n + \omega^{2n}, \qquad 4a = 2n - 3 - (-1)^n.$$

Inserting these values we find that the desired number is

$$\frac{n^2}{12} - \frac{7}{72} - \frac{(-1)^n}{8} + \frac{1}{9}(\omega^n + \omega^{2n}).$$

We may make three cases, mod 6, according as $n \equiv 0, 3; 2, -1; -2, 1$; and we have, respectively,

$$\frac{n^2}{12} - \frac{1}{8}[(-1)^n - 1], \quad \frac{n^2-4}{12} - \frac{1}{8}[(-1)^n - 1], \quad \frac{n^2-1}{12} - \frac{1}{8}[(-1)^n + 1].$$

The proposer stated that the number q of essentially different convex quadrilaterals is given by

$$96q = 2(n+1)(n-1)(n-3) + 3(1 + i^{2n})(3n - 3 + 2i^n).$$

When $n = 2k-1$, then $q = {}_kC_3$.

3896 [1938, 696]. *Proposed by W. B. Clarke, San José, California.*

Let P and Q be isotomic conjugate points with respect to triangle ABC. Find the locus of P if PQ is parallel to a side of the given triangle.

Solution by J. W. Clawson, Ursinus College.

Let P be (p, q, r) in trilinear normal coördinates. Then Q is $(1/a^2p, 1/b^2q, 1/c^2r)$, and the equation of PQ is

$$a^2p(b^2q^2 - c^2r^2)x + \cdots + \cdots = 0.$$

This line is parallel to $x=0$ if $cr(a^2p^2 - b^2q^2) = bq(c^2r^2 - a^2p^2)$. Replacing p, q, r by x, y, z and factoring, we find that the equation of the locus called for is

$$(by + cz)(a^2x^2 - bcyz) = 0.$$

Here $by + cz = 0$ is the straight line through A parallel to BC; also, $a^2x^2 - bcyz = 0$ is the ellipse passing through B, C, G, the centroid of the triangle ABC, and A', the fourth vertex of the parallelogram $BACA'$; and BA and CA are tangents to this ellipse, whose center bisects GA'.

Solved also by Kwan Chao Chih, F. C. Gentry, L. M. Kelly, and O. J. Ramler.

Editorial Note. The solution by Gentry used normal homogeneous coördinates as in the above, while the others used barycentric coördinates. Ramler

remarked that a generalization is obtained by considering the locus of P when P, its isotomic conjugate Q, and a fixed point D, called the "pivot", are collinear. The locus is then a cubic curve circumscribing the triangle and anallagmatic in barycentric point coördinates. More general results, he stated, are given by R. Goormaghtigh in his article *Anallagmatic Cubics*, Mathesis, 1937, p. 255. Kelly gave a very simple derivation of the equation of the locus of the problem. Let AP and AQ cut BC in A_p and A_q; then $AP/PA_p = AQ/QA_q$. Since the coördinates of P and Q are x, y, z and x^{-1}, y^{-1}, z^{-1}, we get $(y+z)/x = (y^{-1}+z^{-1})/x^{-1}$, or $(y+z)(yz-x^2)=0$, *etc.* Perhaps Kelly used the important property that, if x, y, z are barycentric coördinates of P, then P is the centroid of masses x, y, z placed at the vertices A, B, C; and this gives $AP/PA_p = (y+z)/x$.

The given problem is easily handled synthetically. For, BP, CQ, BQ, CP form a complete quadrilateral where the diagonal PQ is parallel to BC. If the first pair of sides meet in P_1 and the second pair in P_2, then the diagonal P_1P_2 passes through the midpoint A_m of BC, the midpoint of PQ, and through A. We thus have the perspective pencils $B(P) \barwedge C(Q)$; and, from the isotomic property, $C(Q) \barwedge C(P)$. Hence $B(P) \barwedge C(P)$, and P describes a conic through B and C tangent to AB and AC. Obviously the conic passes through the centroid G, and the median AA_m is a diameter. It is also clear that the conic passes through A_m', the symmetric of A with respect to A_m. The tangent at A_m' is parallel to BC, and suppose that it cuts AB and AC in B' and C'. Then the conic is an ellipse tangent to the sides of triangle $AB'C'$ at their midpoints. This ellipse has the maximum area of all ellipses inscribed in $AB'C'$, or the minimum area of all ellipses circumscribed about $A_m'CB$. See the solution of 3565 [1933, 372] and of 3718 [1936, 442]. It is obvious that the parallel to BC through A is another part of the locus, where A_p and A_q are at infinity.

In the case of an arbitrarily given pivot D, Clawson's first equation is that of the cubic locus if we regard x, y, z as the coördinates of D and p, q, r as variable coördinates. However, we can learn much about the locus of P without the use of coördinates. If d is a straight line through D, the isotomic conjugate of d is a conic through the vertices A, B, C; and, if this conic cuts d in P and Q, it will be seen easily that P and Q are the isotomic conjugates on d. Moreover, if the isotomic conjugate of D is D', one position of d is DD'; and thus D and D' lie on the locus. Hence every straight line through D cuts the locus in just three points; and it must be a cubic. If DA cuts BC in A', then both A and A' are on the cubic. Obviously the median and three exmedian points lie on the cubic; and we now have twelve of its points. The tangents at the median and exmedian points obviously pass through D; and DD' is the tangent at D. The tangent at A is the isotomic conjugate AA'' of AA' with respect to angle A, and similarly for the tangents at the other two vertices. For, if P is a point on the cubic near A, DP cuts the cubic again in Q where AP and AQ are isotomic conjugates with respect to A. Then as $Q \to A'$, $P \to A$, and the straight line of AP approaches the position of AA''. The synthetic consideration of the asymptotic directions of the cubic is more lengthy and difficult, and this will be reserved for a later problem.

3899 [1938, 696]. *Proposed by V. E. Pound, The University of Buffalo.*

Using elementary vector methods show that the angular precession of an ideal frictionless top, spinning so rapidly that the precession is steady, is given by the well known formula

$$\Omega = \frac{C\omega_3 - (C^2\omega_3^2 - 4mghA\cos\theta)^{1/2}}{2A\cos\theta}.$$

Solution by L. Richardson, The University of British Columbia.

Let the axis $O3$ of the top be inclined at the constant angle θ to the vertical OZ. In the plane $ZO3$ take an axis $O1$ perpendicular to $O3$ and take the axis $O2$ to complete the right-handed system $O123$.

Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ be unit vectors along $O1$, $O2$, $O3$, respectively. Then it is easily seen that the total moment of momentum $\mathbf{H}$ can be written

$$\mathbf{H} = -A\Omega\sin\theta\,\boldsymbol{\alpha} + C\omega_3\boldsymbol{\gamma},$$

where the letters have their usual meaning. Hence

$$d\mathbf{H} = -A\Omega\sin\theta\,d\boldsymbol{\alpha} + C\omega_3 d\boldsymbol{\gamma}. \tag{1}$$

Now since θ is constant and the plane $ZO3$ moves round OZ with constant angular velocity Ω, we have

$$d\boldsymbol{\alpha} = \cos\theta\,\Omega\,dt\,\boldsymbol{\beta}, \tag{2}$$

$$d\boldsymbol{\gamma} = \sin\theta\,\Omega\,dt\,\boldsymbol{\beta}. \tag{3}$$

From (1), (2), and (3) we find

$$\frac{d\mathbf{H}}{dt} = (-A\Omega^2\sin\theta\cos\theta + C\omega_3\Omega\sin\theta)\boldsymbol{\beta}. \tag{4}$$

Since rate of change of moment of momentum about $O2$ is equal to the moment of the impressed forces about that axis we have, by taking moments about $O2$,

$$-A\Omega^2\sin\theta\cos\theta + C\omega_3\Omega\sin\theta = mgh\sin\theta,$$

where h is the distance of the center of gravity of the top from O. Hence

$$C\omega_3\Omega - A\Omega^2\cos\theta = mgh.$$

This leads to the required result.

Solved also by T. C. Esty and the proposer.

Editorial Note. The proposer stated that this problem was suggested by Exs. 25, 26, p. 143 in Weatherburn's *Elementary Vector Analysis*. A few pages of this text give a vector treatment of the fundamental theorems related to this problem. Esty gave, in addition to his solution, a vectorial treatment of the steady motion of a top.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Professor J. S. Gold writes: "The annual Pi Mu Epsilon breakfast will be held Wednesday morning January 1, 1941 at Highland Hall, Louisiana State University. Reservations should be sent to Professor S. T. Sanders, University, La."

Applications for the Benjamin Peirce instructorships in mathematics at Harvard University for the year 1941–42 should be sent to the chairman of the department of mathematics. Candidates should have received the doctorate or have done equivalent work.

Dr. A. D. Bradley of Hunter College has been promoted to an assistant professorship, effective January 1, 1941.

Gertrude M. Cox, Assistant Research Professor of Statistics at Iowa State College has been made Professor of Experimental Statistics and Director of the Statistical Laboratory at North Carolina.

Associate Professor E. P. R. Duval of the University of Oklahoma has been promoted to a professorship.

Dr. E. G. Harrell of the Valley City, N. D., State Teachers College has been appointed to a professorship at the Platteville, Wis., State Teachers College.

Assistant Professor J. J. L. Hinrichsen of Iowa State College has been promoted to an associate professorship.

Professor Lancelot Hogben, well known author of *Mathematics for the Million*, is this year a member of the genetics staff at the University of Wisconsin.

Professor J. B. Linker of the University of North Carolina, who holds a commission in the field artillery, U. S. National Guard, has been granted leave of absence for one year.

George Pólya, hitherto professor of applied mathematics at the Federal Polytechnic Institute at Zürich, has been appointed acting professor of mathematics at Brown University for a two-year term.

Assistant Professor W. C. Randels of the University of Oklahoma has been promoted to an associate professorship.

Dean S. W. Reaves, head of the mathematics department at the University of Oklahoma since 1905 and dean of the College of Arts and Sciences since 1923, has resigned from his administrative duties, but is still professor of mathematics. Assistant Dean E. D. Meacham succeeds him as dean of the College of Arts and Sciences, and Professor J. O. Hassler succeeds him as head of the department of mathematics.

Commander J. C. Van de Carr, U. S. N. retired, has been recalled to active duty in the navy and placed in charge of the University of Oklahoma unit of the Naval R. O. T. C., having been granted a leave of absence as instructor in mathematics.

Dr. Henry Wallman of the Institute for Advanced Study has been appointed assistant professor of mathematics at the University of North Carolina.

Associate Professor Morgan Ward of the California Institute of Technology has been promoted to a professorship.

Dr. L. R. Wilcox, formerly of the University of Wisconsin, has been appointed assistant professor at the Illinois Institute of Technology (an institution recently formed by consolidation of Armour Institute of Technology and Lewis Institute).

The following appointments to instructorships are announced:
Columbia University: Dr. Walter Strodt
Illinois Institute of Technology: Dr. Herbert Busemann, Dr. John De Cicco
Iowa State College: Dr. A. T. Lonseth
University of Kansas: L. R. Shobe
Kansas State College: Dr. H. C. Fryer
Pennsylvania State College: Dr. Morris Bloom, Dr. R. H. Cook
University of Oklahoma: B. S. Whitney
Texas Technological College: Dr. P. W. Gilbert
College of Wooster: M. P. Fobes

J. M. Colaw of Monterey, Va., died February 26, 1940. He was a co-editor of the MONTHLY from its establishment in 1894 until 1902 and was a charter member of the Association.

G. H. K. Strathy, a member of the University of Toronto team which won the William Lowell Putnam Mathematical Competition last spring, was killed in action aboard the *Ajax*, a British warship, on or about October 16.

SYMPOSIUM ON APPLICATIONS OF MATHEMATICS TO EARTH SCIENCES

The addresses presented at the Symposium on Applications of Mathematics in the Earth Sciences, under the joint auspices of Sections A and E of the American Association for the Advancement of Science, the American Mathematical Society, the Mathematical Association of America, and the Geological Society of America, at Columbus, Ohio, December 29, 1939, are to be published as Part A of Volume IV of the Transactions of 1940 of the American Geophysical Union. The addresses tend to further a desirable rapprochement between geophysicists and mathematicians and begin the important task of bringing to each field the knowledge or some suggestions of the contributions which the other field may make. This volume will include also, as Part B, the symposium at Columbus on Hydrologic Problems in the Ohio and Michigan Basins. The volume will make about 100 pages and will be ready for distribution early in November. The regular postpaid price to non-members of the Union is $1.00 but a special price of $0.75 per copy will be made to any members of the Mathematical Association of America who place their orders by December 31, 1940, sending them to 5241 Broad Branch Road, N.W., Washington, D. C.

THE FOURTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The fourth annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 1, 1941. This Competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The examination consists of two three-hour examinations. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from the Secretary of the Association, W. D. Cairns, 97 Elm Street, Oberlin, Ohio, by a postcard request. All applications must be filed with the Secretary not later than February 11, 1941. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made by the Secretary in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in ample time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination the books are to be sent by registered mail or by express to the Secretary of the Association, who will forward them to a qualified reader chosen by the Association.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are \$500, \$300, and \$200 in the order of their rank. In addition, there will be prizes of \$50, \$30, and \$20 awarded to the members of these teams according to the rank of the team, and a prize of \$50 to each of the five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to the three teams next in order after the three winning teams and to the five individuals next in order after the five individual winners. For further encouragement of the Competition, there will be awarded at Harvard University* an annual \$1000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

A more complete announcement of the general regulations under which the plan is operated will be found in the January 1938 issue of the MONTHLY and in the announcement which is being mailed to colleges and universities in the United States and Canada. Reports on the three previous competitions will be found in the MONTHLY for May 1938, May 1939, and May 1940.

* Or at Radcliffe College, in the case of a woman.

CONCERNING JUNIOR COLLEGES

Dr. Walter C. Eells, executive secretary of the American Association of Junior Colleges presents the following statement:

The phenomenal rise of the junior college movement was signalized in September by appearance of *American Junior Colleges*, first edition, a comprehensive handbook of 595 pages published by the American Council on Education, with the coöperation of the American Association of Junior Colleges and the aid of the Carnegie Corporation.

American Junior Colleges is a companion volume to the American Council's *American Universities and Colleges*. Part I contains a concise history of the junior college movement, analysis of its present status, statistical summaries covering 575 institutions, and a discussion of accreditation. The 80 pages on accreditation include the detailed standards and practices of each national, regional, and state accrediting agency, with lists of the junior colleges approved by each.

Detailed reports on 494 accredited junior colleges are given in Part II, comprising 364 pages. The information was carefully assembled from the institutions themselves, and rechecked in proof. It covers such topics as finances, grounds and buildings, admission requirements, departments, staffs, enrollment, degrees, fees, scholarships, and administrative officers.

Thirty-three pages tabulate curricula offered by junior colleges, showing courses parallel to college and university freshman-sophomore work and also the extent of offerings in semiskilled, technical, semiprofessional, and general terminal curricula. There are also classifications of junior colleges under various categories. Indexing is thorough.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-third Summer Meeting, Hanover, N. H., September 9–12, 1940.

Twenty-fifth Annual Meeting, Baton Rouge, La., December 31, 1940–January 2, 1941.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1940 and reported to the Secretary.

Allegheny Mountain, Morgantown, W. Va., April 20; Grove City, Pa., November 2.

Illinois, Bloomington, May 10–11.

Indiana, Richmond, May 3–4.

Iowa, Mt. Vernon, April 19–20.

Kansas, Wichita, March 31.

Kentucky, Lexington, April 27.

Louisiana-Mississippi, Oxford, Miss., March 8–9.

Maryland-District of Columbia-Virginia, Richmond, Va., May 11; Washington, D. C., December 7 or 14.

Michigan, Ann Arbor, April 26–27.

Minnesota, Mankato, May 4.

Missouri, Warrensburg, April 19.

Nebraska, Omaha, May 9–11.

Northern California, Berkeley, January 27.

Ohio, Columbus, April 5.

Oklahoma, Oklahoma City, February 16

Philadelphia, November 30.

Rocky Mountain, Fort Collins, Colo., April 19.

Southeastern, Athens, Ga., March 29–30.

Southern California, Compton, March 2.

Southwestern, Tucson, Ariz., April 22–23.

Texas, Dallas, March 29–30.

Upper New York State, Hamilton, May 11.

Wisconsin, Milwaukee, May 4.

Affiliated Organizations: The New England Association of Teachers of Mathematics, The National Council of Teachers of Mathematics.

MATHEMATICAL PROBLEMS IN AVIATION*

RICHARD VON MISES, Harvard University

It will be very difficult to give you within thirty or forty minutes an outline of the mathematical problems connected with modern airplane design. Both the number and the extent of those problems are immense and it is not at all easy to make them clear to people who have a high standard in pure mathematics but are unfamiliar with the particular topic. I shall have to make a very restricted selection and to confine myself to hints rather than to detailed accounts.

At the start let me make three short statements without giving any arguments: (1) It is true that the amazing development in aircraft performances within the last twenty-five years is due to a large extent to theoretical work which has been carried out by mathematicians in all countries, particularly in England and Germany. (2) It cannot be expected that an isolated solution of a particular mathematical problem is capable of bringing forth an ostensible influence upon practical work; it is rather an accumulative effect, the general level of theoretical knowledge, which determines the intensity and the rate of progress. (3) In order to do useful work in this field it is not sufficient to concentrate upon the pure mathematical aspect of a problem; one has to bear in mind the interconnections with the mechanical theories and the actual physical conditions. The more familiar you are with the whole engineering matter, the more successful will be your mathematical efforts.

Let me turn now to some selected mathematical problems of interest in aircraft design. I shall be concerned mainly with aerodynamical problems which form obviously the specific basis in this field and will content myself, at the end of this report, with giving only very brief hints about problems in structural analysis (which are, however, of greatest importance in design work). As to aerodynamics, I choose for discussion four characteristic domains in which mathematical research has supplied up to now useful results: first we will speak of the so-called two-dimensional wing theory which is an application of the theory of conformal mapping; second, of the three-dimensional wing theory based on the idea of vortex sheets; third, we shall consider the airplane as a whole, as a solid body moving under the influence of air forces; and last I want to call your attention to certain questions in the theory of the so-called boundary layer which may prove to be decisive in the future for certain details in wing design.

1. The airwing in two dimensions. The cross-section through an airplane wing has about the form shown in Figure 1. When the wing is moving through the air with a constant velocity V to the right, or, which is the same, if the wing is at rest and exposed to a uniform flow from the right to the left, the air exerts upon it a certain force depending on the velocity V. In order to find this force, *i.e.*, its magnitude, its direction, and its line of action, we may assume as a

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* Presented at the Hanover meeting of the Mathematical Association of America on September 9, 1940.

first approximation that the flow around the wing is a two-dimensional, steady, potential motion, controlled by the Laplace equation

$$\Delta\phi = 0, \quad \text{where} \quad \frac{\partial\phi}{\partial x} = u, \quad \frac{\partial\phi}{\partial y} = v \tag{1}$$

are the velocity components. The boundary conditions are

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{along the curve;} \quad \frac{\partial\phi}{\partial x} = -V, \quad \frac{\partial\phi}{\partial y} = 0 \quad \text{at infinity.} \tag{2}$$

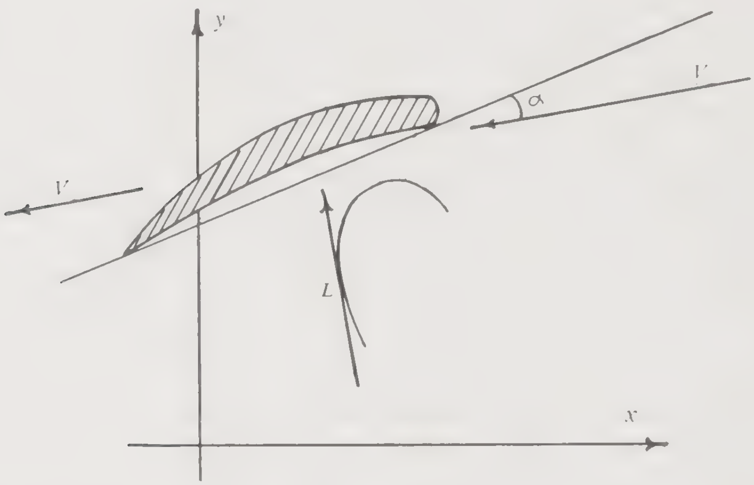

FIG. 1

The trouble is that the outside space around the cross-section of the wing, in the two-dimensional conception, is doubly connected so as to give not a single solution but an infinity of solutions of the potential problem (or a solution depending on a parameter). The decisive idea how to overcome this difficulty was supplied by the Russian mathematician Joukowski in 1910, and this may be considered as the starting-point of the scientific theory of flight. Each profile which can be used in practice as cross-section for a wing or a propeller blade has a singular point at its trailing end, a cusp or a very acute angle. At this point the solution of the boundary problem gives, in general, an infinite velocity and this would mean a negative hydraulic pressure which is physically impossible. Joukowski pointed out that the unknown parameter in the general solution can be uniquely determined by means of the additional condition that the velocity remains finite at the trailing edge. In this way he computed the magnitude and the direction of the airforce for a certain special shape which is today called the Joukowski profile. The general theory embracing all shapes and giving the line of action too was developed in my papers of 1917 and 1920.

It is well known that the general solution of the boundary problem (1), (2) for a circle of radius a is given by

$$w = \phi + i\psi = -V\left(z + \frac{a^2}{z}\right) + \frac{\Gamma}{2\pi i}\log z, \qquad z = x + iy. \tag{3}$$

Here Γ is the constant which has to be determined afterwards by the Joukowski condition. The problem is to find the conformal transformation

$$z = z' + \frac{K_1}{z'} + \frac{K_2}{z'^2} + \cdots \tag{4}$$

which maps the outside of the profile into the outside of the circle and leaves infinity unchanged. When the transformation function is found, we determine the point on the circle into which the cusp is mapped and write down the condition that dw/dz vanish at this point. This equation supplies the value of Γ. Then the reaction of the fluid mass upon the wing can be computed by means of the general laws of dynamics.

From the main results of the theory I may quote the following: (1) The air force acting upon the wing is normal to the velocity vector V, in other terms it is a pure lift or supporting force without resistance or drag. (2) The lift per unit of span is proportional to the velocity V, the density of the air ρ and to the constant Γ which is called the circulation (we will discuss it later):

$$\frac{dL}{ds} = \rho\Gamma V. \tag{5}$$

On the other hand, for small angles of attack α the circulation Γ turns out to be a linear function of α and can be written in the form

$$\Gamma = cV\alpha, \tag{6}$$

where c depends on the shape of the profile only. (3) The lines of action for different angles α envelop in general a parabola. But we can find special forms of profiles for which the lift passes through a fixed point, the so-called lift center; those forms are advantageous from the standpoint of stability.

Much detail work has been done in this field during the past twenty years. Specially practical methods for conformal mapping have been developed and solutions of the so-called inverse problem have been found (special transformation functions which supply suitable forms of cross-sections). But much is still to be done. I mention only as an example the biplane problem for which we have only some general results and a very incomplete approach to a practical solution. It is the same for the airwing combined with a tail surface. In these cases the space is triply connected, we have two circulation constants and two singular points and a conformal mapping into the outside of two circles is required.

2. The airwing of finite span. I turn now to the general three-dimensional theory. The conception of a strictly two-dimensional motion of the air around an airwing is obviously insufficient. It supposes an exactly cylindrical wing of

infinite span. The span of a real wing is finite, about six to eight times the breadth. The most important effect of the finite ends is to reduce the magnitude of the lift force and to produce in addition to the supporting force a resistance or drag force. The three-dimensional wing theory gives a principal account and a very useful numerical estimate of both effects. The basic idea of this theory was outlined by the English automobile manufacturer F. W. L. Lanchester; the mathematical theory which forms today the most efficient help in airplane design has been developed by Prandtl and his disciples since about 1918.

The two-dimensional theory was based upon the fact that the exterior of a closed curve in the plane is doubly connected. But in space the outside of a simple body is a simply connected region. We have a classical theorem, the so-called d'Alembert paradox, which states that there is no dynamic interference between a body moving in a continuous potential velocity field and the surrounding air which fills the simply connected outside region. On the other

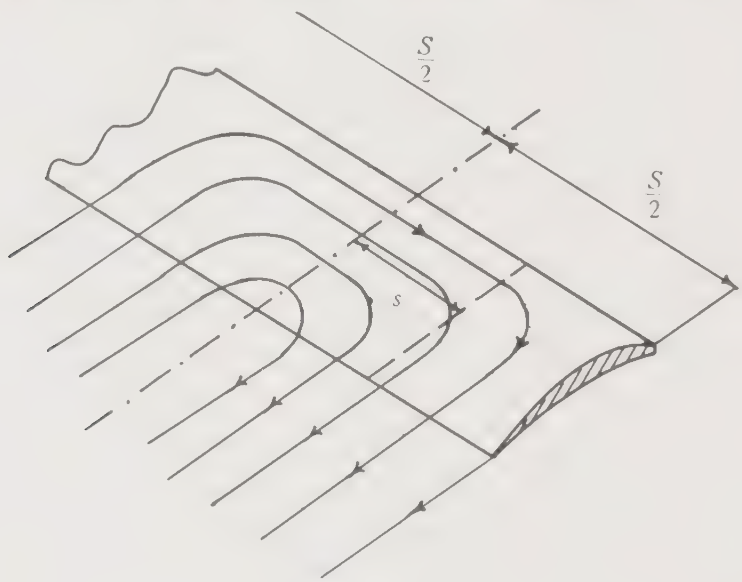

FIG. 2

hand, there must be some continuity between the case of an infinite cylinder and that of a long, but finite cylindrical or almost cylindrical body. The explanation is again connected with the sharp trailing edge of the wing.

We see in equation (3) that in the case of the infinite circular cylinder, Γ is the factor of the logarithmic term in the expression for the complex potential. This term means a circulating flow in parallel circles around the axis of the cylinder. Such a flow can be considered as induced by a straight vortex line, coinciding with the axis, the vorticity of which is exactly Γ (this is why we used the factor 2π in the denominator: so-called rule of Biot-Savart). Or it can be considered as induced by a uniform distribution of parallel vortex lines coinciding with the generators of the cylinder and with a total vorticity Γ. If we pass from the circle to the correct cross-section, this part of the flow

transforms into a motion which likewise can be considered as induced by a continuous distribution of parallel vortex lines of total vorticity Γ. The vortex lines now run on both sides of the wing surface, again normal to the xy-plane. We will remember that the lift force in this two-dimensional problem was proportional to Γ (equation (5)).

Now, if we cut the cylinder off at both sides to have a finite span S, a circulating motion cannot exist in a cross-sectional plane the distance of which from the center is greater than $S/2$. We have to assume that Γ is here a function of the distance s which has a maximum for $s=0$ and vanishes for $s=\pm S/2$. But vortex lines cannot have ends or finite terminal points, or, in other terms, the circulation Γ cannot change its value as long as we do not cut across vortex lines. The only possible assumption therefore is that the vortex lines turn one after the other and leave the wing surface in a backwards direction over the trailing edge, forming thus an infinite vortex sheet behind the wing. This is the famous conception of the so-called horseshoe vortices due to Lanchester (Fig. 2).

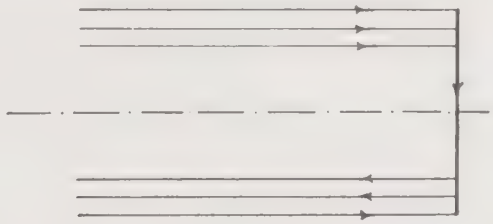

FIG. 3

From a mathematical point of view a vortex sheet is a discontinuity surface of the velocity field. So the d'Alembert paradox is no longer valid here since we have no continuous velocity field over the space. Moreover, Prandtl found a way to utilize this general insight into the character of the motion for a numerical investigation of lift and drag. He simplifies the problem by assuming that the whole cross-section of the wing reduces to a single point, and the wing to what we call a supporting filament. The vortex sheet has then the form of an infinite rectangular strip and the finite parts of the vortex lines concentrate along the supporting line (Fig. 3). In this case the whole velocity field induced by the vortex sheet can be computed by means of the classical formula which the physicist likes to call the law of Biot-Savart. In particular at a point s of the supporting filament we get a small vertical velocity,

$$v(s) = \frac{1}{4\pi}\int_{-S/2}^{S/2} \frac{d\Gamma(\sigma)}{\sigma - s}. \tag{7}$$

This vertical velocity v when added to the horizontal velocity V changes the direction of the latter, and the angle of attack which has been α before, now

equals $\alpha + v/V$ (Fig. 4). If we apply our equation (6) for Γ, using the corrected value of α, we get the fundamental equation of Prandtl,

$$\Gamma(s) = c(s)\left[\alpha(s) + \frac{1}{4\pi V}\int_{-S/2}^{S/2}\frac{d\Gamma(\sigma)}{\sigma - s}\right]V. \tag{8}$$

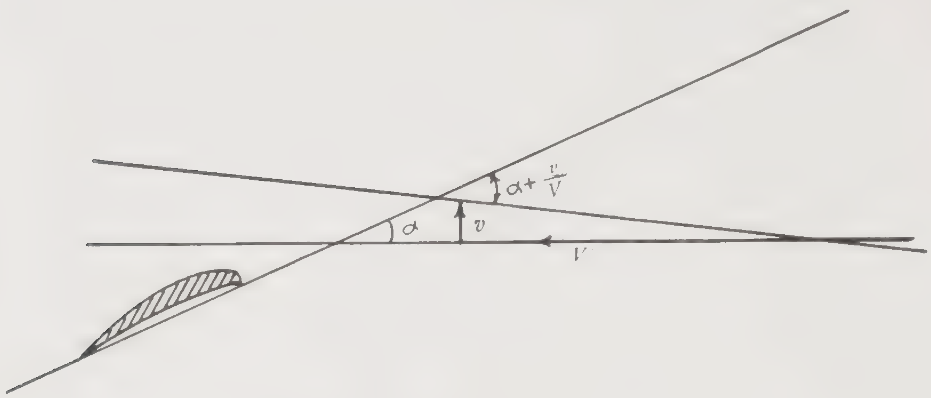

FIG. 4

This is a singular integro-differential equation for the function $\Gamma(s)$, if $c(s)$ and $\alpha(s)$ are supposed to be known from the solution of the two-dimensional problem. If $\Gamma(s)$ has been found, we have immediately the lift force according to (5),

$$L = \rho V \int_{-S/2}^{S/2} \Gamma(s)ds. \tag{9}$$

On the other hand, the total air force in any cross-sectional plane is normal to the direction of the velocity vector which is now composed of the horizontal component V and the vertical v. Therefore we have a horizontal force component opposite to the direction of flight, or a drag D, where

$$D = \rho \int_{-S/2}^{S/2} \Gamma(s)v(s)ds; \tag{10}$$

here v is given by (7). In this way everything depends on the solution of Prandtl's integro-differential equation (8).

Very much successful work has been done in this field during the past twenty years. The best results were reached by using the Fourier development for $\Gamma(s)$ introduced by E. Trefftz,

$$\Gamma(s) = 2SV\sum_n C_n \sin n\theta, \qquad \cos\theta = \frac{2s}{S}\cdot \tag{11}$$

One of the first rather amazing results was the solution supplied by Munk of the problem of variation: To find the function $\Gamma(s)$ which gives the minimum drag D for a given lift L. The answer is that the Fourier series (11) must reduce

to its first term. This leads, if we assume α constant, to a wing of elliptic or semi-elliptic top view. All these results and many others are in best accordance with carefully performed experiments and with actual experience in flying. They supply the basis for various important decisions in design work.

On the other hand, the given equations form only the nucleus of more elaborate theories which discard the restriction to a single supporting line and take into account the complete two-dimensional wing surface. Then Γ has to be considered as a function of two independent variables. The foundation of such a theory has been given by J. Burgers, but his two-dimensional integro-differential equation has not yet found a satisfactory solution. Here we have a vast field for useful research work in applied mathematics. As to the practical application of Prandtl's theory I may mention, in concluding this paragraph, that it applies likewise to the investigation of the characteristic properties of propellers; in fact a propeller blade is, from an aerodynamical point of view, nothing more than a rotating wing. Even some older problems in design of turbines, rotating pumps, and ventilators have been advanced by the use of ideas borrowed from the vortex sheet theory of airwings.

3. Dynamics of an airplane. All these wing theories of which I have spoken consider the airplane in the state of steady motion with uniform velocity, and study the relationship between the wing shape and the forces exerted by the surrounding air. This is essentially an aerodynamic investigation. It would be too difficult and too vast an undertaking to extend this kind of investigation to more general forms of motion. When we are concerned with the question of how the airplane as a whole behaves in curves, how it follows the steering operations, how it reacts to external disturbances, *etc.*, in such cases we assume that the air forces are completely known as functions of time, velocity, angle of incidence, *etc.* To find these functions, we use at first experimental data, then we apply certain consequences of the aerodynamical theory for steady motion, and finally we make reasonable assumptions to be tested by the outcome of the investigation. In any event, by considering the airforces as given functions, the problem we face here becomes a problem of dynamics of a rigid body. But it has a sufficiently specified form here and is so broad that it seems quite justifiable to regard it as a specific airplane problem.

It causes no difficulty to set up the six simultaneous differential equations which control the motion of the airplane as a rigid body. We may use any of the classical forms, the Euler equations, the Lagrange equations with a suitable choice of coördinates, or any other form. The acting forces are the gravity and the air forces which enter into the equations as functions of the angular coördinates, the velocities and eventually the time. The mechanical system is in all cases a non-conservative one. If you remember that mathematicians up to now hardly succeeded in settling the problem of a rigid body that is subjected to gravity only, you will understand that we cannot expect to find complete analytical solutions of the present problem. But useful work has been done in

establishing sufficiently approximate solutions in certain special examples, to allow us to answer various practical questions.

I will not write down here the general equations which you can find in many places, *e.g.*, in my article, *Dynamic problems in engineering*, Encyclopaedie der Mathematischen Wissenschaften, vol. IV, 1911. Let me rather begin by discussing one of the most important special problems which we encounter in this field, the problem of stability of the uniform level flight. According to the well known general procedure in stability investigations, we linearize the equations by studying small deviations from the steady motion the stability of which is to be examined. It turns out that if we consider the airplane as a symmetric body steadily flying parallel to its symmetry plane, then the problem of six simultaneous variables splits into two three-dimensional problems. We can accordingly speak of a longitudinal and of a lateral stability. The first concerns the motion of the center of gravity parallel to the symmetry plane and the rotational motion about an axis normal to this plane. If v is the magnitude of the velocity, θ the inclination of the flight path of the center of gravity, and ϕ the angle between the longitudinal axis of the airplane and the horizontal (Fig. 5), the three equations of motion can be written in this form:

$$\begin{aligned} m\frac{dv}{dt} &= P - D + mg\sin\theta, \\ mv\frac{d\theta}{dt} &= -L + mg\cos\theta, \qquad (12) \\ I\frac{d^2\phi}{dt^2} &= -M. \end{aligned}$$

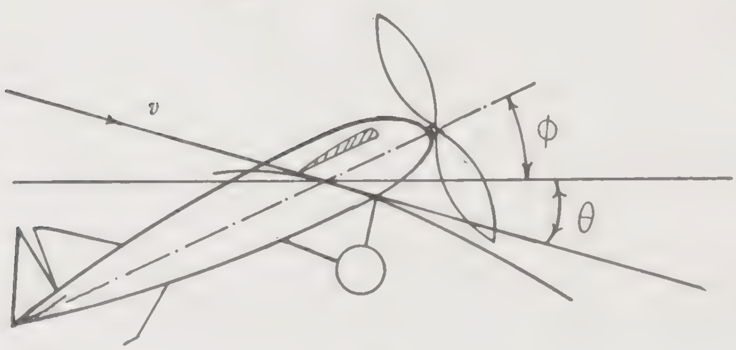

FIG. 5

Here m is the mass, I the moment of inertia of the airplane, L the lift, D the drag, P the propeller force, and M the moment of the air forces with respect to the center of gravity. The so-called natural coördinate system (based on tangential and normal directions) is used in (12). The integration problem for the variables v, θ, ϕ is, as we see, of fourth order. If we linearize and introduce an exponential function $e^{\lambda tg/v_0}$ as a solution, where v_0 is the velocity of level flight, we get for λ the characteristic equation of the fourth degree,

$$\lambda^4 + \left(\frac{1}{\alpha} + \delta\right)\lambda^3 + \left(2 + \sigma + \frac{\delta}{\alpha}\right)\lambda^2 + 2\delta\lambda + 2\sigma = 0. \tag{13}$$

Stability requires that the four roots of this equation have negative real parts which leads to the inequalities

$$\sigma < \delta^2 + \frac{2\alpha\delta}{1 + \alpha\delta}, \qquad \alpha, \delta, \sigma > 0. \tag{14}$$

In this way the conditions for longitudinal stability involve three parameters α, δ, σ which have a well defined significance for the airplane: α is the angle of attack in level flight, δ measures the damping effect produced mainly by the horizontal tail surface, and σ is a parameter which determines the so-called static stability (essentially the rate of change of the moment M with the angle α). It seems that G. A. Bryan was the first to find this result as well as others, in 1911. A similar problem is set by the lateral stability which regards the displacement of the center of gravity normal to the symmetry plane and the rotation about the vertical and the longitudinal axes. Further analogous problems deal with the stability of the flight in curves, the stability of diving motion, *etc.*

In all these cases only small oscillations in the neighborhood of a given motion were considered. I want to give an idea also of the approach to a complete integration of the equations of motion, as attempted by Lanchester as early as 1907. We restrict ourselves again to the case of longitudinal motion with three dependent variables, assuming that the three coördinates which correspond to the lateral displacement, are constantly zero. So we can use the system (12) of differential equations. Now Lanchester assumes first that P equals D constantly, *i.e.*, that the drag is balanced by the propeller force at any instant. Then he simplifies the integration problem by supposing that the moment of inertia I is very small compared with the acting moment M, so that with an infinite angular velocity any deviation of the angle of attack is instantly corrected. By means of these assumptions the equations (12) are reduced to an integrable system with only two dependent variables v, θ. The solution can be written in the form

$$v = \sqrt{2gz}, \qquad \cos\theta = Az + \frac{B}{\sqrt{z}}, \tag{15}$$

where z is the vertical coördinate. The corresponding family of curves which represent the pathway of the airfoil has the character shown in Figure 6. It includes, as you see, some cases of looping. We call this type of airfoil motion phugoid motion. Scores of papers have been published dealing with the motion of an airplane under less restricted conditions which do not lead to closed integrals, but require numerical or graphical methods. You may find here an almost inexhaustible field for practicing the various methods of effective integration as developed in applied mathematics.

4. Boundary layer theory. As the fourth and last branch of aerodynamical research I want to mention the so-called boundary layer theory. Prandtl discovered in 1908 a kind of asymptotic integration of the classical differential equations for viscous fluids. For an infinitely small degree of viscosity Prandtl's integral gives an account of the behavior of the flow in the immediate neighborhood of a rigid body. In airplane design we are interested to a high degree in the question whether on the upper side of an airwing the flow sticks on the wing surface or whether it separates at a certain point as indicated in Figure 7.

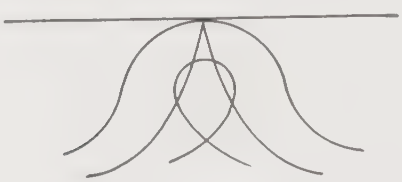

FIG. 6

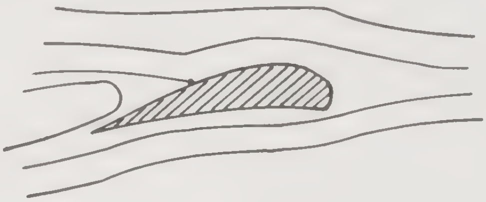

FIG. 7

I showed in 1927 that the Prandtl theory is equivalent to the following mathematical problem (which from a physical standpoint can be considered as a conduction of heat problem): We ask for integration of the partial differential equation of parabolic type

$$\frac{\partial z}{\partial y} = \sqrt{f(x) - z}\,\frac{\partial^2 z}{\partial x^2}, \tag{16}$$

where $f(x)$ is a given function, under the conditions

$$z = f(x) \text{ for } y = 0, \qquad z = 0 \text{ for } y = \infty, \qquad z = z_0(y) \text{ for } x = 0. \tag{17}$$

In the special case of our airwing problem we can assume

$$f(x) = 1 - x, \qquad z_0 = 0, \tag{18}$$

so that the problem includes no parameter, no arbitrary function. The question is: Has the solution $z(x,y)$ of the problem (16), (17), (18) a singular point on the x-axis, $x = x_1, y = 0$, characterized by the fact that $\partial z/\partial y$ vanishes, and, if it has, what is the numerical value of x_1? Recently Kármán and Millikan made an attempt to answer this question by employing methods of numerical approximation. It seems to me that from a mathematical point of view it would be worth while to reconsider the problem in order to give a rigorous proof at least for the existence of the mentioned singularity.

These are the four typical problems of aerodynamics I wanted to present to you. In the first we had to do with conformal mapping, in the second with an integral equation of singular type, in the third case it was a system of ordinary differential equations, and in the last a boundary problem of a partial differential equation of second order. You see that there is no lack of variety, and it would be easy to multiply the topics by enumerating further questions

which we encounter in aerodynamic practice. On the other hand, as I told you at the beginning of this lecture, the modern aircraft engineer is often more busy with problems which do not belong to aerodynamics at all, but to a senior sister branch of applied mechanics, I mean the elasticity theory or strength of materials or, as we say today, structural analysis. One could think that the high standard reached in mechanical and in civil engineering constructions must supply sufficiently elaborate fundamentals for settling the structural problems which are met in aircraft design. But the point is this, that it is compulsory in aviation to spare on weight to the utmost. We cannot use methods of computation based on superficial approximations and then multiply the dimensions by a big factor for the sake of safety, as is often done in the older branches of engineering. Even where the problems are quite the same, we need more careful and more detailed investigations. I want to give two examples.

5. Thin sheet constructions. In the design of airwings we notice a continuous trend from the very beginning up to now. The first wings were complicated systems of supporting beams and cross girders covered by a non-supporting skin which supplied the aerodynamically required shape—an imitation in a certain sense of the methods customary in bridge design. Now we tend more and more to a sort of "monolith" pattern where the whole wing must be computed as a self-supporting elastic shell or plate of very complicated form. The classical theories of elastic plates and shells with their linearized differential equations are not always sufficient and must be replaced by equations which involve certain terms of second order. Further problems arise in the design of the fuselage and similar constituents of an airplane, and I may mention here a new type of a two-dimensional elastic continuum which might suggest some interesting mathematical research work. It was introduced some years ago by Herbert Wagner in connection with the increasing use of thin metal sheets in airplane design. Wagner considers a strain and stress field in the xy-plane where the usual equilibrium conditions

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \qquad \frac{\partial \tau}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0, \tag{19}$$

(σ_x, σ_y, τ stress components) are fulfilled. But, instead of the usual two strain stress relations it is assumed that one of the two principal stresses,

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\tau^2 + \left(\frac{\sigma_x - \sigma_y}{2}\right)^2}, \tag{20}$$

say σ_2, is constantly zero and σ_1 alone is proportional to the expansion in its direction,

$$\sigma_2 = 0, \qquad \sigma_1 = E\epsilon_1. \tag{21}$$

This system of equations (19), (20), (21) determines completely a manifold of two-dimensional stress fields, as does, for example, the classical theory of plane

fields. It turns out that in the new case one of the two families of principal stress lines consists of straight lines and that we can therefore represent the general solution by means of four arbitrary functions. On the other hand, what would be essential, the corresponding boundary problem is not yet solved, even not yet formulated in a sufficiently general way.

6. Elastic stability problems. The last group of problems I want to mention here is concerned with elastic stability and is connected with the questions of vibrations which are a decisive factor in high speed aircraft. Starting with the famous stability investigation of Euler for a loaded column, mathematicians for two centuries have developed various solutions of stability problems, most of them concerned with one-dimensional continua. The large application of thin sheets in airplane design suggested many investigations in the field of elastic disks, plates, and shells. To give an example of one of the finest achievements

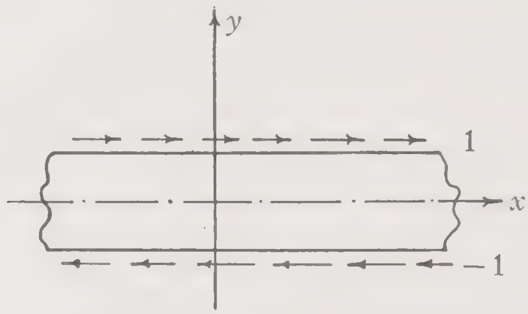

FIG. 8

I quote the problem solved by R. V. Southwell in 1924. A plane, infinite, rectangular strip is charged by shearing forces along its boundaries (Fig. 8) and it is asked under what load the strip loses its plane form. Mathematically speaking, we have the homogeneous partial differential equation of fourth order

$$\Delta\Delta u = \lambda \frac{\partial^2 u}{\partial x \partial y}, \tag{22}$$

and ask for the smallest Eigenwert (proper value) λ when the boundary conditions are

$$u = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad y = \pm 1. \tag{23}$$

The results of the mathematical theory of elastic stability are generally in accordance with experiments as far as plane sheets are concerned. Considerable discrepancies (not quite explained) are found in cases of cylindrical and spherical shells. Recently Th. v. Kármán and his disciples made a hopeful attempt to clear up these difficulties by enlarging the fundamentals of the theory. At any rate, a large field of mathematical research useful in aviation opens here.

Let me conclude, with these hints, my very incomplete and in many respects

most unsatisfactory report. I only hope that I have given you the impression that there is something for mathematicians to do, if they want to help in future progress in aeronautical engineering.

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THE RESOLVENTS OF A POLYNOMIAL*

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1. Introduction. The present paper is to be regarded as a synthesis of methods for solving the quartic and for proving the fundamental theorem of algebra. For a polynomial $f(z)$ with complex coefficients we define two resolvents R_x, R_y and find a simple relation between their roots and those of f (Theorem 5.1). Either of the resolvents may be employed to prove the fundamental theorem and to solve the quartic. We give the detailed discussion for R_x. For the reduced quartic, R_x is converted by the transformation $k=2x$ into the resolvent cubic appearing in Descartes's solution. Moreover, R_y can similarly be transformed into the equation which Lagrange† defined and applied in particular to the quartic. In the case of real coefficients Gordan‡ has employed a polynomial closely related to R_x in simplifying Gauss's second proof of the fundamental theorem of algebra. The proof of the fundamental theorem given here is believed to be simpler than Gordan's.

2. Definition of the resolvents. Let $f(z)$ be a polynomial of degree n with complex coefficients. Write $z=x+iy$ and

$$f(z) = g(x, y) + iyh(x, y), \tag{2.1}$$

where the terms of odd degree in y have been segregated so that g, h are polynomials in x, y^2. We denote by R_x the resultant of g, h written as polynomials in y^2 and by R_y the resultant of g, h written as polynomials in x. The polynomials R_x, R_y will be called respectively the *x-resolvent* and the *y-resolvent* of f.

First we wish to find the degree of R_x, R_y. By Taylor's theorem,

$$\begin{aligned} f(z) &= \sum_{j=0}^{n} \frac{f^{(j)}(x)}{j!}(iy)^j \\ &= \sum_{k=0}^{[n/2]} f_{n-2k}y^{2k} + iy\sum_{k=0}^{[(n-1)/2]} f_{n-2k-1}y^{2k}, \end{aligned}$$

where the brackets mean "the largest integer in," and where f_{n-l} is the polynomial of degree $n-l$ in x given by

$$f_{n-l} = (-1)^{(1/2)l(l-1)}f^{(l)}(x)/l!.$$

Thus for n even and equal to $2p$ we have

$$\begin{aligned} g(x, y) &= f_0y^n + f_2y^{n-2} + \cdots + f_n, \\ h(x, y) &= f_1y^{n-2} + f_3y^{n-4} + \cdots + f_{n-1}, \end{aligned} \tag{2.2}$$

* Presented at the meeting of the Southeastern Section of the Mathematical Association of America at Athens, Georgia, March 29, 1940.

† J. L. Lagrange, Oeuvres, vol. 8, Paris, 1879, pp. 24–28, 67–68.

‡ P. Gordan, Über den Fundamentalsatz der Algebra, Mathematische Annalen, vol. 10, 1876, pp. 572–575; Invariantentheorie, vol. 1, Leipzig, 1885, pp. 166–174.

and for n odd and equal to $2p+1$ we have

$$(2.3)\qquad \begin{aligned} g(x, y) &= f_1 y^{n-1} + f_3 y^{n-3} + \cdots + f_n, \\ h(x, y) &= f_0 y^{n-1} + f_2 y^{n-3} + \cdots + f_{n-1}. \end{aligned}$$

The term of maximum degree in R_x will result from selecting the term of highest degree in each element of the determinant form of R_x, provided the resulting coefficient is not zero. All terms in the resultant have fixed weight, even if the coefficients have the indices above instead of the conventional $0, 1, 2, \cdots$ used in defining the resultant. Hence we need examine only one term for degree. In the even case, $n=2p$, there is a term in $f_n^{p-1}f_1^p$ whose weight in the f's is $n(p-1)+p=p(2p-1)=\frac{1}{2}n(n-1)$. Hence R_x is of grade* $\frac{1}{2}n(n-1)$ in x. To find the coefficient of the highest power of x, replace in R_x each f_i by its leading term to get a determinant D all of whose terms have grade $\frac{1}{2}n(n-1)$. To show that $D\neq 0$, we evaluate it for $x=1$:

$$D(1) = \begin{vmatrix} e & -e & e & \cdots & -1 & 1 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e & e & -e & \cdots & 1 & 0 & 0 & \cdots \end{vmatrix},$$

where $e=\pm 1$. It is seen that $D(1)$ is the resultant of

$$\begin{aligned} P &= ex^p - ex^{p-1} + \cdots - x + 1, \\ Q &= \qquad - ex^{p-2} + \cdots - x + 1. \end{aligned}$$

If $D(1)$ were zero, P, Q would have a common factor which would have to divide $P-Q=ex^p$. As x divides neither P nor Q, we see that $D\neq 0$.

If $n=2p+1$, we consider the term $f_n^p f_0^p$, whose weight and degree in x are $np=\frac{1}{2}n(n-1)$, and reach the same conclusion.

A similar argument is used to prove that R_y is of degree $\frac{1}{2}n(n-1)$ in y^2.

3. Conjugate polynomials. If a is a complex number, the conjugate complex number is denoted by $\bar{a}$. Similarly, z, $\bar{z}$ represent conjugate indeterminates. The polynomial f^* arising from f by replacing each coefficient (but not z) by its conjugate is called the *polynomial conjugate* to f. The polynomial f has real coefficients if and only if $f=f^*$. Clearly

$$(3.1)\qquad \overline{f(z)} = f^*(\bar{z}).$$

By use of this relation we readily obtain

$$(3.2)\qquad (f+g)^* = f^* + g^*, \qquad (fg)^* = f^*g^*,$$

that is, the operation of taking the conjugate polynomial is distributive over both sums and products.

If ϕ is the highest common factor of f, f^* and if the initial (that is, leading co-

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* The grade in x is the maximum exponent of x appearing in the polynomial.

efficient) of ϕ is unity, simple manipulations based on the second of (3.2) show that ϕ has real coefficients and that we may write

$$f = \phi g, \qquad f^* = \phi g^*, \tag{3.3}$$

where every pair from ϕ, g, g^* is relatively prime.

Now let F be a polynomial with real coefficients divisible by f. Then $F=\psi\phi g$ implies $F=\psi^*\phi g^*$ because of (3.2). Hence F is divisible by g^* and, since g^* is relatively prime to g, F is divisible by the product gg^*, whose coefficients are real and whose degree is at most double that of f. Accordingly, we have the following:

THEOREM 3.1. *If a polynomial with real coefficients has a factor whose coefficients are complex and whose degree is k, it has a factor whose coefficients are real and whose degree is at most $2k$.*

4. Proof of the fundamental theorem. If f has complex coefficients, as we have seen ff^* has real coefficients. Moreover, if z is a root of ff^*, then z is a root of f or of f^*. If z is a root of f^*, from (3.1) rewritten as

$$\overline{f(\bar{z})} = f^*(z),$$

we see that $\overline{f(\bar{z})}=0$. Hence $f(\bar{z})=0$ and $\bar{z}$ is a root of f. Consequently, if every polynomial of positive degree with real coefficients has a root, every polynomial of positive degree with complex coefficients has a root.

Accordingly, let f be a polynomial with *real* coefficients. The coefficients of g, h in (2.1) and those of R_x, R_y are then also real.

Let the degree of f be $n=2^k \cdot q$, where q is odd. The pair of numbers (k, q) will be called the *index* of f. If (l, r) is the index of a second polynomial g, then f is said to have higher index than g provided either (i) $k>l$ or (ii) $k=l$ and $q>r$.

We wish to prove by induction on the index that every equation with positive degree and with real coefficients has a complex root. As a starting point, we have the case of index $(0, q)$, that is, the equations of odd degree. Each of these has a real root by the well known theorem whose proof is based on the axiom of continuity. Accordingly, we consider a polynomial f with real coefficients, even degree, and index (k, q) and assume that every polynomial with real coefficients, even degree, and lower index has a complex root.

The problem of finding the roots of f is equivalent to finding the roots of the simultaneous system

$$g(x, y) = 0, \qquad yh(x, y) = 0. \tag{4.1}$$

The roots z we wish can all be obtained from real values of x, y. It is clear, however, that a solution (x, y) of (4.1) leads to a root z of f even if x or y is imaginary. It is convenient not to make the restriction that x, y be real. Consequently, there will be several pairs (x, y) which will give the same root z. Moreover, we shall ultimately see that the factor y in the second equation of (4.1) can be omitted without losing any roots of f. In any event, it is certainly *sufficient* to prove the existence of a root (x, y) of

$$g(x, y) = 0, \qquad h(x, y) = 0. \tag{4.2}$$

Regarded as equations in y^2, these equations have resultant R_x. Moreover, since g contains iy raised to the even power n, the initial of g is $\pm$ the initial of f and is not zero. Consequently, if $R_x=0$ has a root x, the polynomials g, h evaluated for that x have a common factor of positive degree in y^2.

Now R_x has degree $\frac{1}{2}n(n-1)$. Since $n-1$ is odd, the index of R_x is $(k-1, n-1\ q)$, and this index is obviously lower than that of f. By assumption, R_x has a complex root $\alpha+\beta i$. For this x the polynomials g, h have a common factor, which from (2.1) is seen to divide $f(\alpha+\beta i+yi)$, so that we may write

$$f(\alpha + \beta i + yi) = \phi(y^2)\psi(y). \tag{4.3}$$

We wish first to dispose of the case when both ϕ and ψ are of degree $\frac{1}{2}n$ in y. Then ϕ is of degree $\frac{1}{4}n$ in y^2 and $\phi\phi^*$ is a polynomial which has real coefficients, which has degree $\frac{1}{2}n$ in y^2 and which therefore has index $(k-1, q)$. By assumption, $\phi\phi^*$ has a complex root y^2. As we saw at the beginning of this section this root leads to a root y^2 of ϕ and hence by the extraction of a square root to a root of f. Accordingly, in proving that f has a root, we may assume that ϕ or ψ has degree $<\frac{1}{2}n$ in y.

Changing the indeterminate to $z=\alpha+\beta i+yi$ in (4.3) gives

$$f(z) = f_1(z)f_2(z), \tag{4.4}$$

where f_1 has complex coefficients and degree less than $\frac{1}{2}n$. Theorem 3.1 shows that f has a factor whose coefficients are real and whose degree is less than n. Changing the notation, we may therefore interpret the f_1, f_2 in (4.4) as having real coefficients and positive degree less than n. Let the indices of f_1, f_2 be (k_1, q_1), (k_2, q_2). If both k_1, k_2 were greater than k, then n, which equals the sum of the degrees of f_1, f_2, would be divisible by a higher power of 2 than the kth. Hence the notation can be chosen so that $k_1 \leqq k$. If $k_1<k$, f_1 has a root because its index is less than (k, q). If $k_1=k$, the fact that the degree of f_1 is less than n gives $2^{k_1}q_1<2^k q$ or $q_1<q$, and the index of f_1 is again less than (k, q).

Hence every polynomial with real coefficients and with index (k, q) has a root, the induction is complete, and the fundamental theorem of algebra is fully established.

5. Roots of the resolvents. We return now to the consideration of the resolvent R_x when f has complex coefficients. Let z_1, z_2 be arbitrary complex numbers. By the formulas

$$x = \frac{1}{2}(z_1 + z_2), \qquad y = \frac{i}{2}(z_2 - z_1), \tag{5.1}$$

we define complex numbers x, y so that

$$z_1 = x + iy, \qquad z_2 = x - iy.$$

From (2.1) we then have

$$f(z_1) = g(x, y) + iyh(x, y),$$
$$f(z_2) = g(x, -y) - iyh(x, -y)$$
$$= g(x, y) - iyh(x, y).$$

Solving these for g, h and substituting from (5.1), we have the identities

$$\tfrac{1}{2}[f(z_1) + f(z_2)] = g[\tfrac{1}{2}(z_1 + z_2), \tfrac{1}{2}i(z_1 - z_2)],$$
$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = h[\tfrac{1}{2}(z_1 + z_2), \tfrac{1}{2}i(z_1 - z_2)]. \tag{5.2}$$

Now if z_1, z_2 are distinct roots of f, we see from (5.2) that $[\frac{1}{2}(z_1+z_2), \frac{1}{2}i(z_1-z_2)]$ is a root of (4.2). Consequently, $\frac{1}{2}(z_1+z_2)$ is a root of R_x and $-\frac{1}{4}(z_1-z_2)^2$ is a root of R_y. If f, R_x, and R_y have no multiple roots, we accordingly have the truth of the following:

THEOREM 5.1. *The roots of R_x are the $\frac{1}{2}n(n-1)$ arithmetic means of the roots of f taken in pairs. If the roots of R_y are multiplied by -4, they become the $\frac{1}{2}n(n-1)$ squares of the differences of the roots of f.*

To show the truth of the foregoing theorem in the general case, we proceed as follows. Let the initial of f be unity and let its roots be the indeterminates $z_1, z_2, \cdots, z_n$. The coefficients of the x-resolvent of f are symmetric polynomials in the z's. Let Q_x be a polynomial in x which has the same initial as R_x and which has the $\frac{1}{2}n(n-1)$ roots $\frac{1}{2}(z_j+z_k)$, $(j\neq k, j, k=1, 2, \cdots, n)$. Our task is to prove that the polynomial

$$Q_x - R_x \tag{5.3}$$

is identically zero in $x, z_1, \cdots, z_n$. We already know that (5.3) vanishes identically in x when the roots of f and of R_x are simple. Choose $z_1, \cdots, z_n$ successively as positive integers satisfying

$$z_1 + \cdots + z_j < z_{j+1}. \tag{5.4}$$

Then the values z_j, $\frac{1}{2}(z_j+z_k)$ are all distinct. Consequently, (5.3) has for coefficients polynomials in $z_1, \cdots, z_n$ which are zero for all positive integers z chosen in accordance with (5.4).

To proceed, we find it convenient to have the following:

THEOREM 5.2. *A polynomial in $z_1, \cdots, z_n$ whose grade in the indeterminate z_j is g_j is identically zero if and only if it has the roots $(z_1, z_2, \cdots, z_n)$, where each z_j ranges independently over a complete residue system* mod g_j+1.

The proof, made by induction and based on the theorem that a polynomial in a single indeterminate is zero if it has distinct roots greater in number than its grade, is left to the reader.

Since each residue class for any modulus contains arbitrarily large integers, it is clear that we can find integers $(z_1, \cdots, z_n)$ satisfying both (5.4) and the

condition of Theorem 5.2. Hence (5.3) has coefficients which are identically zero in $z_1, z_2, \cdots, z_n$, and Theorem 5.1 is completely established, in so far as it concerns R_x. The treatment of R_y is similar and is omitted.

6. Solution of the x-resolvent for the quartic. If $n=2$, R_x is linear and can be used to solve the quadratic. For $n=3$, R_x is cubic; and every cubic is the x-resolvent of a second cubic.

If $n>3$, however, $\frac{1}{2}n(n-1)>n$ and Theorem 5.1 shows that R_x is an equation of degree $\frac{1}{2}n(n-1)$ with special properties since the roots of R_x can be found by solving the equation f of degree n.

It is instructive to use these special properties to solve R_x in the case of a quartic

$$f(z) = z^4 - E_1z^3 + E_2z^2 - E_3z + E_4. \tag{6.1}$$

The x-resolvent R_x satisfies

$$R_x = (2x)^6 - F_1(2x)^5 + F_2(2x)^4 - F_3(2x)^3 + F_4(2x)^2 - F_5(2x) + F_6,$$

where

$$F_1 = 3E_1, \qquad F_2 = 3E_1^2 + 2E_2, \qquad F_3 = E_1^3 + 4E_1E_2,$$
$$F_4 = 2E_1^2E_2 + E_1E_3 + E_2^2 - 4E_4, \qquad F_5 = E_1^2E_3 + E_1E_2^2 - 4E_1E_4,$$
$$F_6 = -E_1^2E_4 + E_1E_2E_3 - E_3^2.$$

Denoting the roots of f, R_x by z_j, x_k, we write

$$\begin{aligned} z_1 + z_2 &= 2x_1, & z_1 + z_3 &= 2x_2, & z_1 + z_4 &= 2x_3, \\ z_3 + z_4 &= 2x_6, & z_2 + z_4 &= 2x_5, & z_2 + z_3 &= 2x_4. \end{aligned} \tag{6.2}$$

We then have the obvious relations

$$x_1 + x_6 = x_2 + x_5 = x_3 + x_4 = \tfrac{1}{2}E_1. \tag{6.3}$$

Next evaluate the elementary symmetric polynomials for the x's. We have

$$\begin{aligned}\sum x_1x_2 = {} & x_1x_6 + x_2x_5 + x_3x_4 + (x_2 + x_5)(x_3 + x_4) \\ & + (x_3 + x_4)(x_1 + x_6) + (x_1 + x_6)(x_2 + x_5).\end{aligned}$$

Each of the first three terms is the product of two letters constituting a pair in (6.3). The others are products of letters from different pairs. Write similar expressions for the other elementary symmetric polynomials in the x's. Denote the elementary symmetric polynomials in the three products x_1x_6, x_2x_5, x_3x_4 by G_1, G_2, G_3. We then have, by use of (6.3),

$$F_1 = 3E_1, \qquad F_2 = 4G_1 + 3E_1^2,$$
$$F_3 = 8E_1G_1 + E_1^3, \qquad F_4 = 16G_2 + 4E_1^2G_1,$$
$$F_5 = 16E_1G_2, \qquad F_6 = 64G_3.$$

Hence G_1, G_2, G_3 are polynomials in the E's, which are easy to write from the above formulas.

The three products x_1x_6, x_2x_5, x_3x_4 are the roots of

$$t^3 - G_1t^2 + G_2t - G_3. \tag{6.4}$$

The roots of this cubic can be found by Cardan's method. These values equated to x_1x_6, x_2x_5, x_3x_4 and (6.3) give the equivalent of three quadratics whose roots are the pairs

$$x_1,\ x_6; \qquad x_2,\ x_5; \qquad x_3,\ x_4. \tag{6.5}$$

The order in which the roots of the cubic are identified with x_1x_6, x_2x_5, x_3x_4 makes no difference because of the symmetry of (6.3).

7. Use of R_x to solve the quartic. Now suppose $x_1, \cdots, x_6$ found, and seek the roots of the quartic. First the x's must be grouped into three pairs (6.5) each of which has sum equal to $\frac{1}{2}E_1$. It is, moreover, an easy exercise in symmetric polynomials to show that (6.1), (6.2) imply also

$$(x_1 - x_6)(x_2 - x_5)(x_3 - x_4) = -f'(\tfrac{1}{4}E_1). \tag{7.1}$$

Suppose $x_1, \cdots, x_6$ have been identified with the roots of R_x in such a way that both (6.3) and (7.1) are satisfied. Because of (6.3), system (6.2) comprising six equations is consistent in the four unknown z's and has the unique solution

$$\begin{aligned} z_1 &= x_1 + x_2 - x_4, & z_3 &= -x_1 + x_2 + x_4, \\ z_2 &= x_1 - x_2 + x_4, & z_4 &= -x_1 + x_3 + x_5. \end{aligned} \tag{7.2}$$

The following statements are readily verified: (i) if the pairs (6.5) are permuted, the order in each pair being preserved, the values (7.2) are simply permuted; (ii) if the order is inverted in two pairs simultaneously, the values (7.2) are again permuted; (iii) if the order is inverted in any single pair, the roots (7.2) are changed into a single set, which in general is different from (7.2). The last inversion, however, preserves (7.1) only if $f'(\frac{1}{4}E_1)=0$. When $f'(\frac{1}{4}E_1)=0$, it is seen from (7.1) that the two members of at least one pair, say 1, 6, are equal. The transposition (16) will not alter (7.2), whereas the transposition (25) can be replaced by (16)(25) which has been seen to permute (7.2). Hence if the x's are roots of R_x satisfying (6.3) and (7.1), the only possible roots of (6.1) are given by (7.2). Since the fundamental theorem states the existence of four roots, the numbers (7.2) must accordingly be roots of f.

Since $\sum x = \frac{3}{2}\sum z$, the same translation will make the sum of the x's and the sum of the z's simultaneously zero. This translation leaves (6.3), (7.1), (7.2) invariant. We now interpret the formulas as applying to the reduced quartic. The sum of each pair (6.5) is zero. Hence if the quartic is

$$z^4 + Qz^2 + Rz + S, \tag{7.3}$$

the solution is

$$z_1 = x_1 + x_2 + x_3, \qquad z_3 = -x_1 + x_2 - x_3,$$
$$z_2 = x_1 - x_2 - x_3, \qquad z_4 = -x_1 - x_2 + x_3,$$

where x_1, x_2, x_3 are roots of the resolvent

$$R_x = (2x)^6 + 2Q(2x)^4 + (Q^2 - 4S)(2x)^2 - R^2$$

which satisfy

$$8x_1x_2x_3 = -R.$$

The solution is readily recognized as Descartes's with the answer put in Euler's form,* and the resolvent as the resolvent cubic in $k = 2x$.

We note in passing that the quartic

$$z^4 + Qz^2 - Rz + S,$$

which is distinct from (7.3) if $R \neq 0$, has the same resolvent as (7.3). Its roots are the negatives of those of (7.3) and satisfy (7.1) only in the exceptional case $R = 0$.

8. Use of resolvents in the general case. From §5 we know that for an equation of degree n we may write

$$z_j + z_k = 2x_{jk}, \qquad (j < k), \tag{8.1}$$

where x_{jk} are the roots of R_x. When R_x has been solved, it is possible to order the roots x so that (8.1) are consistent and give the roots of f. We shall not deduce the criteria like (6.3) and (7.1) for identifying the roots with x_{jk}, but simply remark that when identification has been properly made, a unique solution is

$$z_1 = x_{12} + x_{13} - x_{23},$$
$$z_j = 2x_{1j} - x_{12} - x_{13} + x_{23}, \qquad (j = 2, 3, \cdots, n).$$

Similarly, a solution of f in terms of the roots of R_y is

$$z_1 = \frac{1}{n}\left(E_1 + 2i\sum_{k=2}^{n} y_{1k}\right),$$
$$z_j = \frac{1}{n}\left(E_1 + 2i\sum_{k=2}^{n} y_{1k}\right) - 2i\, y_{1j}, \qquad (j = 2, 3, \cdots, n).$$

We wish next to verify that system (4.2) gives all the roots of f, as was stated in §4. If all the roots of f equal z_1, by Theorem 5.1 all the roots of R_x equal z_1 and all the roots of R_y equal zero. In this case the only root of (4.2) is $(z_1, 0)$ and it gives the root $z = z_1 + 0 \cdot i$ of f. If z_1, z_2 are distinct roots of f, from (5.2) we see that $[\frac{1}{2}(z_1+z_2), \pm\frac{1}{2}i(z_1-z_2)]$ are roots of (4.2) and they give, respectively,

$$z = \tfrac{1}{2}(z_1 + z_2) + \tfrac{1}{2}i(z_1 - z_2)i = z_2,$$
$$z = \tfrac{1}{2}(z_1 + z_2) - \tfrac{1}{2}i(z_1 - z_2)i = z_1$$

* *Cf.*, for example, J. M. Thomas, Theory of Equations, New York, 1938, pp. 115, 116.

as the roots of f. Since every other root of f is different from z_1 or from z_2, this argument can be repeated to show that (4.2) gives all the roots of f.

9. Alternative resolvents in the complex case. If we write

$$f = g + ih,$$

where g, h are real, and let S_x, S_y be the resultants of g, h written as polynomials in y, x respectively, we have alternative resolvents for the complex case. Moreover, if all initials are made unity, and if f has real coefficients, then

$$S_x = fR_x^2, \qquad S_y = y^n R_y.$$

In any case, the n^2 roots of S_x, S_y have, respectively, the forms $\frac{1}{2}(z_j+w_k)$, $(z_j-w_k)/2i$, where z_j is a root of f, w_k is a root of f^*, and j, k range independently over $1, 2, \cdots, n$.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. J. WALKER, Fine Hall, Princeton, N. J.

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

THE GNOMONIC PROJECTION OF THE SPHERE

A. D. BRADLEY, Hunter College

The important topic of map projections receives little emphasis in courses in mathematics. It is the purpose of this note to present the development of the transformation for the gnomonic projection as an exercise in plane trigonometry and analytic geometry. Since the gnomonic projection is one of the most venerable developments of the sphere, no particular originality can be claimed for the material presented here. However the transformation in its polar form (formulas I below) seems not to be given in the standard treatises on projections, and the rectangular form of the transformation here given involves only one auxiliary angle rather than the two usually given.*

The gnomonic or great circle projection of the sphere has the property that great circles on the sphere are mapped by straight lines, and, that straight lines on the map are projections of great circles on the sphere. The importance of this projection in navigation is obvious. The gnomonic projection belongs to the general class of perspective projections, and is, therefore, a projection in the geometric sense. The principle of the perspective projections is that the points of the sphere are projected on the plane of the map by straight lines through some

* See, for example, the treatment given in the following:

A. Germain, Traité des Projections des Cartes Géographiques, Paris, 1866.

Norbert Herz, Lehrbuch der Landkarten Projektionen, Leipzig, 1885.

Deetz and Adams, Elements of Map Projection, U. S. Coast and Geodetic Survey, Special Publication No. 68, fourth edition, 1934.

given point. In the gnomonic projection the center of projection is the center of the sphere. From this it is obvious that the parallels of latitude are mapped as conics and the great circle property mentioned above is also obvious.

In the works of Germain and Herz, formulas are developed for the general perspective projection, and the gnomonic is obtained as a special case. Adams and Deetz obtain the formulas for the gnomonic transformation directly by using spherical trigonometry. The following development consists in finding the polar form of the transformation by plane trigonometry and transforming this to obtain the rectangular form.

In Figure 1, C is the center of a sphere, O' its north pole, and $A'B'$ an arc of its equator; the sphere is to be mapped on a plane, OAB, which is tangent to the prime meridian, $O'TA'$, at a point T of given latitude ϕ_1. It is required to determine the coördinates of P in the plane OAB as functions of the latitude, ϕ,

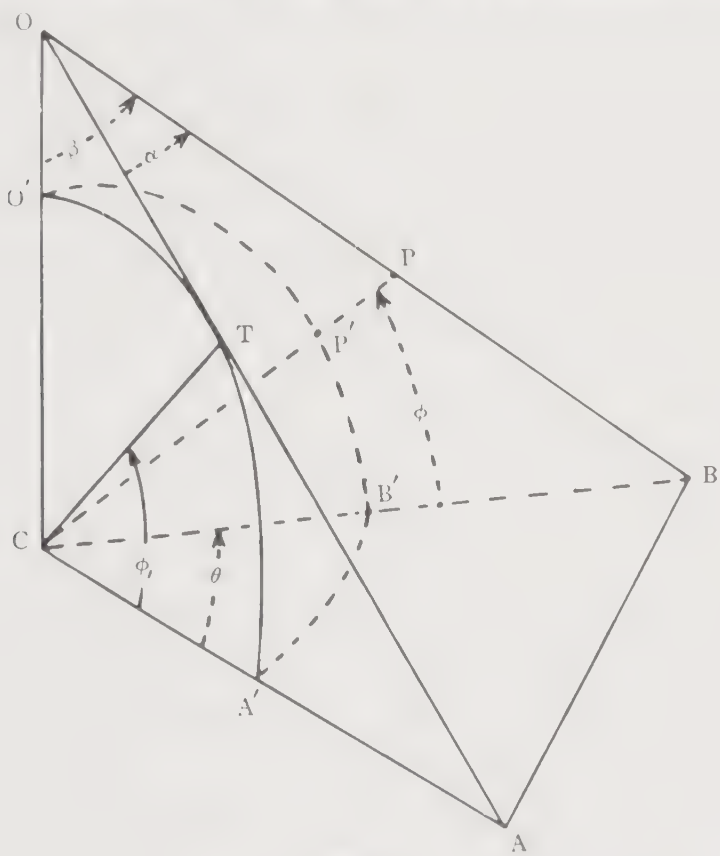

FIG. 1

and longitude, θ, of the corresponding point P' of the sphere. If polar coördinates are used, and if the projection of the prime meridian, OTA, is taken as the axis, then the coördinates are $\alpha=$angle AOB and $\rho=OP$. It is convenient to use an auxiliary angle, $\beta=$angle COB, the angle between the polar axis and the projection of the meridian $O'P'B'$. Expressions for α, β, and ρ are easily found by trigonometry, since OC is perpendicular to the plane ABC; BA to the plane OAC; and, CT to the plane OAB.

$$\text{Triangle } OCA: \quad \begin{aligned} CO &= r \csc \phi_1, \\ CA &= r \sec \phi_1, \\ OA &= r \csc \phi_1 \sec \phi_1. \end{aligned}$$

$$\text{Triangle } CBA: \quad \begin{aligned} AB &= r \sec \phi_1 \tan \theta, \\ CB &= r \sec \phi_1 \sec \theta. \end{aligned}$$

$$\text{(Ia)} \qquad \text{Triangle } OAB: \quad \tan \alpha = \frac{r \sec \phi_1 \tan \theta}{r \csc \phi_1 \sec \phi_1} = \sin \phi_1 \tan \theta.$$

$$\text{(Ib)} \qquad \text{Triangle } OCB: \quad \tan \beta = \frac{r \sec \phi_1 \sec \theta}{r \csc \phi_1} = \tan \phi_1 \sec \theta.$$

$$\text{(Ic)} \qquad \text{Triangle } OCP: \quad \rho = \frac{r \csc \phi_1 \cos \phi}{\cos (\phi - \beta)} = r \csc \phi_1 \cos \phi \sec (\phi - \beta).$$

If a gnomonic chart is not to include the polar regions, it is convenient to express the transformation in rectangular coördinates. If the projection of the prime meridian is taken as the X-axis and the projection of the meridian of 90° as the Y-axis, the formulas for α and ρ are readily transformed to expressions for x and y.

From (Ib) we have

$$\begin{aligned} \sec^2 \beta \cos^2 \phi_1 &= (\tan^2 \beta + 1) \cos^2 \phi_1 \\ &= (\tan^2 \phi_1 \sec^2 \theta + 1) \cos^2 \phi_1 \\ &= \sin^2 \phi_1 \sec^2 \theta + \cos^2 \phi_1 \\ &= \sin^2 \phi_1 \tan^2 \theta + 1. \end{aligned}$$

From (Ia) we have

$$\begin{aligned} \sec^2 \alpha = \tan^2 \alpha + 1 &= \frac{x^2 + y^2}{x^2} \\ &= \sin^2 \phi_1 \tan^2 \theta + 1. \end{aligned}$$

Hence

$$\text{(IIa)} \qquad \frac{x^2 + y^2}{x^2} = \sec^2 \beta \cos^2 \phi_1.$$

From (Ic) we have

$$\text{(IIb)} \qquad \rho^2 = x^2 + y^2 = r^2 \csc^2 \phi_1 \cos^2 \phi \sec^2 (\phi - \beta).$$

Solution of equations II yields

$$\text{(IIIa)} \qquad x = \frac{2r \cos \phi \cos \beta}{\sin 2\phi_1 \cos (\phi - \beta)},$$

$$\text{(IIIb)} \qquad y = \frac{r \cos \phi \cos \beta \tan \theta}{\cos \phi_1 \cos (\phi - \beta)}.$$

If the point of tangency is taken as the origin and the X- and Y-axes are respectively the projections of the prime meridian and the great circle orthogonal to it at T, the transformation becomes

$$\text{(IV)} \qquad \begin{aligned} x' &= x - r \cot \phi_1, \\ y' &= y. \end{aligned}$$

The transformation (IV) is of course equivalent to that given by Deetz and Adams. The formulas of Deetz and Adams involve functions of two auxiliary angles, while the above transformation requires only the auxiliary angle β. In the actual construction of a map it is sufficient to calculate the coördinates of only two points in order to determine a given meridian. The parallels may then be plotted by finding one coördinate of their intersection with the meridians.

The parallels are mapped as conics with eccentricity $e = \cos \phi_1 \csc \phi$.

From (IIa) we have

$$\begin{aligned} \cos \beta &= \cos \alpha \cos \phi_1, \\ \rho \cos \beta &= \rho \cos \alpha \cos \phi_1 = x \cos \phi_1, \\ \rho^2 \sin^2 \beta &= \rho^2(1 - \cos^2 \phi_1 \cos^2 \alpha) \\ &= \rho^2(1 - \cos^2 \alpha + \sin^2 \phi_1 \cos^2 \alpha) \\ &= y^2 + x^2 \sin^2 \phi_1. \end{aligned}$$

Equation (Ic) may be written in the form,

$$\rho \tan \phi \sin \beta = r \csc \phi_1 - \rho \cos \beta.$$

Squaring this equation and applying the above relations, we obtain the equation of the parallel of $\phi°$,

$$\text{(V)} \qquad (\sin^2 \phi_1 \tan^2 \phi - \cos^2 \phi_1)x^2 + 2r \cot \phi_1 x + \tan^2 \phi y^2 = r^2 \csc^2 \phi_1.$$

If

$$f = \sin^2 \phi_1 \tan^2 \phi - \cos^2 \phi_1 = \sin^2 \phi_1 \sec^2 \phi - 1,$$

then

$$f \cot^2 \phi = 1 - e^2, \quad \text{and} \quad f \csc^2 \phi_1 + \cot^2 \phi_1 = \tan^2 \phi.$$

For $f \neq 0$, equation (V) may be written,

$$\text{(VI)} \qquad (1 - e^2)\left(x + \frac{r \cot \phi_1}{f}\right)^2 + y^2 = (1 - e^2)\frac{r^2}{f^2} \tan^2 \phi.$$

The parallels in the interval $90° - \phi_1 < \phi < 90°$ are mapped as ellipses, for in this case $0 < f$ and $e < 1$. Similarly, if $-(90° - \phi_1) < \phi < 90° - \phi_1$, $f < 0$ and $1 < e$, and

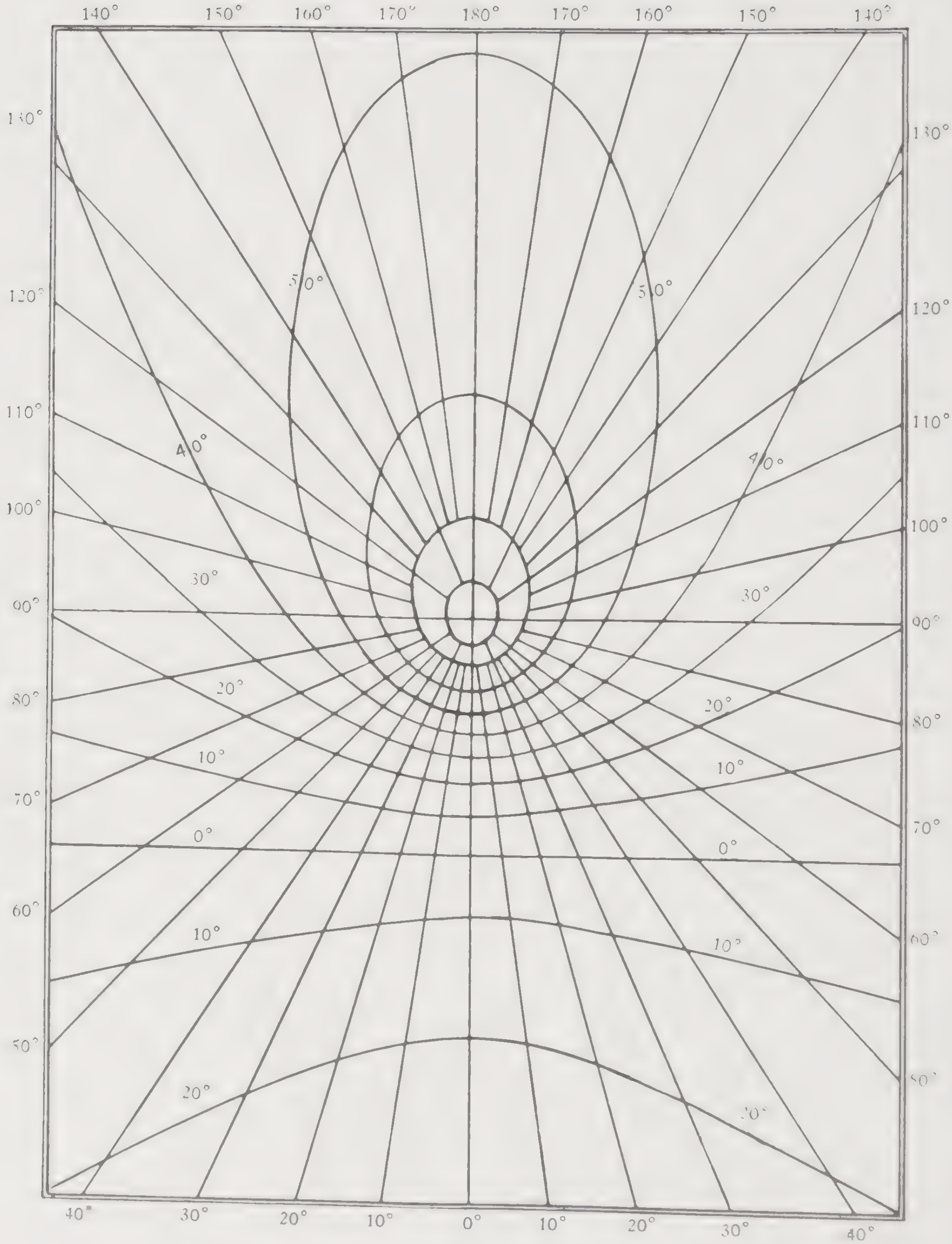

FIG. 2. Gnomonic projection on a plane tangent at latitude 50°.

these parallels are mapped as hyperbolas. If $\phi=90°-\phi_1$, then $f=0$ and $e=1$, and equation (V) becomes

$$y^2 = -2r \cot \phi x + r^2 \csc^2 \phi,$$

so that the parallel of $90°-\phi_1$ is mapped by a parabola.

Two special cases of the gnomonic projection are of interest though not of great practical importance. If the plane of the map is tangent at the pole of the sphere, the parallels are mapped as circles. If the plane of the map is tangent at the equator the parallels are mapped as hyperbolas. Many elementary exercises of rather interesting character may be based on these special cases.* While the gnomonic projection has little practical application other than navigation, it was in use long before the relatively late development of great circle sailing, and tradition would make it one of the first projections used by the Greeks. The history of this projection has been fully treated by Günther and d'Avenzac.†

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Non-Euclidean Geometry or Three Moons in Mathesis. Second edition. By Lillian R. Lieber; drawings by H. G. Lieber, Lancaster, Pa., The Science Press, 1940. 40 pages. $1.25.

Mathematisch-Astronomische Blätter, mit über 100 Figuren auf 11 Tafeln. Heft 1. March, 1940. (Herausgegeben von der Mathematischen-Astronomischen Sektion der Freien Hochschule für Geisteswissenschaft am Goethaenum, Dornach, Switzerland. Winterthur, A. Vogel, 1940. 96 pages. Fr. 3.50 (Swiss).

Wiley Trigonometric Tables. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Limited, 1940. 3+81 pages. $0.75.

REVIEWS

Advanced Algebra. By S. Barnard and J. M. Child. London, Macmillan and Company, 1939. 10+278 pages. $4.00.

The *Advanced Algebra* by Barnard and Child is a continuation of the *Higher Algebra* by the same authors. Its scope has been determined by what is neces-

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* Appropriate references may be found in Club Topics, No. 32, this MONTHLY, vol. 46, 1939, p. 650. See also W. H. Roever, Some drawings and graphic solutions in navigation, this MONTHLY, vol. 25, 1918, p. 415.

† d'Avenzac, Coup d'Oeil Historique sur la Projection des Cartes Géographiques, Paris, 1863. S. Günther, Die gnomonische Kartenprojektion, Zeitschrift der Gesellschaft für Erdkunde zu Berlin, Band 18, 1883.

sary for Honors Degrees at the Universities. Consequently it deals with the subjects set forth in the following list of Chapter headings.

I. The homographic relation. II. The quadratic and systems of quadratics. III. Double series. IV. Uniform convergence. V. The complex variable. VI. Exponential and logarithmic functions. VII. Elimination. VIII. Probability. IX. Continued fractions. X. Quadratic residues. XI. Indeterminate equations of the second degree. XII. Primitive roots. XIII. The equation $x^n = 1$, where n is an odd prime.

MAX ZORN

Mathematics and the Imagination. By Edward Kasner and James Newman, with drawings and diagrams by Rufus Isaacs. New York, Simon and Schuster, 1940. 14+380 pages.

The authors of this readable book have set themselves the task "to extend the process of '*haute vulgarisation*' to those outposts of mathematics which are mentioned, if at all, only in a whisper; which are referred to, if at all, only by name; to show by its very diversity something of the character of mathematics, of its bold, untrammeled spirit, of how, as both an art and a science, it has continued to lead the creative faculties beyond even imagination and intuition" (p. xiv). They have succeeded in that task by writing of many varied aspects of mathematics with wit and clarity. At the same time, they bring the reader repeatedly face to face with the dangers of inaccurate reasoning and the need for clearly stated assumptions.

In keeping with the title, Kasner and Newman have laid greatest emphasis upon those aspects of modern mathematics which seem so mysterious to the layman because of his inability to imagine objects to fit the concepts. Thus they include discussions of infinite classes and transfinite numbers in a chapter called "Beyond the Googol," and chapters on "Assorted Geometries—Plane and Fancy," and "Rubber Sheet Geometry." Following the same theme, paradoxes of various kinds are discussed at various points throughout the book and there is a special chapter entitled "Paradox Lost and Paradox Regained."

In addition to these topics which seem highly dramatic, others more likely to be familiar to those who have had a moderate exposure to mathematics are also included. Thus, transcendental and complex numbers, chance and probability, and the calculus are all treated in the same readable fashion.

While the book is full of amusing illustrations and engrossing puzzles, it is not as superficial as one might expect a book at the popular level to be. For example, the description of the invention of the term "googol" by Dr. Kasner's nine-year-old nephew is amusing, but underlying it is a very serious attempt to show how misused is the term "infinite" when applied to large and finite numbers. Again, in the discussion of four-dimensional and non-euclidean geometries the reader is given a clear notion of the fundamental assumptions underlying these developments. In addition he is clearly shown the great danger in confusing geometrical with physical space. Altogether, a great many famous and im-

portant mathematical problems, fallacies, and sources of error are covered by the authors.

Another contribution which the present reviewer, at least, regards as important is that the discussion deliberately leaves the reader dissatisfied at various points. For example, the concepts underlying the theory of point sets and the theorems of Hausdorff, Banach, and Tarski are presented at just enough length to create a desire for more. In addition to footnotes at the end of each chapter, the authors append a selected bibliography at the end of the book. As the authors say, "It is not (nor is it intended to be) full, authoritative, or exhaustive. It is merely an expression of personal tastes which may be helpful to the reader whose curiosity has been stimulated" (p. 363). A useful feature is that each book listed is described briefly and its relative difficulty is assessed.

No listing of topics, even in greater detail than the list presented above, can convey the flavor of Kasner and Newman's accomplishment. Quotation is not satisfactory either, since much of the humor in their examples and comments depends upon the setting in which they appear. With this warning, however, we may venture one quotation. In the discussion of infinite and finite numbers mentioned above we find the following example: "Though people do a great deal of talking, the total output since the beginning of gabble to the present day, including all baby talk, love songs, and congressional debates, totals about 10^{16} . . . . Contrary to popular belief, this is a larger number of words than is spoken at the average afternoon bridge" (p. 21).

It is difficult to guess what the audience of this book will be. The present reviewer approaches it as a layman with an interest in mathematics and some knowledge of the standard branches of the discipline. The reactions recorded above are therefore to be taken in that light. Certainly the reviewer's interest and knowledge gained from the reading. It seems that the book should appeal to anyone who wishes to learn something of modern mathematics. Perhaps, also, some will be led to discover an interest which they did not know existed.

T. A. RYAN

Mathematico-Deductive Theory of Rote Learning. A Study of Scientific Methodology. By C. L. Hull, C. I. Hovland, R. T. Ross, M. P. Hall, D. T. Perkins, and F. B. Fitch. (Published for the Institute of Human Relations.) New Haven, Yale University Press; London, Humphrey Milford and Oxford University Press, 1940. 12+331 pages. $3.50.

The coördinated efforts of a number of Yale specialists have resulted in this study in scientific methodology. Three of the authors, Marshall Hall, Robert T. Ross, and Donald T. Perkins, are mathematicians while the others come from the field of psychology. Psychologists have experimented with the theory of rote learning and demonstrated certain facts concerning it. Professor Hull made the experiment some years ago of applying formal theoretical methodology to this psychological problem. From the earlier study of a system consisting of only eleven theorems and from the present conviction that the time was

right to attempt a more complete experiment, this book has developed.

In order to make the problem clear there is a chapter explaining the psychological experiments in rote learning. Briefly, the *subject* is confronted with a rotating drum on which a series of meaningless syllables is placed. As each syllable is presented the subject recalls the succeeding one. The number of failures before complete mastery of the series, the last failure on each syllable, and the first successful recall of each syllable are noted. From such data the psychologist has been able to reach some conclusions concerning the advantages of distributed over massed practice and evidence that there are optimal lengths of learning tasks.

The system itself is constructed to conform to scientific procedure. Thus there are 16 undefined terms including *syllable exposure*, *subject*, *reaction*, *stimulus trace*, *excitatory potential*, *inhibitory potential*, *reaction threshold*, and *basic reaction latency*. These are used in defining over 80 terms such as *syllable presentation cycle*, *syllable presentation number*, *rote learning*, *massed practice*, and *compound stimulus trace*. This list of definitions is followed by 18 postulates and 54 theorems with corollaries.

For the benefit of different classes of readers the authors have used double proofs and explanations. In stating the definitions and postulates after careful wording one finds restatement in symbolic logic. Further, each mathematical proof of a theorem or corollary is followed by a paragraph in ordinary language rewording the proof for non-mathematicians. This again in most cases is followed by data bearing on the validity of the proposition as found by experiment.

The contents are impressive evidence that coöperating specialists in related fields can supply sounder concepts of behavior than it is possible in any single field alone. The numerical results from experiments on rote learning are analyzed to yield mathematical and logical corroboration of inferences from recorded observations. It is an advantage to be able to express amounts of inhibition mathematically. And the concept of *compound stimulus trace* is more clearly interpreted by techniques of mathematics and symbolic logic than is possible in mere verbal description.

Some of the theorems are particularly notable. Theorem XI, page 122, gives proof concerning retention as evidenced by the economy of relearning compared to new learning. Theorem XIII, page 127, supports the advantage of distributed over massed practice by demonstrating that it is a linear function of the inhibition induced by one repetition. Further, Theorem XIV states that the advantage of distributed practice is an increasing function of the number of repetitions needed for learning by massed practice. The relation between difficulty of learning and length of series is brought out by Theorem XXVI, page 201. This and subsequent theorems show that the increase in the number of repetitions needed for learning the longer series follows a mathematical equation. Theorem XXI and its corollaries seem to find proof not only of the greater difficulty of learning syllables in central positions in the series but also of the relatively greater advantage of distributed over massed practice for this portion of the series.

Since most of the results obtained here merely corroborated experiments, the contribution of this book seems to be a pointing in the direction towards which mathematicians and psychologists can reduce controversy and promote more significantly planned and therefore more productive experimentation.

CAROLINE A. LESTER

Plane and Spherical Trigonometry. Second edition. By L. M. Kells, W. F. Kern, and J. R. Bland. New York and London, McGraw-Hill Book Company, Inc., 1940. 16+402 pages. \$2.00. (With tables, \$2.75.)

This book is a revision of a well known text-book that appeared in 1935 and was reviewed by me in this MONTHLY, vol. 43, 1936, page 39.

The second edition is considerably longer than the first, which ran to 270 pages (exclusive of tables). Although the first edition already covered a wide scope of topics the new one contains in addition three appendices dealing with (1) the use of the mil in military science, (2) the range finder, and (3) stereographic projection. The expansion of the text-book is, however, only in small part attributable to these appendices; it is due primarily to pedagogical improvements resulting, as the authors state in the preface, from suggestions and criticism derived from classroom experience. To achieve a better psychological development, the first chapter and many sections throughout the book have been rewritten and amplified with a view to stimulating interest and motivating new concepts. Definitions and explanations have been clarified by fuller treatment and by the addition of numerous helpful diagrams and pictures. A feature notable in the first edition and emphasized in the second is the skill of the authors in anticipating students' difficulties; for example, in the new edition the passage from the definitions of the trigonometric functions of acute angles to the general definitions, which appear in a later chapter, is ingeniously bridged.

The lists of exercises, though adequate in the first edition, have been considerably amplified, and sets of review exercises have been added. Following the practice observed in the earlier edition, the new exercises are graded and in many cases they are accompanied by provocative diagrams. A notable feature of the new lists of problems is the inclusion of numerous applications to navigation, engineering, military science, and objects of everyday experience. Bright students will find many challenging exercises throughout the book. Answers to exercises are provided at the end of the book.

It is regrettable, in the reviewer's opinion, that the authors have retained symbols such as $\sin^{-1}x$ for the inverse trigonometric functions. Another criticism that should be pointed out is that the authors have on occasion, though less often than is usual in books of this kind, followed traditional treatment at the expense of rigor; while some such lapses are inevitable in a text-book intended for freshmen, others can and should be avoided; for instance, in the derivation of the formula (page 130), $\tan \phi/2 = (1-\cos \phi)/\sin \phi$, the authors of this book, like those of most trigonometry text-books seen by the reviewer, drop a minus

sign without explanation. Two typographical errors were noted by the reviewer, namely, $a-bi$ for $a+bi$ on page 201, and the spelling *De Moirve* for *De Moivre* on page 208. The reviewer does not intend, however, by his criticism to detract from the high merits of this excellent text-book.

J. M. FELD

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N. J.

Reports received indicate that a number of clubs sponsor a service for helping students, particularly freshmen, who may be experiencing difficulties in their study of mathematics. Some colleges have found it worth while to place in the hands of every freshman, directions on "How to Study." An example is the "Guide to Study for DePauw Freshmen" which has been published by DePauw University and which includes suggestions for study of each of the major subjects. The material for the mathematics student has been prepared by Professor W. C. Arnold, and we are including it here in the belief that it may prove of aid to clubs as well as departments of mathematics faced with this problem.

HOW TO STUDY MATHEMATICS

W. C. ARNOLD, DePauw University

This paper has been written to help you, the student enrolled in freshman mathematics. It has for its specific purpose your instruction in the *technique of studying mathematics*. Studying is to be your career for four years. Take sufficient professional pride to be as efficient as possible in your career.

The subject will be discussed under the heads: general instructions, how to study the text, how to solve problems, how to retain what is learned, how the teacher can help.

A. GENERAL INSTRUCTIONS

1. First of all go easy on extra-curricular activities until you have tried yourself out on college work. Make your adjustments to your studies and then if all goes well you can enter other activities later.

2. Get down to work the very first days of the semester. In mathematics a good start is very important. It gives you the impetus to carry you along.

3. Take part in the class discussions yourself. Studying is an *active* process and not a passive one. It is not sufficient to read the book or listen to the teacher. It is like hearing someone use a word that is new to you. The word does not belong to you until you have used it several times in conversation or in writing.

4. Study any assignment just as soon as possible after it is given to you. You are fresh from class instruction then and the lesson is easier to get. Review the lesson and recite it to yourself shortly before going to class.

5. Free yourself at once from any bad habits in mathematics which you may have acquired from high school. Pay close attention to the teacher when he is talking about this.

6. Find out as soon as possible if you have a command of the fundamentals necessary for the course. If you do not, have a conference with the teacher about it.

7. An essential point in studying mathematics is to begin. There is more to this business of beginning than you may realize and many students fail on just this point. They think they are studying if they just sit in front of a book. Choose a time to study your mathematics as soon as possible after the assignment. This time should be when you are rested and your mind is alert and

ready to pounce on the material. Of course you cannot always have such an ideal situation. Do not wait for the proper mood to strike you—by going through the motions of study, you may acquire the proper mood. Turn off the radio and announce to your roommate that you have some heavy work to do, pull up a straight chair and sit down in an erect position at your study table, have paper and pencil handy, be sure to open the book, then—oblivious of everything else go after the lesson just as you would rush into an athletic contest. Concentrate on the lesson. If there are unavoidable distractions, overcome them, and do not use them as excuses for not doing your work. Hereafter the sum total of these activities will be referred to as assuming the *aggressive attitude* for the purpose of studying mathematics.

8. Your work will be better motivated if you understand the aims of freshman mathematics. They are as follows:

(1) To learn how to study mathematics effectively.

(2) To learn to appreciate and to use logical reasoning, accurate thinking, and a careful and precise way of making statements.

(3) To acquire ability with the processes learned so that you can use them in other subjects where they have applications, such subjects as advanced mathematics, astronomy, actuarial science, physics, chemistry, surveying and navigation, civil and electrical engineering, the statistical phases of many subjects like psychology, education, sociology, biology, and economics. The world does not need a large group of mathematics teachers but it does need a great many individuals who can use mathematics expertly.

(4) To learn to appreciate and understand the statement: "Mathematics is the science of necessary conclusions."

(5) To learn and to appreciate the fascination of some topics of mathematics which are interesting in themselves, although they may not have any application to earning a living.

B. HOW TO STUDY THE TEXT

1. Get acquainted with the arrangement of subject-matter in the book. Look through the tables and any other reference material present. Learn how to use the book and its helps.

2. A definite method to be used in studying text-book material is presented here. It immediately puts the student into action and gives him something to do about the text rather than to read it passively.

(1) Assume the aggressive attitude.

(2) Read through the text assignment with the specific purpose of discovering the central idea of the author. Do not try to work out all the details of the material; that can be done later. Read at a moderate rate—mathematics must not be read too rapidly. If you come across terms that are not clear, look up their definitions by using the index.

(3) Read the material again, this time carefully working out all details. Copy the equations line for line, filling in any omitted steps. Be sure that you see why each step follows as a necessary result of what has gone before. If some point is not clear, do not puzzle over it too much but make a note to ask the teacher. However, hold to a minimum the points on which you must ask for help. When you work out a point for yourself, you build up your confidence.

(4) After this second reading, write an outline of the material; this outline can be brief and should not tell you too much. It should be just adequate to suggest what is covered so that you can recite the lesson to yourself. It has several uses, to be described later.

(5) Practice reproducing the material. Close the book and see if you can reproduce the discussion using only the outline you have made for a guide. Recite to yourself the parts which can be covered orally—write out the parts which are more involved. This is an important part of our method. It shows whether you have mastered the lesson. You need not use the exact language of the book. Making statements in your own words is good practice; but if the text-book is good, no one should be able to improve on the definitions of the mathematical terms used.

(6) Look for applications of the theory you have studied. Of course theory sometimes leads on to more theory and the applications come later. Sometimes the applications are problems to be solved. Proficiency in this will furnish motivation in your work. Frequently the teacher must

furnish the applications but you are ahead when you can do it for yourself. Once in a while you should be willing to work through a theory because it is interesting and will add to your life to know about it. Not all of our time each day is spent in earning a living.

(7) Review the lesson just before class. Use your outline and see if you have forgotten anything. Take time to clinch all points.

(8) If you have entered into the spirit of this method you will experience a feeling of satisfaction resulting from your efforts. The certainties which one encounters in mathematics are pleasing to most people. The conclusions are not matters of opinion—you can know when you are right.

C. HOW TO SOLVE PROBLEMS

1. Assume the aggressive attitude.

2. Carefully study the theory preceding the problems and the illustrative problems in the text or those furnished by the instructor.

3. Work the easier problems first and lead up gradually to the harder ones. If a problem gives trouble, drop it for the time and come back to it later, making a new start so as not to repeat your former errors.

4. Practice at working accurately. If you find you are continually making numerical errors, check over every step of the work as you go. Success in working problems is tested by your getting right answers the first time you work them. At first, do not work hurriedly and under pressure—allot yourself plenty of time for solving problems. But as you improve in accuracy and your success is greater, increase the speed with which you work. You should finally be able to do your work quickly and expertly.

5. In most problems it is possible to check your solutions. Get the habit of doing this rather than relying on the answer book. It is a great source of satisfaction on an examination.

6. When solving a written problem by the use of algebra, first read the problem carefully. Then *state the problem in your own words.* This is absolutely necessary if you are to be successful. Use letters in place of the unknown quantities and state all the relationships given in terms of these letters. When you have done this, you will have as many equations as you have unknowns and can proceed to solve the equations. If the type of equations found does not seem to fall under the theory just studied, you have probably made a mistake somewhere, although this is not always the case.

7. When solving a problem in geometry by the use of algebra, first state clearly what is known and what is to be done. Then make a rough approximate drawing, putting into it all you know from the statement of the problem. As you go along, you will find you know more about the situation all the time. Then state the geometrical relationships in algebraic form and proceed to solve the equations set up.

8. If the problem in geometry is to be solved by purely geometrical methods (without the use of algebra), state carefully the hypotheses and what is to be proved and draw the figure. Usually the best thing to do next is to assume as true what is to be proved and see if you can work back to the hypotheses. After you are successful at this, then reverse what you have just done.

9. The solutions of problems in trigonometry are better described as the course develops, although much of what precedes will be helpful. The student should realize that a great deal of algebra is used in this subject.

10. Remember that the letters used stand for numbers. If you are uncertain as to whether a certain manipulation is correct, try it on numbers. A common error is to use $a+b$ as the equal of $\sqrt{a^2+b^2}$. The error is seen when you try $\sqrt{16+9}$ which is equal to 5.

11. Many students have trouble with fractions. A simple but very important rule about fractions is contained in the equation:

$$\frac{a}{b} = \left(\frac{a}{b}\right) \cdot 1.$$

As an illustration where c/c is used as 1, we have the equation:

$$\frac{a}{b}=\frac{a}{b}\cdot\frac{c}{c}=\frac{ac}{bc}\,.$$

This rule has many applications.

D. HOW TO RETAIN WHAT IS LEARNED

Often a student has conscientiously prepared each day's lesson only to find a week later that he has forgotten most of what he has learned. This is fatal in several ways. You will find that every course in mathematics is accumulative. Each day's work depends on what was done on the preceding days. Examinations will become a great difficulty to you and this will worry you and destroy your efficiency. Finally, if you are to have anything worth while from the course, the subject-matter and the procedures learned must stay with you.

It is too often the case that a freshman fails on the first formal examination. This affects him in one of two ways. It may serve as a warning to stimulate him to get down to work as he should have been working from the first of the semester. It may make him despondent and disgruntled with the subject. In either case the situation is not the best it might be and is much better avoided.

1. The answer to all this is that you must continually review. Devote some of the time you have budgeted for the preparation of your daily work to reviewing what you have learned. The proportion of time to be spent depends on how much you know. When you first review, say twenty pages of the text, go over all the principles and see what you have retained. The next time spend your time on those points in which you found you were weak the first time. Use the outlines you have made to suggest questions to ask yourself. When you once get this program going you will find you are more resourceful at preparing your daily lessons and you will spend less time on the assignments.

2. When reviewing problems do not work at the most difficult ones. You will get tangled up in one that is extra hard or long. In a review you want to cover the ground. Work several easy problems and several moderately hard ones in each list. Select those which give a variety of types.

E. HOW THE TEACHER WILL HELP

1. The teacher will try to discover at once whether you are prepared in the fundamentals leading to the course.

2. He will give demonstrations of the technique described in this paper in the course and will try to convince you that it works.

3. He will give conferences to you liberally. In return for this, do not bring an apple to the conference but instead have with you notes on specific questions you wish to ask.

4. He will give a practice examination early in the course which will not count on the semester grade. This will let you know the type of thing that is expected of you. The papers will be handed back to the class and thoroughly discussed with the class.

5. He will give clear explanations of the text material if it is especially involved and of problems which have given difficulty. These explanations will be clear to you only if you have mastered the subject-matter to date.

6. He will frequently furnish applications of the material in the course in order to give the student motivation.

7. He will not lecture. He will invite the student to get into the game with him.

8. Frequently the teacher will use short quizzes to find out whether the student understands the study methods described in this paper and to find out whether he is putting them into practice. These ten- to twenty-minute quizzes will help the teacher find out which of the following remarks applies to you: (1) The student is not proficient in the fundamentals required for this course, (2) he does not study, (3) he tries to study but does not yet know how, (4) he does not retain what he has learned, (5) he is successful according to his ability, (6) he is doing satisfactory work.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 446. *Proposed by L. S. Johnston, University of Detroit.*

In an exercise on the piano, n notes, $k_1, k_2, \cdots, k_n$, are played in the order $k_1, k_2, \cdots, k_{n-1}, k_n, k_{n-1}, \cdots, k_2, k_1, k_2, \cdots$, in measures of m. If k_1 is the first note of the first measure, in what measure will the jth note be k_i?

E 447. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Find the locus of the center of a variable sphere which passes through a given point and touches two given planes.

E 448. *Proposed by Ruth Mason Ballard, Chicago.*

How many bridge hands are there with which it would be impossible to take a trick, no matter how the other cards are distributed or played, (a) if no trump were the bid, (b) if spades are trumps?

E 449. *Proposed by Esther Szekeres, Shanghai, China.*

Given n positive numbers a_i, with

$$a_1 + a_2 + \cdots + a_n = 2n,\ a_i \geqq 1, \qquad (i = 1, 2, \cdots, n),$$

define $a_{n+j} = a_j$, $(j = 1, 2, \cdots)$ and

$$s_{i,\nu} = a_i + a_{i+1} + \cdots + a_{i+\nu}, \qquad (\nu = 0, 1, \cdots).$$

Prove that, for any $A \geqq 0$, there is an $s_{i,\nu}$ with $A < s_{i,\nu} \leqq A+2$.

E 450. *Proposed by Virgil Claudian, Roumanian Mathematical Institute.*

Evaluate

$$\lim_{n\to\infty} \left\{ \frac{n^4}{e}\left(1+\frac{1}{n}\right)^n - n^4 + \frac{n^3}{2} - \frac{11n^2}{24} + \frac{7n}{16} \right\}.$$

SOLUTIONS

E 411 [1940, 175]. *Proposed by J. H. Butchart, Phillips University.*

Prove that if the sides of a triangle form an arithmetic progression, the line joining the centroid to the incenter is parallel to one side.

Solution by D. L. MacKay, Evander Childs High School, New York.

If a, b, c are in arithmetic progression, we have, in the usual notation, $s=3b/2$, and

$$r = \Delta/s = 2\Delta/3b = h_b/3.$$

Thus the incenter and centroid are equidistant from the side b.

Also solved by Michael Aissen, W. E. Buker, W. B. Clarke, T. C. Esty, Albert Furman, Emanuel Mehr, P. W. A. Raine, Hazel Schoonmaker, E. P. Starke, Herbert Tate, C. W. Trigg, Jeanette Van Os, and the proposer.

Editorial Note. The sides being in a. p., so are the sines of the angles. Hence the above result is analogous to E 259 [1937, 54], where it is shown that, if the *tangents* of the angles are in a. p., the line joining the centroid to the *circumcenter* is parallel to one side. For W. B. Clarke's refinement of the present problem, see the *National Mathematics Magazine*, vol. 12, 1938, pp. 194, 249. It has been shown by R. A. Johnson (*Modern Geometry*, pp. 149, 225) that, in any triangle, the line GI contains the Nagel point or "verbicenter" V, the centroid trisecting the segment IV (just as it trisects the segment NO of the Euler line). This becomes obvious when we use areal coördinates (with sum s):

$$I(a/2,\ b/2,\ c/2),\qquad G(s/3,\ s/3,\ s/3),\qquad V(s-a,\ s-b,\ s-c).$$

The line is, of course,

$$(b-c)x + (c-a)y + (a-b)z = 0,$$

which is concurrent with $y=0$ and $x+y+z=0$ if $a-b=b-c$. It is unfortunate that N. A. Court (*College Geometry*, p. 105) uses the term "Nagel point" for a different point (collinear with the incenter and circumcenter).

E 412 [1940, 175]. *Proposed by R. A. Johnson, Brooklyn College.*

Solve the simultaneous equations

$$x_0x_1 = x_2,\ x_1x_2 = x_3,\ \cdots,\ x_{n-2}x_{n-1} = x_0,\ x_{n-1}x_0 = x_1.$$

Solution by E. P. Starke, Rutgers University.

It is obvious that the vanishing of one x_j entails the vanishing of all, and gives the solution $(0, 0, \cdots, 0)$. We therefore restrict our attention to non-zero values. Replacing x_2 in the second equation by its value from the first, we have $x_0x_1^2=x_3$; replacing both x_2 and x_3 in the next equation, we have $x_0^2x_1^3=x_4$. Continuing in this manner, we express each x_j in terms of x_0 and x_1:

$$x_0^{f_{j-1}}x_1^{f_j} = x_j, \qquad (j = 2, 3, \cdots), \tag{1}$$

where f_j is the jth term of the Fibonacci sequence $1, 1, 2, 3, 5, 8, \cdots$, so that

$$f_1 = f_2 = 1,\qquad f_{j+1} = f_{j-1} + f_j,\qquad f_j^2 = f_{j-1}f_{j+1} - (-1)^j. \tag{2}$$

The above elimination will terminate with the two equations

$$(3)\qquad x_0^{f_{n-1}}x_1^{f_n}=x_0,\qquad x_0^{f_n}x_1^{f_{n+1}}=x_1.$$

Let d denote the highest common divisor of f_n and $f_{n-1}-1$ (or of f_n and $f_{n+1}-1$). Putting $f_{n-1}-1=dr$, $f_n=ds$, so that $f_{n+1}-1=d(r+s)$, we have from (3),

$$(4)\qquad x_0^{dr}x_1^{ds}=1,\qquad x_0^{ds}x_1^{d(r+s)}=1,$$

with r and s co-prime. We then obtain

$$x_0^{dr(r+s)}x_1^{ds(r+s)}=1,\qquad x_0^{ds^2}x_1^{ds(r+s)}=1,$$

so that, upon eliminating x_1, we have $x_0^p=1$, where

$$(5)\qquad p=(s^2-rs-r^2)d=2r+s+\{1-(-1)^n\}/d$$

since, by (2), $d^2s^2=(rd+1)\{(r+s)d+1\}-(-1)^n$. Now, s and $r+s$ being co-prime, integers u and v may be found such that

$$us-v(r+s)=1.$$

(These are not uniquely determined, but different choices all lead to the same general solution.) We then obtain from (4),

$$x_0^{urd}x_1^{usd}=1=x_0^{vsd}x_1^{v(r+s)d},$$

whence $x_1^d=x_0^{d(sv-ru)}$. Let ϵ and ω be arbitrary pth and dth roots of unity, respectively. Then the complete non-zero solution is given by

$$x_0=\epsilon,\qquad x_1=\omega\cdot\epsilon^{sv-ru},$$

along with (1). The first five cases are tabulated below:

n	d	r, s	p	u, v	$sv-ru$	Solution	
3	2	0, 1	2	1, 0	0	$\epsilon,\ \omega,\ \epsilon\omega,$	$(\epsilon^2=\omega^2=1)$
4	1	1, 3	5	3, 2	3	$\epsilon,\ \epsilon^3,\ \epsilon^4,\ \epsilon^2,$	$(\epsilon^5=1)$
5	1	2, 5	11	3, 2	4	$\epsilon,\ \epsilon^4,\ \epsilon^5,\ \epsilon^9,\ \epsilon^3,$	$(\epsilon^{11}=1)$
6	4	1, 2	4	2, 1	0	$\epsilon,\ \omega,\ \epsilon\omega,\ \epsilon\omega^2,\ \epsilon^2\omega^3,\ \epsilon^3\omega,$	$(\epsilon^4=\omega^4=1)$
7	1	7, 13	29	-3, -2	-5	$\epsilon,\ \epsilon^{24},\ \epsilon^{25},\ \epsilon^{20},\ \epsilon^{16},\ \epsilon^7,\ \epsilon^{23},$	$(\epsilon^{29}=1)$

E 413 [1940, 175]. *Proposed by H. T. R. Aude, Colgate University.*

The graph of a cubic function $y=x^3+ax^2+bx+c$ crosses the x-axis at three distinct points, two of which are A and B. The lines AP and BQ are drawn tangent to the humps of the curve, the points of contact being P and Q. Show that the ratio of the distance AB to the horizontal distance from P to Q is constant.

Solution by E. P. Starke, Rutgers University.

Let A, B, P have coördinates $(\alpha, 0)$, $(\beta, 0)$, (x_1, y_1), respectively, and let the

cubic cross the x-axis also at $C(\gamma, 0)$, so that the equation of the cubic takes the form

$$(1) \qquad y = (x - \alpha)(x - \beta)(x - \gamma).$$

Now any line $y = m(x-\alpha)$ through A meets (1) in two other points whose abscissas are given by $(x-\beta)(x-\gamma) = m$. If these points coincide at P, we must have

$$m = -(\beta - \gamma)^2/4, \qquad x_1 = (\beta + \gamma)/2.$$

Thus the projection $(x_1, 0)$ of P on the x-axis is midway between B and C. Similarly the projection of Q on the x-axis is midway between A and C. It thus follows that the distance AB is always twice the horizontal distance from P to Q, but in the opposite direction. The required ratio is then -2, independent of the coefficients of the cubic.

Also solved by Albert Furman, Hazel E. Schoonmaker, C. W. Trigg, and the proposer.

E 414 [1940, 175]. *Proposed by V. Thébault, Le Mans, France.*

Find a number of the form *abbbb* whose square, diminished by unity, has ten digits, all different.

Solution by C. W. Trigg, Los Angeles City College.

For such a number N, we have $N^2 - 1 \equiv 0 \pmod 9$, so that

$$N \equiv a + 4b \equiv \pm 1 \pmod 9.$$

Neither a nor b may be zero, nor may b be a multiple of 3, for if it were, the last four digits of N^2-1 would be alike. Thus the possible values of N are reduced to twelve, given by

$$a = 3,\ 5,\ 6,\ 3,\ 6,\ 9,\ 1,\ 4,\ 7,\ 2,\ 8,\ 9;$$

$$b = 4,\ 8,\ 1,\ 8,\ 5,\ 2,\ 4,\ 1,\ 7,\ 2,\ 5,\ 7.$$

The last four, and in general the first three digits of N^2 can be read from a table of the squares of $N < 10000$. Appearance of duplicate digits eliminates the first six possibilities. When the other eligible values of N are squared, only the last two are found to meet the conditions, namely

$$85555^2 - 1 = 7319658024,$$

$$97777^2 - 1 = 9560341728.$$

Since the value of N was not restricted further, we have incidentally shown that there are no other N's of the given form for which N^2-1 is composed of the different digits from 0 through 8 or of the different digits from 1 through 9.

Also solved by M. L. Constable, Daniel Finkel, P. R. Hill, E. P. Starke, and the proposer. One of the two values of N was obtained by R. K. Allen, W. E.

Buker, Wm. Douglas, and B. C. Zimmerman. Finkel remarks that these are special solutions for the more general problem 3926 [1939, 515].

E 415 [1940, 175]. *Proposed by Cezar Coşniţă, Focşani, Roumania.*

In a triangle of sides a, b, c, prove that the distance from the centroid G to the incenter I is given by the formula

$$3(a+b+c)GI = \left\{\sum a^2(b-c)^2 - \sum(b^2+c^2-a^2)(c-a)(a-b)\right\}^{1/2}.$$

Solution by T. C. Esty, Amherst College.

If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be the position vectors of the vertices, and $\mathbf{G}$ and $\mathbf{I}$ those of the centroid and incenter, we have

$$\mathbf{G} = (\mathbf{A}+\mathbf{B}+\mathbf{C})/3, \qquad \mathbf{I} = (a\mathbf{A}+b\mathbf{B}+c\mathbf{C})/(a+b+c).$$

Hence

$$\begin{aligned} 3(a+b+c)(\mathbf{I}-\mathbf{G}) &= 3(a\mathbf{A}+b\mathbf{B}+c\mathbf{C}) - (a+b+c)(\mathbf{A}+\mathbf{B}+\mathbf{C}) \\ &= (2a-b-c)\mathbf{A} + (-a+2b-c)\mathbf{B} + (-a-b+2c)\mathbf{C}. \end{aligned}$$

Squaring this vector, and making the substitution

$$2\mathbf{B}\cdot\mathbf{C} = \mathbf{E}^2 + \mathbf{C}^2 - (\mathbf{B}-\mathbf{C})^2 = \mathbf{B}^2 + \mathbf{C}^2 - a^2,$$

with two similar expressions for $2\mathbf{C}\cdot\mathbf{A}$ and $2\mathbf{A}\cdot\mathbf{B}$, we find that all terms in $\mathbf{A}^2$, $\mathbf{B}^2$, $\mathbf{C}^2$ vanish, leaving

$$\begin{aligned} (1) \quad 9(a+b+c)^2GI^2 &= -\sum a^2(a-2b+c)(a+b-2c) \\ &= -\sum a^4 + \sum a^3b + 4\sum a^2b^2 - 5abc\sum a \\ &= \sum a^2(b-c)^2 - \sum(b^2+c^2-a^2)(c-a)(a-b). \end{aligned}$$

Also solved by W. E. Buker, Albert Furman, D. L. MacKay, P. W. A. Raine, Hazel E. Schoonmaker, E. P. Starke, and C. W. Trigg. MacKay gives the alternative formula

$$3GI = (s^2 + 5r^2 - 16rR)^{1/2};$$

Buker and Starke give

$$9(a+b+c)GI^2 = -\sum a^3 + 2\sum a^2b - 9abc.$$

Editorial Note. Equation (1) is an immediate consequence of the formula $-\sum a^2(y-y')(z-z')$ for the square of the distance between the points whose actual areal coördinates are (x, y, z) and (x', y', z'), *i.e.*, whose position vectors are $x\mathbf{A}+y\mathbf{B}+z\mathbf{C}$ and $x'\mathbf{A}+y'\mathbf{B}+z'\mathbf{C}$.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3975. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

The orthopoles of a straight line parallel to one of the axes of an equilateral hyperbola as to the four triangles formed by any three of four points given on that hyperbola are on a straight line.

3976. *Proposed by W. O. Pennell, Exeter, N. H.*

Through a point P in the plane of a central conic, a line CC' is drawn parallel to the diameter conjugate to the diameter located by a line joining P with the center of the conic. Draw two lines through P intersecting the conic in A and B, and A' and B', respectively. Prove that AB' and $A'B$ (extended if necessary) intersect CC' at points equidistant from P. Likewise, AA' and BB' intersect CC' at points equidistant from P.

3977. *Proposed by Aaron Herschfeld, Washington, D.C.*

Denote the nth iterate of $f(x)$, a real function of a real variable x, by the symbol $f_n(x)$. Thus $f_0(x)\equiv x$, $f_1(x)\equiv f(x)$, $f_2(x)\equiv f[f(x)]$, $\cdots$, $f_n(x)\equiv f[f_{n-1}(x)]$. Prove the following:

(1) If $f(x)=x^2-2x$, then

$$\lim_{n\to\infty}\{f_n(x)\}^{2^{-n}}=\tfrac{1}{2}\{|x-1|+\sqrt{x^2-2x-3}\}>1,\text{ when } |x-1|>2,$$

$$\text{while } |f_n(x)|\leqq 3,\text{ for all } n,\text{ when } |x-1|\leqq 2.$$

(2) If $f(x)=2x^2-1$, then

$$\lim_{n\to\infty}\{f_n(x)\}^{2^{-n}}=|x|+\sqrt{x^2-1}>1,\text{ when } |x|>1,$$

$$\text{while } |f_n(x)|\leqq 1,\text{ for all } n,\text{ when } |x|\leqq 1.$$

3978. *Proposed by V. Thébault, Tennie, Sarthe, France.*

Given an orthocentric tetrahedron and the spheres which are loci of points such that the ratio of the squares of their distances to the extremities of one of the edges of the tetrahedron is equal to the ratio of the sum of the squares of the edges of the faces containing the opposite edge. Show that the sum of the powers

of the respective extremities of the latter edge with respect to one or the other of the two spheres is constant.

SOLUTIONS

3898 [1938, 696]. *Proposed by Otto Dunkel, Washington University.*

A point is chosen on a rectangular hyperbola. In how may ways, and under what conditions, may two other points on the curve be selected so that the centroid of the three points will lie also on the curve?

I. *Solution by G. A. Williams, Oregon State College.*

Let $xy=c>0$ be the equation of the hyperbola, and $(k, c/k)$, $(k>0)$ the chosen point. Let $(\alpha, c/\alpha)$ and $(\beta, c/\beta)$ be two other points on the curve.

The centroid of the three points will lie on the curve if

$$(\alpha + \beta + k)\left(\frac{c}{\alpha} + \frac{c}{\beta} + \frac{c}{k}\right) = 9c,$$

or

$$(\beta + k)\alpha^2 + (\beta^2 - 6k\beta + k^2)\alpha + \beta k(\beta + k) = 0.$$

The discriminant of this last equation as a quadratic in α is

$$(\beta - k)^2(\beta^2 - 14k\beta + k^2).$$

Hence α will be real if $\beta=k$ or if

$$\beta^2 - 14k\beta + k^2 \geqq 0,$$

that is, if

$$(7 + 4\sqrt{3})k \geqq \beta \geqq (7 - 4\sqrt{3})k.$$

Hence if a first point $(k, c/k)$ is chosen, then a second point $(\beta, c/\beta)$ may be chosen (β satisfying the above condition) and a third point $(\alpha, c/\alpha)$ may be determined so that the conditions of the problem are satisfied.

There will be two values of α for each value of β except $\beta=(7-4\sqrt{3})k$, $\beta=(7+4\sqrt{3})k$, $\beta=k$. In the latter case, all three points will coincide.

II. *Solution by E. P. Starke, Rutgers University.*

Let A be the given point on the hyperbola H (not necessarily rectangular) whose center is O. On the same branch of H as A, determine points P and Q such that the distance of each from an asymptote is one-third that of A. Choose for the centroid an arbitrary point G of H not within the arc PAQ. Produce AG half its length to M, which is then to be the midpoint of side BC of the triangle. Hence let any chord of H be drawn parallel to OM and join its midpoint R to O. Draw BC parallel to OR and terminated by H. Then ABC is the required triangle, chosen in an infinite variety of ways.

The proof is obvious from the fact that a diameter of a central conic is the locus of midpoints of chords parallel to the conjugate diameter. Here OM and OR are conjugate diameters. It is evident also that no real chord of H has a

midpoint within either of the V-shaped areas bounded by H and its asymptotes. The restrictions above on G are necessary to prevent M falling in one of these areas.

Solved also by W. B. Campbell, C. R. Cassity, O. J. Ramler, and C. W. Trigg.

Editorial Note. Starke gave also an analytical proof of his construction. Ramler gave a geometrical construction different from the above; the rest were analytic, Trigg's solution being similar to the above and the others different from each other and the above. The analysis in Williams's solution is valid for any hyperbola, since no point of it requires the axes of coördinates to be rectangular. However, the rectangular hyperbola is of more interest, since in that case the curve passes not only through the vertices and centroid of each of the required triangles but also through its orthocenter and the incenter and three excenters of the triangle formed by the midpoints of its sides; see the solution of 3797 [1938, 487]. For each required triangle the hyperbola is the isogonal conjugate of its circumdiameter through its symmedian point, and it is called Kiepert's hyperbola for that triangle; see the solutions of 3882, 3883 [1940, 403].

3900 [1939, 52]. *Proposed by J. E. Trevor, Cornell University.*

Two players carried out an experiment with a heap of freshly minted twenty-five cent pieces. The first player discarded one coin from the heap, and then one-half of the remainder, which was easily done by use of an available balance. In the second move, beginning with the undiscarded coins, the second player discarded 2^r coins, and also one-half of the coins then remaining. Here r is an arbitrary positive integer. In general, in the xth move, x^r coins and one-half of those then remaining were discarded. If n coins were given, and y coins remained at the close of the xth move, n is a function of x, y, r. For successive values $r=1, 2, 3, \cdots$ of the parameter r, obtain the formulation

$$n = 2^x[y + \phi_r(x)] + c_r,$$

where $\phi_r(x)$ is a polynomial of $r+1$ terms, and c_r is a constant.

Solution by K. W. Miller, Chicago.

Let $y_{(x-k)}$ be the remainder after the $(x-k)$th move. Then, according to the conditions of the problem,

$$y_{(x-k)} = [y_{(x-k-1)} - (x - k)^r]/2. \tag{1}$$

Multiplying through by $2^{(x-k)}$ and rearranging gives

$$2^{(x-k)}y_{(x-k)} - 2^{(x-k-1)}y_{(x-k-1)} + 2^x2^{-(k+1)}(x - k)^r = 0.$$

Summing from $k=0$ to $k=(x-1)$, all but two of the terms resulting from the first two terms cancel, and we have

$$2^xy_x - 2^0y_0 + 2^x \sum_{k=0}^{k=x-1} 2^{-(k+1)}(x - k)^r = 0. \tag{2}$$

In the proposer's notation, the last remainder y_x is y and the "remainder" y_0 after the zeroth move is the initial number n. Further, by substituting $(z+1)$ for $(n-k)$ in the summation, simplifying, and replacing z by k, it may be rewritten in simpler equivalent form. Making these substitutions and rearranging, we obtain

$$n = 2^x y + \sum_{k=0}^{k=x-1} 2^k (k+1)^r. \tag{3}$$

It is a very easy matter by letting x be the first few integers in equation (3) to find numerically, for $r=1, 2, 3$, and 4, that

$$n = 2^x[y + \phi_r(x)] - \phi_r(0), \tag{4}$$

where

$$\begin{aligned} \phi_1 &= x - 1, \\ \phi_2 &= x^2 - 2x + 3, \\ \phi_3 &= x^3 - 3x^2 + 9x - 13, \\ \phi_4 &= x^4 - 4x^3 + 18x^2 - 52x + 75. \end{aligned} \tag{5}$$

It remains to find a general expression for $\phi_r(x)$. We assume the equality

$$\sum_{k=0}^{k=x-1} 2^k (k+1)^r = 2^x \phi_r(x) + c_r. \tag{6}$$

Replacing x by $(x-1)$ in equation (6) and subtracting the result from equation (6) member by member, dividing out the common factor 2^{x-1} and rearranging, we obtain

$$2\phi_r(x) = x^r + \phi_r(x-1). \tag{7}$$

Expressing ϕ_r explicitly in polynomial form, we have

$$\phi_r(x) = \sum_{i=0}^{i=r} a_i x^i \tag{8}$$

which, substituted into equation (7), expanding and equating coefficients of equal powers of x, yields

$$\begin{aligned} a_r &= 1 \\ a_{r-k} &= \sum_{j=0}^{j=k-1} (-1)^{(k-j)} \frac{(r-j)!}{(k-j)!(r-k)!} a_{r-j}, \qquad k \geqq 1; \end{aligned} \tag{9}$$

from this we can solve consecutively for the coefficients a_n, obtaining

$$a_r = 1, \quad a_{r-1} = -r, \quad a_{r-2} = \frac{3}{2} r(r-1), \quad a_{r-3} = -\frac{13}{6} r(r-1)(r-2), \textit{ etc.} \tag{10}$$

Thus, the polynomial formulation assumed in equation (6) is valid provided coefficients are selected in accordance with conditions (9) or (10). Moreover, setting $x=0$ in equation (6) determines c_r as $-\phi_r(0)$. Lastly, from the form of equations (9) and (10), it is apparent that no coefficient for $r-k\geqq 0$ vanishes, while all others do, so that there are $(r+1)$ terms in all. This completes the proof that equation (4) is true in general and verifies the proposer's remarks concerning it, except ϕ_r can be obtained for any value of r without previously finding ϕ for all preceding values of r. This is certainly a very ingenious generalization of the problem of "the monkey and the coconuts."

Solved also by the proposer.

Editorial Note: The above solution may be set in a familiar form. In order to solve the non-homogeneous linear difference equation (1) $2y_x-y_{x-1}=-x^r$, we solve first $2\bar{y}_x-\bar{y}_{x-1}=0$. The solution is obviously $\bar{y}_x=c(1/2)^x$. We then add to the right member of this equation any particular solution of (1), say $y_x=-\phi_r(x)$, and we then have the general solution of (1) in the form (2) $y_x=c(1/2)^x-\phi_r(x)$, where (3) $2\phi_r(x)-\phi_r(x-1)=x^r$. For $\phi_r(x)$ we can easily find a polynomial in x whose degree must be r, and it will be clear that the coefficient of x^r must be unity and that the rest must be integers. Since $y_0=n$, we have $n=c-\phi_r(0)$, and then

$$(4) \qquad n = 2^x[y+\phi_r(x)] - \phi_r(0), \qquad y \equiv y_x.$$

The coefficients of $\phi_r(x)$ are given by (9) of the above solution.

The proposer worked out in a different manner the explicit polynomials, $\phi_r(x)$, for $r=1, 2, 3, 4, 5$, and observed from these the following rule for deducing the coefficients of $\phi_r(x)$ from those of $\phi_{r-1}(x)$:

$$(5) \qquad a_{r,i} = \frac{r}{i}\, a_{r-1,i-1}, \quad i>0; \qquad a_{r,r}=1; \qquad a_{r,0} = (-1)^r\sum_{j=1}^{r} |\, a_{rj} \,|.$$

We shall prove that this rule is true for all positive integral values of r; that no $a_{r,i}$ is zero; and that they alternate in sign. From $2\phi_r(0)-\phi_r(-1)=0$, we have at once

$$(6) \qquad a_{r,0} = (-1)^r\sum_{j=1}^{r}(-1)^{r-j}a_{r,j},$$

and from the above solution,

$$(7) \qquad a_{r,t} = \sum_{j=t+1}^{r}(-1)^{j-t}{}_jC_t\, a_{r,j}, \qquad t<r, \qquad a_{r,r}=1.$$

This gives $a_{r,r-1}=-r$, $a_{r,r-2}=3_rC_2$; and it is easily seen that the first part of (5) is true for $i=r, r-1, r-2$. Assume that this first part is true for $i=r, r-1, \cdots, t+1\geqq 2$; it will be shown that it must also be true for $i=t\geqq 1$. It will then follow that, if no coefficient of $\phi_{r-1}(x)$ is zero and if they alternate in sign, this will also

be true for $\phi_r(x)$ excluding $a_{r,0}$. Thus $(-1)^{r-j}a_{r,j}>0$, $j\geqq 1$. Then using (6) we shall have $(-1)^r a_{r,0}>0$, and therefore no coefficient of $\phi_r(x)$ is zero and they alternate in sign. In the cases $r=1, 2, 3$ all of this is true, and we can then conclude by another induction that for every r all of these statements are true. From (7) we have, by the first induction,

$$a_{r,t} = \frac{r}{t}\sum_{j=t+1}^{r}(-1)^{j-t}{}_jC_t\,\frac{t}{r}\,\frac{r}{j}\,a_{r-1,j-1} = \frac{r}{t}\sum_{j=t+1}^{r}(-1)^{j-t}{}_{j-1}C_{t-1}\,a_{r-1,j-1},$$
$$= \frac{r}{t}\sum_{j=t}^{r-1}(-1)^{j-t+1}{}_jC_{t-1}\,a_{r-1,j} = \frac{r}{t}\,a_{r-1,t-1};$$

and the above multiple induction completes the proof.

The proposer's rule may be stated in the form

$$\phi_r(x) = r\left[\int_0^x \phi_{r-1}(x)dx + \int_0^{-1}\phi_{r-1}(x)dx\right], \qquad \phi_0(x) = 1.$$

It is quite easy to prove, independently of the above, that every $\phi_r(x)$ thus obtained satisfies the equation (3).

3901 [1939, 52]. *Proposed by V. Thébault, Le Mans, France.*

Given the positive integers a, b, c such that $a^2=b^2+c^2$, the positive numbers m and n may be determined so that (1) $a=m+n+(2mn)^{1/2}$. Conversely, if $2mn$ is a perfect square, then a in (1) is such that its square is the sum of two squares. Show that, if $x=2(m+n)/mn$, the numbers

$$(bx+1)^2-1, \qquad (cx+1)^2-1, \qquad (ax-1)^2-1$$

are also the sides of a right triangle, the last term being the length of the hypotenuse. Determine the smallest integral value of x so that the sides may be expressed as integers.

Solution by H. E. Vaughan, University of Illinois.

Let (a, b, c) be any P.T. (Pythagorean Triple, *i.e.*, a, b, and c integers, $b>c$, and $a^2=b^2+c^2$), and let $a'=(ax-1)^2-1$, $b'=(bx+1)^2-1$, $c'=(cx+1)^2-1$. We wish to find all non-zero values of x such that (a', b', c') is a P.T. By direct calculation,

$$a'^2-b'^2-c'^2 = (a^4-b^4-c^4)x^4 - 4(a^3+b^3+c^3)x^3 + 4(a^2-b^2-c^2)x^2.$$

Setting this equal to zero, noting that $a^2-b^2-c^2=0$, and dividing by x^3, we find that $(a^4-b^4-c^4)x-4(a^3+b^3+c^3)=0$. But

$$a^4-b^4-c^4 = (a^2-b^2)(a^2+b^2)-c^4 = c^2(a^2+b^2)-c^4 = c^2(a^2+b^2-c^2)$$
$$= 2b^2c^2 = 2(a^2-b^2)(a^2-c^2),$$

and

$$\begin{aligned} a^3 + b^3 + c^3 &= a^3 + (b^2 - bc + c^2)(b + c) = a^3 + (a^2 - bc)(b + c) \\ &= a^3 + a^2c + a^2b - bc^2 - (a^2 - c^2)c \\ &= (a + c)[a^2 + b(a - c) - c(a - c)] \\ &= (a + c)[a^2 + ab - c(a + b) + a^2 - b^2] \\ &= (a + c)(a + b)[(a - c) + (a - b)]. \end{aligned}$$

Hence

$$x(a, b, c) = 2[(a-b)+(a-c)]/(a-b)(a-c) = 2[1/(a-b)+1/(a-c)].$$

If we set $m=a-b$ and $n=a-c$, it is readily verified that

$$\begin{aligned} m + n + (2mn)^{1/2} &= (a - b) + (a - c) + [2(a - b)(a - c)]^{1/2} \\ &= a + (a - b - c) + [2(a^2 - ab + bc - ac)]^{1/2} \\ &= a + (a - b - c) + (a^2 + b^2 + c^2 - 2ab + 2bc - 2ac)^{1/2} \\ &= a + (a - b - c) + [(a - b - c)^2]^{1/2} = a, \end{aligned}$$

since from $a^2-b^2-c^2=0$, it follows that $a-b-c<0$. We note also that $b=n+(2mn)^{1/2}$, $c=m+(2mn)^{1/2}$, and $x(a, b, c)=2(m+n)/mn$. It is easily verified that $a^2\equiv b^2+c^2$ for the given values of a, b, c in terms of m and n, and hence the converse stated in the problem is true.

Going back to the second expression for $x(a, b, c)$ we see that, since $a-b\geqq 1$ and $a-c\geqq 2$, we must have $x(a, b, c)\leqq 3$, and $x(a, b, c)=3$ if and only if $(a, b, c)=(5, 4, 3)$. Similarly, $x(a, b, c)\neq 2$, since equality would imply that $a-b=a-c=2$. Finally, $x(a, b, c)=1$ only if $a-b=3$ and $a-c=6$ which necessitate that $(a, b, c)=(15, 12, 9)$.

The smallest integral value of x such that a', b', and c' are integers is, therefore, 1 and $(a', b', c')=(195, 168, 99)$. There is no smallest positive value of x for which a', b', and c' are integers, since, if we take $(a, b, c)=(5k, 4k, 3k)$, where k is any integer, $x(a, b, c)=3/k$ and again the corresponding $(a', b', c')$is (195, 168, 99).

Solved also by F. L. Dennis, E. P. Starke, and the proposer.

Editorial Note. If m and n are positive integers such that $2mn$ is a perfect square, then $m=2r^2d$ and $n=s^2d$ (or the reverse), where d is the highest common divisor of m and n. Then, from the values of a, b, c in terms of m and n, we have

$$a = d[(r + s)^2 + r^2], \qquad b = d[(r + s)^2 - r^2], \qquad c = 2d(r + s)r,$$

which is the familiar form for a P.T.

3902 [1939, 53]. *Proposed by V. Thébault, Le Mans, France.*

A point M is chosen arbitrarily on the circumcircle of the triangle ABC, and the chords MA', MB', MC' are drawn parallel to BC, CA, AB. Show that the orthopoles of the circumcircle diameters through A', B', C' are the vertices of a triangle equal to the orthic triangle of ABC. Generalize.

Solution by the Proposer.

It is evident that the triangles $A'B'C'$ and ABC are equal, and that suitable angles between the circumdiameters through A', B', C' are equal to the angles of the orthic triangle $A_hB_hC_h$ of ABC. Moreover, the orthopoles ω_a, ω_b, ω_c of the above diameters are on (N), the nine-point circle of ABC; and the arcs $\omega_b\omega_c$, $\omega_c\omega_a$, $\omega_a\omega_b$ subtend on (N) angles equal to the above mentioned angles between the diameters. Hence the two triangles $A_hB_hC_h$ and $\omega_a\omega_b\omega_c$ inscribed in (N) are equal.

We obtain a generalization by supposing that MA', MB', MC' turn through the same given angle and thus cut BC, CA, AB at the same angle θ. It is easily shown that the result is the same whatever the angle θ.

Editorial Note. If the circumdiameter turns through a given angle, its orthopole rotates on (N) through the same angle in contrary sense with respect to any chosen point on (N). See the solutions of 3882, 3883 [1940, 403].

3903 [1939, 111]. *Proposed by Simon Mowshowitz, New York, N. Y.*

Denote $[(n!)!]!$ by $n(!)^3$, *etc.*, $n(!)^0=n$. Prove that for $k\geqq 2$,

$$\frac{n(!)^k}{(n!)^{[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!}}$$

is an integer.

Solution by Kwan Chao Chih, Yenching University, Peking, China.

We know that the product of any n consecutive integers is divisible by $n!$.

Now $n(!)^k$ is a product of $n(!)^{k-1}$ consecutive integers. We divide these numbers into groups of n consecutive integers. Then we have

$$[n-1]![n!-1]![n(!)^2-1]!\cdots[n(!)^{k-2}-1]!, \qquad k\geqq 2,$$

groups, since

$$\begin{aligned} n(!)^{k-1} &= n(!)^{k-2}[n(!)^{k-2}-1]! \\ &= n(!)^{k-3}[n(!)^{k-3}-1]![n(!)^{k-2}-1]! \\ &= \cdots \\ &= n(n-1)![n!-1]!\cdots[n(!)^{k-2}-1]!. \end{aligned}$$

Thus, $n(!)^k$ is divisible by

$$(n!)^{[n-1]![n!-1]!\cdots[n(!)^{k-2}-1]!}.$$

Solved also by Fritz John and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The following persons are to be at the Institute for Advanced Study for all or part of the current academic year: Dr. Warren Ambrose, Dr. Valentin Bargmann, Assistant Professor D. G. Bourgin, Dr. A. T. Brauer, Dr. C. L. Critchfield, Professor Orrin Frink, Dr. Guido Fubini, Dr. Kurt Gödel, Dr. P. R. Halmos, Dr. M. H. Heins, Professor Shizuo Kakutani, Dr. Dorothy Maharam, Assistant Professor Gordon Pall, Professor Wolfgang Pauli, Dr. Abraham Schwartz, Dr. C. E. Shannon, Dr. Seymour Sherman, Professor C. L. Siegel, Assistant Professor A. H. Taub, Dr. R. M. Thrall, and Professor W. J. Trjitzinsky.

Dr. Frances E. Baker of Mount Holyoke College has been promoted to an assistant professorship.

Professor H. H. Dalaker of the University of Minnesota has been given the title emeritus.

Associate Professor P. A. DeVore of Central Missouri State Teachers College has been granted a leave of absence for the current academic year.

Lloyd Dulmage of the University of Toronto has been appointed lecturer in actuarial science at the University of Manitoba.

L. T. Dunlap, W. O. Gordon, and Dr. Beatrice L. Hagen of Pennsylvania State College have been promoted to assistant professorships.

Dr. H. C. Fryer has been appointed assistant professor of mathematics at Kansas State College.

L. B. Hedge of Brown University has been appointed to an assistant professorship at The Citadel.

Assistant Professor H. A. Jordan of Georgetown University has been promoted to an associate professorship.

Dr. J. F. Kubis of Fordham University was promoted to an assistant professorship in June 1939.

Dr. S. B. Littauer of the U. S. Naval Academy has been promoted to an assistant professorship.

Assistant Professors G. C. Munro and D. T. Sigley of Kansas State College have been promoted to associate professorships.

Assistant Professor N. N. Royall of The Citadel has been appointed to an associate professorship at Winthrop College.

Associate Professor I. M. Sheffer of Pennsylvania State College has been promoted to a professorship.

Dr. S. S. Smith of the University of Utah has been promoted to an assistant professorship.

Associate Professor R. C. Stephens of Knox College has been promoted to a professorship.

The following appointments to instructorships are announced:
Brooklyn College: Saul Gorn, Dr. D. C. Harkin
College of the City of New York: Dr. G. N. Garrison
Cooper Union: Dr. Alvin Sugar
Eastern Washington College of Education: R. F. Bell
Kansas State College: B. H. Buikstra, Frank Faulkner, part-time
University of Nevada: E. M. Beesley
Oklahoma Agricultural and Mechanical College: Dr. P. E. Lewis
Pennsylvania State College: Dr. A. B. Cunningham
Southern Methodist University: Dr. D. W. Starr
Texas Technological College: Dr. F. D. Rigby
Vanderbilt University: Dr. John Dyer-Bennet

Dr. H. G. Titt, dean and professor of mathematics at Huron College, died August 21, 1940.

MATHEMATICAL REVIEWS AND A READING MACHINE FOR MICROFILM

When the microfilm service of *Mathematical Reviews* was introduced, it was realized that its usefulness would depend to a large extent upon the availability of reading machines. The Committee on Scientific Aids to Learning, a committee of the National Research Council, is promoting, among other things, the use of microfilm. As a result of its efforts, a reading machine is being manufactured which will be sold at a retail price of $32.00. A grant from the Committee on Scientific Aids to Learning has made it possible for *Mathematical Reviews* to distribute a limited number of these machines on the following terms.

Terms of offer. A reading machine for microfilm will be given—as long as the available supply lasts—to any person who has paid his subscription, *at the rate to which he is entitled*, to *Mathematical Reviews* in advance for three years beginning January, 1941. The person who receives a reading machine must pay express charges and import duty, if any, from Buffalo, New York. Until January 1 this offer is being made only to the present subscribers to *Mathematical Reviews*. After that date, however, it will be extended to new subscribers also. Since only a limited number of machines is available, anyone who desires one should place an order early.

The purpose of the Committee on Scientific Aids to Learning in giving the financial support which makes possible the distribution of reading machines to subscribers to *Mathematical Reviews* was two-fold, to promote the use of microfilm and to give financial aid to *Mathematical Reviews*.

History of Students Microfilm Reader. The history of the reader, known as the Students Microfilm Reader, begins in the fall of 1939. At that time an advisory group on microphotography to the Committee on Scientific Aids to Learning, composed of Mr. Keyes D. Metcalf, Director of the Harvard University Library, (chairman), Professors Ralph D. Bennett and Ernest I. Huntress of Massachusetts Institute of Technology, Dr. Vernon D. Tate of the National Archives, and Dr. Irvin Stewart, Director of the Committee on Scientific Aids to Learning, (*ex officio*), was requested to consider the possibilities of designing and making available a simple, inexpensive microfilm

reading machine for the use of the individual scholar. The problem was discussed at length and did not appear insoluble. Several designs were suggested, and three models were constructed. Each of these models was thoroughly tested both in the laboratory and in actual use; a set of plans and specifications embodying the final accepted design was prepared for distribution to manufacturers specializing in equipment of this type.

Bids for the manufacture of the reading machine were received from a number of companies, and the Spencer Lens Company was authorized to build a pilot model. It was built, tested and inspected, and the Committee on Scientific Aids to Learning has now signed a contract for a number of these machines. In addition, they will be placed on the market by the Spencer Lens Company.

Purpose of Students Microfilm Reader. Emphasis has been placed throughout on suitability for the purpose for which the reader was designed: simplicity and low cost. No claims are made for extreme convenience, beauty, ready-portability, or universality.

The benefits of microphotography in assembling research data of all types are well known. Facilities exist in the principal libraries, archives, and other institutions for the reproduction of their holdings. Many individuals have secured equipment and microphotographed extensive files. The greatest difficulty to date has not been to secure material on microfilm but rather to obtain adequate utilization equipment. In sponsoring the development of a simple, inexpensive microfilm reader, the Committee on Scientific Aids to Learning has considered solely the requirements of the individual. Excellent equipment developed primarily for commercial and library use is already available on the market. In most cases, it is entirely satisfactory (except for price) for individual use. The Students Microfilm Reader is not intended to compete with any existing reader equipment. It was developed specifically to permit the individual scholar or scientist to utilize in his own study or laboratory microphotographic copies which he may have made personally or procured from one of the existing sources of supply.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-fourth Summer Meeting, Chicago, Illinois, September 1–3, 1941.

Twenty-sixth Annual Meeting, Bethlehem, Pa., December 29, 1941–January 2, 1942.

The following is a list of the Sections of the Association, with dates of those Section meetings which have been scheduled for 1941 and reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh.
ILLINOIS, Peoria, May 9–10.
INDIANA, Indianapolis, May 2–3.
IOWA, Indianola, April 25–26.
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, New Orleans, La., March 7–8.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Annapolis, Md., May.
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA, Lincoln, May.
NORTHERN CALIFORNIA, San Francisco, January 25.
OHIO, Columbus, April 3 or 4.
OKLAHOMA
PHILADELPHIA, Swarthmore, November 29.
ROCKY MOUNTAIN, April.
SOUTHEASTERN, Chapel Hill, N. C., March.
SOUTHERN CALIFORNIA, Redlands, March 8.
SOUTHWESTERN, Lubbock, Tex., April 28–29.
TEXAS, Denton, March 28–29.
UPPER NEW YORK STATE, Ithaca, May 3.
WISCONSIN, Beloit.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS, THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

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Numbers in black face type refer to problems, those in light face to pages.

NEWS AND NOTICES

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PERSONAL MENTION

This section contains the names of persons taking any active part in meetings, newly elected members, officers of the Association and of the various Sections, and persons mentioned in the department of News and Notices.

NECROLOGY